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# 平面グラフにおける辺遊離操作と辺連結度増大問題について

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あらまし 本論文では、$k$辺連結な平面グラフ $G$ (ただし、$k$は偶数あるいは $k=3$) の任意の点 $s$ において、グラフの $k$辺連結性と平面性を同時に保つような完全な辺遊離操作が存在することを示し、そのような辺遊離操作を求める $O(n^3 \log n)$ 時間のアルゴリズムを与える。ここで, $n$ は点の数。$k$が5以上の奇数の場合にはそのような辺遊離操作を持たないグラフの例が存在する。新しい辺遊離操作を用いると、外平面グラフに最小数の辺を加え平面性を保ったまま $k$辺連結に最適に増大させる問題を $O(n^3 \log n)$ 時間で解くことができる (ただし、$k$は偶数あるいは $k=3$).

**キーワード** 無向グラフ, 多重グラフ, 平面グラフ, 外平面グラフ, 辺連結度, 辺遊離操作, 最小カット

# Edge-Splitting and Edge-Connectivity Augmentation in Planar Graphs

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Abstract In this paper, we prove that, for a $k$-edge-connected planar graph $G$ where $k$ is an even integer or $k = 3$, there exists a complete splitting at $s$ such that the resulting graph $G'$ is still $k$-edge-connected and planar, and present an $O(n^3 \log n)$ time algorithm for finding such a splitting, where $n = |V|$. However, for every odd $k \geq 5$, there is a planar graph $G$ with a vertex $s$ which has no such complete splitting. As an application of this result, we show that the problem of augmenting the edge-connectivity of an outerplanar graph to an even integer can be solved in $O(n^3 \log n)$ time.

**key words** undirected graph, multigraph, planar graph, outerplanar graph, edge-connectivity, edge splitting, minimum cut

## 1 Introduction

Let $G = (V, E)$ stand for an undirected multigraph, where an edge with end vertices $u$ and $v$ is denoted by $(u, v)$. For a subset[1] $S \subseteq V$ in $G$, $G[S]$ denotes the subgraph induced by $S$. For two disjoint subsets $X, Y \subset V$, we denote by $E_G(X, Y)$ the set of edges $(u, v)$ with $u \in X$ and $v \in Y$, and by $c_G(X, Y)$ the number of edges in $E_G(X, Y)$. The set of edges $E_G(u, v)$ may alternatively be represented by a single *link* $(u, v)$ with multiplicity $c_G(u, v)$. In this way, we also represent a multigraph $G = (V, E)$ by an edge-weighted simple graph $N = (V, L_G, c_G)$ (called a *network*) with a set $V$ of vertices and a set $L_G$ of links weighted by $c_G : L_G \to Z^+$, where $Z^+$ is the set of non-negative integers. We denote $|V|$ by $n$, $|E|$ by $e$ and $|L_G|$ by $m$. A *cut* is defined as a subset $X$ of $V$ with $\emptyset \neq X \neq V$, and the *size* of the cut $X$ is defined by $c_G(X, V - X)$, which may also be written as $c_G(X)$. If $X = \{x\}$, $c_G(x)$ denotes the degree of vertex $x$. For a subset $X \subseteq V$, define its *inner-connectivity* by $\lambda_G(X) = \min\{c_G(X') \mid \emptyset \neq X' \subset X\}$. In particular, $\lambda_G(V)$ (i.e., the size of a minimum cut in $G$) is called the *edge-connectivity* of $G$. For a vertex $v \in V$, a vertex $u$ adjacent to $v$ is called a *neighbor* of $v$ in $G$. Let $\Gamma_G(v)$ denote the set of neighbors of $v$ in $G$.

Let $s \in V$ be a *designated vertex* in $V$. A cut $X$ is called *s-proper* if $\emptyset \neq X \subset V - s$. The size $\lambda_G(V - s)$ of a minimum $s$-proper cut is called the *s-based-connectivity* of $G$. Hence $\lambda_G(V) = \min\{\lambda_G(V - s), c_G(s)\}$. A *splitting operation at s* replaces two edges $(s, u)$ and $(s, v)$ incident to $s$ with a single edge $(u, v)$. A set of splitting operations at $s$ is called *complete* if there is no edge incident to $s$ in the resulting graph. A splitting at $s$ is *$(k, s)$-feasible* if $\lambda_{G'}(V - s) \geq k$ holds for the resulting graph $G'$. Lovász [6] showed the following important property:

**Theorem 1.1** [2, 6] *Let $G = (V, E)$ be a multigraph with a designated vertex $s \in V$ with even $c_G(s)$, and $k$ be an integer with $2 \leq k \leq \lambda_G(V - s)$. Then there is a complete $(k, s)$-feasible splitting.* □

Since a complete $(k, s)$-feasible splitting effectively reduces the number of vertices in a graph while preserving its $s$-based-connectivity, it plays an important role in solving many graph connectivity problems (e.g., see [1, 2, 8]).

In this paper, we prove a new type of extension of Lovász's edge-splitting theorem, aiming to solve the edge-connectivity augmentation problem with an additional constraint that preserves the planarity of a given planar graph. Firstly, we consider the following type of splitting; for a multigraph $G = (V, E)$ with a designated vertex $s$, let $\Gamma_G(s) = \{w_0, w_1, \ldots, w_{p-1}\}$ ($p = |\Gamma_G(s)|$) of neighbors of $s$, and assume that a cyclic order $\pi = (w_0, w_1, \ldots, w_{p-1})$ of $\Gamma_G(s)$ is given. We say that two edges $e_1 = (w_h, w_i)$ and $e_2 = (w_j, w_\ell)$ are *crossing* (with respect to $\pi$) if $e_1$ and $e_2$ are not adjacent and the four end vertices appear in the order of $w_h, w_j, w_i, w_\ell$ along $\pi$ (i.e., $h + a = j + b = i + c = \ell \pmod p$ holds for some $1 \leq c < b < a \leq p - 1$). A sequence of splittings at $s$ is called *noncrossing* if no two split edges resulting from the sequence are crossing. We prove that there always exists a complete and noncrossing $(k, s)$-feasible splitting for even integers $k$, and such a splitting can be found in $O(n^2(m + n \log n))$ time.

Next we consider a planar multigraph $G = (V, E)$ with a vertex $s \in V$ of even degree. A complete splitting at $s$ is called *planarity-preserving* if the resulting graph from the splitting remains planar. Based on the result of noncrossing splitting, we prove that, if $k$ is an even integer with $k \leq \lambda_G(V - s)$, then there always exists a complete $(k, s)$-feasible and planarity-preserving splitting, and the splitting can be found in $O(n^3 \log n)$ time. For $k = 3$, we prove by a separate argument that there exists a complete $(k, s)$-feasible and planarity-preserving splitting if the resulting graph is allowed to be re-embedded in the plane.

**Example 1** **(a)** Fig. 1(a) shows a graph $G_1 = (V, E)$ with $c_{G_1}(s, w_i) = 1$ and $c_{G_1}(w_i, w_{i+1}) = a$, $0 \leq i \leq 3$ for a given integer $a \geq 1$. Clearly, $\lambda_{G_1}(V - s) = k$ for $k = 2a + 1$. For a cyclic order $\pi = (w_0, w_1, w_2, w_3)$, $G_1$ has a unique complete $(k, s)$-feasible splitting (i.e., splitting pair of $(s, w_0), (s, w_2)$ and a pair of $(s, w_1), (s, w_3)$), which is crossing with respect to $\pi$. This implies that, for every odd $k \geq 3$, there is a graph $G$ with a designated vertex $s$ and a cyclic order of $\Gamma_G(s)$ which has no complete and noncrossing $(k, s)$-

[1] A singleton set $\{x\}$ may be simply written as $x$, and " $\subset$ " implies proper inclusion while " $\subseteq$ " means " $\subset$ " or " $=$ ".

feasible splitting. Note that the planar $G_1$ has a complete and planarity-preserving $(k, s)$-feasible splitting (by putting one of the split edges in the inner area of cycle $C_1 = \{w_0, w_1, w_2, w_3\}$).

**(b)** Fig. 1(b) shows a planar graph $G_2 = (V, E)$ with $c_{G_2}(w_i, w_{i+1}) = a \pmod{12}$ for $0 \le i \le 11$ and $c_{G_2}(e) = 1$ otherwise for an integer $a \ge 1$, which satisfies $\lambda_{G_2}(V - s) = k$ for $k = 2a + 1$. The $G_2$ has a unique complete $(k, s)$-splitting, which is not planarity-preserving unless the embedding of subgraph $G_2[V - s]$ is not changed; if $G_2[V - s]$ is re-embedded in the plane so that block components $\{w_2, w_3, w_4\}$ and $\{w_8, w_9, w_{10}\}$ of $G_2[V - s]$ are flipped and two vertices $w_3$ and $w_9$ share the same inner face, then the complete $(k, s)$-splitting is now planarity-preserving. From this, we see that for every odd $k \ge 3$, there is a planar graph $G$ with a designated vertex $s$ which has no complete and planarity-preserving $(k, s)$-feasible splitting (unless the embedding of $G$ is re-embedded).

**(c)** Let $a \ge 2$ be an integer, and consider the graph $G_3 = (V, E)$ in Fig. 1(c), where $c_{G_3}(w_i, w_{i+1}) = a$ for $i \in \{1, 7\}$, $c_{G_3}(w_i, w_{i+1}) = a \pmod{12}$ for $i \in \{0, 1, \ldots, 11\} - \{1, 7\}$, and $c_{G_3}(e) = 1$ otherwise. Clearly, $\lambda_{G'_3}(V' - s) = k$ for $k = 2a + 1$ $(\ge 5)$. It is easily observed that the unique complete $(k, s)$-feasible splitting is not planarity-preserving for any choice of re-embedding of $G_3$ in the plane. This implies that for every odd $k \ge 5$, there exists a graph which has no complete and planarity-preserving $(k, s)$-feasible splitting even if re-embedding after splitting is allowed. □

# 2 Preliminaries

## 2.1 Computing $s$-based connectivity

The vertex set $V$ of a multigraph $G = (V, E)$ are denoted by $V(G)$. We say that a cut $X$ *separates* two disjoint subsets $Y$ and $Y'$ of $V$ if $Y \subseteq X \subseteq V - Y'$ (or $Y' \subseteq X \subseteq V - Y$ ). The *local edge-connectivity* $\lambda_G(x, y)$ for two vertices $x, y \in V$ is defined to be the minimum size of a cut in $G$ that separates $x$ and $y$. A cut $X$ *crosses* another cut $Y$ if none of subsets $X \cap Y$, $X - Y$, $Y - X$ and $V - (X \cup Y)$ is empty.

An ordering $v_1, v_2, \ldots, v_n$ of all vertices in $V$ is called a *maximum adjacency* (MA) *ordering* in $G$ if it satisfies $c_G(\{v_1, v_2, \ldots, v_i\}, v_{i+1}) \ge c_G(\{v_1, v_2, \ldots, v_i\}, v_j)$, $1 \le i < j \le n$.

**Lemma 2.1** [7] *Let $G = (V, E)$ be a multigraph, and $v_1$ be a vertex in $V$.*
(i) *An MA ordering $v_1, v_2, \ldots, v_n$ of vertices in $G$ can be found in $O(m + n \log n)$ time.*
(ii) *The last two vertices $v_{n-1}$ and $v_n$ satisfy $\lambda_G(v_{n-1}, v_n) = c_G(v_n)$.* □

**Lemma 2.2** *For a multigraph $G = (V, E)$ with a designated vertex $s \in V$, a cut $X^*$ such that $c_G(X^*) = \lambda_G(V - s) < \lambda_G(X^*)$ can be computed in $O(n(m + n \log n))$ time.* □

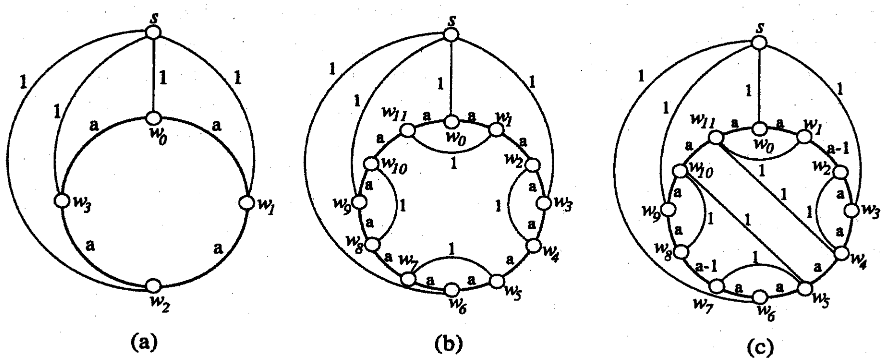

Figure 1: Example of three planar graphs (a) $G_1$, (b) $G_2$ and (c) $G_3$.

### 2.2 Splitting edges for a pair of neighbors

Given a multigraph $G = (V, E)$, a designated vertex $s \in V$, vertices $u, v \in \Gamma_G(s)$ (possibly $u = v$) and a non-negative integer $\delta \leq \min\{c_G(s,u), c_G(s,v)\}$, we construct graph $G' = (V, E')$ from $G$ by deleting $\delta$ edges from $E_G(s,u)$ and $E_G(s,u)$, respectively, and adding new $\delta$ edges to $E_G(u,v)$. We say that $G'$ is obtained from $G$ by *splitting* $\delta$ pairs of edges $(s,u)$ and $(s,v)$ by *size* $\delta$), and denote the resulting graph $G'$ by $G/(u,v,\delta)$. Clearly, for any $s$-proper cut $X$, we see that

$$c_{G/(u,v,\delta)}(X) = \begin{cases} c_G(X) - 2\delta & \text{if } u, v \in X \\ c_G(X) & \text{otherwise.} \end{cases} \quad (2.1)$$

Given an integer $k$ satisfying $0 \leq k \leq \lambda_G(V-s)$, we say that splitting $\delta$ pairs of edges $(s,u)$ and $(s,v)$ is *$(k,s)$-feasible* if $\lambda_{G/(u,v,\delta)}(V-s) \geq k$.

For an integer $k$, let $\Delta_G(u,v,k)$ be the maximum $\delta$ such that splitting edges $(s,u)$ and $(s,v)$ with size $\delta$ is $(k,s)$-feasible in $G$. In this subsection, we show how to compute $\Delta_G(u,v,k)$. An $s$-proper cut $X$ is called *$(k,s)$-semi-critical* in $G$ if it satisfies $c_G(s,X) > 0$, $k \leq c_G(X) \leq k+1$ and $\lambda_G(X) \geq k$.

An algorithm, called MAXSPLIT$(u,v,k)$, for computing $\Delta_G(u,v,k)$ is described as follows.

**1.** Let $\delta_{max} = \min\{c_G(s,u), c_G(s,v)\}$ if $u \neq v$, and $\delta_{max} = \lfloor c_G(s,u)/2 \rfloor$ if $u = v$, and let $G_{max} = G/(u,v,\delta_{max})$.

**2.** Compute $\lambda_{G_{max}}(V-s)$ and an $s$-proper cut $X$ with $c_{G_{max}}(X) = \lambda_{G_{max}}(V-s) < \lambda_{G_{max}}(X)$ (such $X$ exists by Lemma 2.2). If $\lambda_{G_{max}}(V-s) \geq k$, then $\Delta_G(u,v,k) = \delta_{max}$, where at least one of $u$ and $v$ is no longer a neighbor of $s$ in $G_{max}$ in the case $u \neq v$, or $c_{G_{max}}(s,u) \leq 1$ in the case $u = v$.

**3.** If $k - \lambda_{G_{max}}(V-s) \geq 1$, then $u, v \in X$ (for otherwise $c_G(X) = c_{G_{max}}(X) < k$ would hold). Output $\Delta_G(u,v,k) = \delta_{max} - \lceil \frac{1}{2}(k - \lambda_{G_{max}}(V-s)) \rceil$ and the $s$-proper cut $X$ as such a $(k,s)$-semi-critical cut with $u, v \in X$.

The correctness of step 2 is clear. In step 3, we see from (2.1) that $G' = G/(u,v,\delta)$ with $\delta = \delta_{max} - \lceil \frac{1}{2}(k - \lambda_{G_{max}}(V-s)) \rceil$ satisfies $\lambda_{G'}(V-s) = k$ or $k+1$. This implies that $\Delta_G(u,v,k) = \delta$. We show that the $X$ has a property that

$$c_{G'}(Z) > c_{G'}(X) \text{ for any } Z \text{ with } u, v \in Z \subset X,$$

where we call such a $(k,s)$-semi-critical cut $X$ with $u, v \in X$ *admissible* (with respect to $u, v$) in $G'$. For any $Z$ with $u, v \in Z \subset X$, we have $c_{G'}(Z) = c_{G_{max}}(Z) + 2\lceil \frac{1}{2}(k - \lambda_{G_{max}}(V-s)) \rceil > c_{G'}(X)$, since $\lambda_{G_{max}}(X) > \lambda_{G_{max}}(V-s)$ implies $c_{G_{max}}(Z) > c_{G_{max}}(X)$. By summarizing this, we have the next result.

**Lemma 2.3** *For a multigraph $G = (V, E)$ with a designated vertex $s \in V$, and vertices $u, v \in \Gamma_G(s)$ (possibly $u = v$), let $k$ be a nonnegative integer with $k \leq \lambda_G(V-s)$, and let $G' = G/(u,v,\delta)$ for $\delta = \Delta_G(u,v,k)$. Then:*
(i) *If $c_{G'}(s,u) > 0$ and $c_{G'}(s,v) > 0$ in the case $u \neq v$ or if $c_{G'}(s,u) \geq 2$ in the case $u = v$, then $G'$ has an admissible cut $X$.*
(ii) *The cut $X$ in* (i) *(if any) and $\Delta_G(u,v,k)$ can be computed in $O(mn + n^2 \log n)$ time.* □

## 3 Noncrossing Edge Splitting

For a cyclic order $\pi = (w_0, w_1, \ldots, w_{p-1})$ of $\Gamma_G(s)$, a sequence of splittings at $s$ is called *noncrossing* (with respect to $\pi$) if no two split edges $(w_h, w_i)$ and $(w_j, w_\ell)$ are crossing with respect to $\pi$ (see Section 1 for the definition). In this section, we show that for any even $k \leq \lambda_G(V-s)$, there always exists a complete and noncrossing $(k,s)$-splitting. However, as observed in Example 1(a), for every odd $k \geq 3$, there is a graph that has no such splitting.

### 3.1 $(k,s)$-semi-critical collections

Before computing a complete $(k,s)$-feasible splitting, we first find a family $\mathcal{X}$ of subsets of $V-s$ (by performing some noncrossing edge splittings at $s$) as follows. For a multigraph $G = (V, E)$ and $s \in V$, a family $\mathcal{X} = \{X_1, X_2, \ldots, X_r\}$ of disjoint subsets $X_i \subset V - s$ is called a *collection* in $V-s$. A collection $\mathcal{X}$ may be empty. A collection $\mathcal{X}$ is called *covering* if $\sum_{i=1}^{r} c_G(s, X_i) = c_G(s)$. A collection $\mathcal{X}$ in $V-s$ is called *$(k,s)$-semi-critical* in $G$ either if $\mathcal{X} = \emptyset$ or if all $X_i \in \mathcal{X}$ are $(k,s)$-semi-critical. We can easily see that a $(k,s)$-semi-critical covering collection in $G$ with $c_G(s) > 0$ satisfies $|\mathcal{X}| \geq 2$ [8].

Let $X$ be an $s$-proper cut with $c_G(X) \leq k+1$. Clearly, splitting two edges $(s,u)$ and $(s,v)$ such that $u, v \in X$ is not $(k,s)$-feasible. Then the size of any cut $Z \subseteq X$ remains unchanged after any

$(k,s)$-feasible splitting in $G$. We say that two $s$-proper cuts $X$ and $Y$ *$s$-cross* each other if $X$ and $Y$ cross each other and $c_G(s, X \cap Y) > 0$. It is not difficult to prove the following properties by using submodularity of cut function $c_G$ (the detail is omitted).

**Lemma 3.1** *Let $G = (V, E)$ be a multigraph with a designated vertex $s$, and $k$ be an integer with $k \le \lambda_G(V - s)$. Then:*
(i) *If two $(k,s)$-semi-critical cuts $X$ and $Y$ $s$-cross each other, then $c_G(X) = c_G(Y) = k+1$, $c_G(X - Y) = c_G(Y - X) = k$ and $c_G(X \cap Y, V - (X \cup Y)) = 1$.*
(ii) *Let $X_i$ be an admissible cut with respect to $u, u' \in V - s$ (possibly $u = u'$), and $Y$ be a $(k,s)$-semi-critical cut. If $X$ and $Y$ cross each other, then $c_G(X) = c_G(Y) = k+1$ and $c_G(Y - X) = k$.*
(iii) *Let $X_i$ (resp., $X_j$) be admissible cuts with respect to $u_i, u'_i$ (resp., with respect to $u_j, u'_j$), where possibly $u_i = u'_i$ or $u_j = u'_j$ holds, but $u_j \ne u_i \ne u'_j$ and $u_j \ne u'_i \ne u'_j$. Then two cuts $X_i$ and $X_j$ do not cross each other.* □

We now describe an algorithm, called COLLECTION, which computes a $(k,s)$-semi-critical covering collection $\mathcal{X}$ in a graph $G^*$ obtained from $G$ by a noncrossing sequence of $(k,s)$-feasible splittings. Let $\pi = (w_0, w_1, \ldots, w_{p-1})$ be a cyclic order of $\Gamma_G(s)$ and initialize $\mathcal{X}$ to be $\emptyset$.

**1. for** each $w_i$, $i := 0, \ldots, p-1$ **do**
**if** $w_i$ is not in any cut $X \in \mathcal{X}$ **then** execute MAXSPLIT$(w_i, w_i, k)$ to compute $G' = G/(w_i, w_i, \delta)$ with $\delta = \Delta_G(w_i, w_i, k)$ and an admissible cut $X_{w_i}$ in $G'$ (if $c_{G'}(s, w_i) \ge 2$); let $G := G'$;
**if** $c_G(s, w_i) \ge 2$ **then** $\mathcal{X} := \mathcal{X} \cup \{X_{w_i}\}$, discarding all $X \in \mathcal{X}$ with $X \subset X_{w_i}$ from $\mathcal{X}$.
**end** { for }
**2. for** each $w_i$ such that $c_G(s, w_i) = 1$, $i := 0, \ldots, p-1$ **do**
**if** $w_i$ is not in any cut $X \in \mathcal{X}$ **then** execute MAXSPLIT$(w_i, w_j, k)$ for $w_i$ and its nearest neighbor $w_j$ in the current $G$ to compute $G' = G/(w_i, w_j, \delta)$ with $\delta = \Delta_G(w_i, w_j, k)$ and an admissible cut $X_{w_i}$ in $G'$ (if $c_{G'}(s, w_i) = 1$); let $G := G'$;
**if** $c_G(s, w_i) = 1$ **then** $\mathcal{X} := \{X - X_{w_i} \mid c_G(s, X - X_{w_i}) > 0,\ X \in \mathcal{X}\} \cup \{X_{w_i}\}$. **else** (if $c_G(s, w_i) = 0$) remove any cut $X$ with $c_G(s, X) = 0$ from $\mathcal{X}$.
**end** { for }
Output $G^* := G$. □

Clearly, the resulting sequence of splitting is $(k,s)$-feasible and noncrossing.

**Lemma 3.2** *Algorithm* COLLECTION *correctly computes a $(k,s)$-semi-critical covering collection $\mathcal{X}$ in the output graph $G^*$.*

**Proof:** Let $\mathcal{X}$ be the set of cuts obtained after step 1. If two cuts $X_{w_i}, X_{w_j} \in \mathcal{X}$ $(0 \le i < j \le p-1)$ has a common vertex $v$, then $w_j \notin X_{w_i}$ and $X_{w_i} - X_{w_j} \ne \emptyset$ (otherwise, $X_{w_i}$ must have been discarded). However, this implies that $X_{w_i}$ and $X_{w_j}$ cross each other, contradicting Lemma 3.1(iii). Thus, the $\mathcal{X}$ is a $(k,s)$-semi-critical collection.

Now we prove by induction that $\mathcal{X}$ is a $(k,s)$-semi-critical collection during step 2. Assume that MAXSPLIT$(w_i, w_j, k)$ is executed to compute $G' = G/(w_i, w_j, \delta)$ with $\delta = \Delta_G(w_i, w_j, k)$. If $c_{G'}(s, w_i) = 0$, then a cut $X \in \mathcal{X}$ with $w_j \in X$ may satisfy $c_{G'}(s, X) = 0$ after the splitting. However, such a cut will be removed from $\mathcal{X}$. If $c_{G'}(s, w_i) = 1$, then an admissible cut $X_{w_i}$ in $G'$ is found. Clearly, any $X \in \mathcal{X}$ satisfies one of the followings: (i) $X \cap X_{w_i} = \emptyset$, (ii) $X \subset X_{w_i}$, and (iii) $X \cap X_{w_i} \ne \emptyset \ne X - X_{w_i}$. Since $\mathcal{X}$ is updated to $\{X - X_{w_i} \mid c_G(s, X - X_{w_i}) > 0,\ X \in \mathcal{X}\} \cup \{X_{w_i}\}$, it is sufficient to show that $c_{G'}(X - X_{w_i}) = k$ holds in the case (iii) (note that $\lambda_{G'}(X - X_{w_i}) \ge k$ follows from $\lambda_{G'}(X) \ge k$). Since two cuts $X$ and $X_{w_i}$ cross each other in the case (iii), $c_{G'}(X - X_{w_i}) = k$ follows from Lemma 3.1(ii). This proves that $\mathcal{X}$ remains to be a $(k,s)$-semi-critical collection, which becomes covering after step 2. □

## 3.2 Algorithm for noncrossing edge-splitting

In this subsection, $k$ is assumed to be a positive even integer. We can prove the next property by Lemma 3.1(i) and the evenness of $k$ (the detail is omitted).

**Lemma 3.3** *Let $G = (V, E)$ be a multigraph with a designated vertex $s$, and $k$ be an even integer with $k \le \lambda_G(V - s)$. Further, let $X$ be a $(k,s)$-semi-critical cut, and $Y$ and $Y'$ be $(k,s)$-semi-critical cuts with $Y \cap Y' = \emptyset$. Then $X$ can $s$-cross at most one of $Y$ and $Y'$.* □

Using the lemma, we now describe an algorithm that constructs a complete and noncrossing $(k,s)$-feasible splitting from a given $(k,s)$-semi-critical covering collection $\mathcal{X}$ in a graph $G$.

If $s$ has at most three neighbors, then any complete $(k,s)$-feasible splitting is noncrossing (with respect to any cyclic order of $\Gamma_G(s)$) and such a splitting can be found by applying MAXSPLIT at most three times. In what follows, we assume that $|\Gamma_G(s)| \geq 4$.

First, we define a notion of segment. For a given covering collection $\mathcal{X}$ with $|\mathcal{X}| \geq 2$ in a multigraph $G$ with a designated vertex $s$ and a cyclic order $\pi = (w_0, w_1, \ldots, w_{p-1})$ of $\Gamma_G(s)$, a subset $P \subset \Gamma_G(s)$ of neighbors of $s$ which are consecutive in the cyclic order is called *segment* if there is a cut $X \in \mathcal{X}$ with $P \subset X$ such that $P$ is maximal subject to this property. Note that there may be two segments $P$ and $P'$ with $P \cup P' \subseteq X$ for the same cut $X \in \mathcal{X}$. A segment $P$ with $|P| = 1$ is called *trivial*. We now describe the two cases.

**Case-1:** There is a nontrivial segment $P = \{w_i, w_{i+1}, \ldots, w_j\}$ (with respect to $\mathcal{X}$). We execute MAXSPLIT$(w_{i-1}, w_i, k)$ and then MAXSPLIT$(w_j, w_{j+1}, k)$. If one of $w_{i-1}, w_i, w_j, w_{j+1}$ is no longer a neighbor of $s$ in the resulting graph $G''$, then the number of neighbors of $s$ decreases at least by one (in this case, a cut $X \in \mathcal{X}$ with $c_{G''}(s,X) = 0$ (if any) is removed from $\mathcal{X}$). Let us consider the case where all of $w_{i-1}, w_i, w_j, w_{j+1}$ remain neighbors of $s$ in $G''$. Thus, the resulting graph $G''$ has admissible cuts $X_i$ and $X_j$ (with respect to $w_{i-1}, w_i$ and $w_j, w_{j+1}$, respectively). By Lemma 3.1(iii), two cuts $X_i$ and $X_j$ do not cross each other. Let $Y_1$, $Y_2$ and $Y_3$ be the cuts in $\mathcal{X}$ such that $w_{i-1} \in Y_1$, $\{w_i, \ldots, w_j\} \subseteq Y_2$ and $w_{j+1} \in Y_3$ (possibly $Y_1 = Y_3$). There are two subcases (a) $X_i \cap X_j = \emptyset$ and (b) $X_i \subseteq X_j$ or $X_j \subseteq X_i$.

**(a)** $X_i \cap X_j = \emptyset$. We prove that $Y_1 \neq Y_3$ and $Y_1 \cup Y_2 \cup Y_3 \subseteq X_i \cup X_j$. Since $(k,s)$-semi-critical cuts $X_i$ and $Y_2$ $s$-cross each other, $c_{G''}(Y_2 - X_i) = c_{G''}(X_i - Y_2) = k$ by Lemma 3.1(i). Note that $Y_2 - X_i$ is a $(k,s)$-semi-critical cut. Thus $Y_2 - X_i$ cannot cross another admissible cut $X_j$ (otherwise $c_{G''}(Y_2 - X_i) = k$ would contradict Lemma 3.1(ii)), and hence $Y_2 \subset X_i \cup X_j$. By Lemma 3.3, $X_i$ which already crosses $Y_2$ cannot $s$-cross $Y_1$, and thus $Y_1 \subset X_i$. Similarly, we have $Y_3 \subset X_j$. Therefore, $Y_1 \neq Y_3$ and $Y_1 \cup Y_2 \cup Y_3 \subseteq X_i \cup X_j$. There may be some cut $X \in \mathcal{X} - \{Y_1, Y_2, Y_3\}$ which crosses $X_i$. By Lemma 3.1(iii), $c_{G''}(X - X_i) = k$. We see that $c_{G''}(s, X - X_i) = c_{G''}(s, X) \geq 1$, because if $c_{G''}(X \cap X_i) \geq 1$ (i.e., $X$ and $X_i$ $s$-cross each other) then $c_{G''}(X_i - X) = k < k+1 = c_{G''}(X_i)$ by Lemma 3.1(i), contradicting the admissibility of $X_i$ (note $\{w_{i-1}, w_i\} \subseteq X_i - X$). Thus $X - X_i$ is a $(k,s)$-semi-critical cut in $G''$. Similarly, if some cut $X \in \mathcal{X} - \{Y_1, Y_2, Y_3\}$ crosses $X_j$ then $X - X_j$ is a $(k,s)$-semi-critical cut in $G''$. Therefore, we can update $\mathcal{X}$ by $\mathcal{X} := \{X - X_i - X_j \mid c_G(s, X - X_i - X_j) > 0,\ X \in \mathcal{X}\} \cup \{X_i, X_j\}$.

**(b)** $X_i \subseteq X_j$ or $X_j \subseteq X_i$. Without loss of generality, assume $X_j \subseteq X_i$. Since $c_{G''}(s, Y_2 \cap X_i) \geq 2$ holds by $w_i, w_j \in Y_2 \cap X_i$, we see by Lemma 3.1(i) that $Y_2$ cannot cross $X_i$ (hence, $Y_2 \subset X_i$). By Lemma 3.3, at most one of $Y_1$ and $Y_3$ can $s$-cross $X_i$. Thus $Y_1 \subset X_i$ or $Y_3 \subset X_i$. For any cut $X \in \mathcal{X} - \{Y_3\}$ which crosses $X_i$, we can show that $X - X_i$ is a $(k,s)$-semi-critical cut in $G''$ using similar reasoning as for Case-1(a). Therefore, we can update $\mathcal{X}$ by $\mathcal{X} := \{X - X_i \mid c_G(s, X - X_i) > 0,\ X \in \mathcal{X}\} \cup \{X_i\}$.

Note that the number of cuts in $\mathcal{X}$ in cases (a) and (b) decreases at least by one after updating.

**Case-2:** All segments are trivial. We choose a neighbor $w_i$ of $s$ and the neighbor $w_j$ of $s$ nearest to $w_i$, and execute MAXSPLIT$(w_i, w_j, k)$. Assume that MAXSPLIT$(w_i, w_j, k)$ finds an admissible cut $X_i$ (otherwise, the number of neighbors of $s$ decreases at least by one). Let $Y_1$ and $Y_2$ be the cuts in $\mathcal{X}$ which contain $w_i$ and $w_j$, respectively. We see that $Y_1 \subset X_i$ or $Y_2 \subset X_i$, because otherwise both $Y_1$ and $Y_2$ would $s$-cross $X_i$ (contradicting Lemma 3.3). If $Y_1$ $s$-crosses $X_i$, then we see that $Y_1 - X_i$ is a $(k,s)$-semi-critical cut if $c_{G''}(s, Y_1 - X_i) \geq 1$. The case where $Y_2$ $s$-crosses $X_i$ is similar. For any cut $X \in \mathcal{X} - \{Y_1, Y_2\}$ which crosses $X_i$, we can show that $X - X_i$ is a $(k,s)$-semi-critical cut in $G''$ using similar reasoning as for Case-1(a). We update $\mathcal{X}$ by $\mathcal{X} := \{X - X_i \mid c_G(s, X - X_i) \geq 1,\ X \in \mathcal{X}\} \cup \{X_i\}$. In this case, the number of cuts in $\mathcal{X}$ never increases, but it may not decrease, either. However, $X_i$ contains a nontrivial segment in the new $\mathcal{X}$, and we can apply the above argument of Case-1.

By applying the above argument to Case-1, at least one vertex is no longer a neighbor of $s$ or the number of cuts in a collection $\mathcal{X}$ is decreased at least by one. After applying the argument of Case-2, at least one vertex is no longer a neighbor of $s$ or a nontrivial segment is created. Therefore, by executing MAXSPLIT at most $4(|\Gamma_G(s)| + |\mathcal{X}|) = O(|\Gamma_G(s)|)$ times, we can obtain a complete $(k, s)$-feasible splitting of a given graph $G$, which is obviously noncrossing.

**Theorem 3.1** *Given a multigraph $G = (V, E)$ with a designated vertex $s \in V$ of even degree, a positive even integer $k \leq \lambda_G(V - s)$, and a cyclic order $\pi$ of neighbors of $s$, a complete and noncrossing $(k, s)$-feasible splitting can be found in $O(|\Gamma_G(s)|n(m + n \log n))$ time.* □

# 4 Planarity-preserving splitting

In this section, we assume that a given graph $G$ with a designated vertex $s$ of even degree and an integer $k \leq \lambda_G(V - s)$ is *planar*, and consider whether there is a complete and planarity-preserving $(k, s)$-feasible splitting. We prove that such splitting always exists if $k$ is even or $k = 3$, but may not exist if $k$ is odd and $k \geq 5$, as observed in Example 1(c). We initially fix an embedding $\psi$ of $G$ in the plane, and let $\pi_\psi$ be the order of neighbors of $s$ that appear around $s$ in the embedding $\psi$ of $G$.

## 4.1 The case of even $k$

Clearly, a complete splitting at $s$ is planarity-preserving if it is noncrossing with respect to $\pi_\psi$. Therefore, if $k$ is an even integer, then we establish the next theorem by Theorem 3.1 and the fact that $m$ is $O(n)$ in a planar graph $G$.

**Theorem 4.1** *Given a planar multigraph $G = (V, E)$ with a designated vertex $s \in V$ of even degree, and a positive even integer $k \leq \lambda_G(V-s)$, there exists a complete and planarity-preserving $(k, s)$-feasible splitting (which also preserves the embedding of $G[V - s]$ in the plane), and such splitting can be found in $O(|\Gamma_G(s)|n^2 \log n)$ time.* □

## 4.2 The case of $k = 3$

For $k = 3 \leq \lambda_G(V - s)$, we can prove that there is a complete and planarity-preserving $(k, s)$-feasible splitting. However, as observed in Example 1(b), in this case we may need to re-embed the subgraph $G[V - s]$ in the plane to obtain such a splitting.

**Theorem 4.2** *Given a planar multigraph $G = (V, E)$ with a designated vertex $s \in V$ of an even degree, and $\lambda_G(V - s) \geq 3$, there exists a complete and planarity-preserving $(3, s)$-feasible splitting, and such a splitting can be found in $O(n^2)$ time.*

**Proof:** (Sketch) By a procedure similar to COLLECTION, for the cyclic order $\pi_\psi$ of $\Gamma_G(s)$, we can find a noncrossing sequence of $(k, s)$-feasible splittings (which may not be complete) such that the resulting graph $G^*$ satisfies the following (i)-(ii):
(i) $\lambda_{G^*[V-s]}(V-s) = 2$ (i.e., the induced subgraph $G^*[V-s]$ is 2-edge-connected), where a minimum cut $X$ in $G^*[V - s]$ is called a *2-cut*, and is called *minimal* if $c_{G^*[V-s]}(Z) > 2$ for all nonempty and proper subsets $Z \subset X$.
(ii) There is a bijection between the $M(G^*[V-s])$ of all minimal 2-cuts in $G^*[V - s]$ and the set $E_{G^*}(s)$ in the following sense. For each edge $e = (s, w) \in E_{G^*}(s)$ there is a minimal 2-cut $X$ with $w_i \in X$. Conversely, for any minimal 2-cut $X$, there is exactly one edge $e = (s, w) \in E_{G^*}(s)$ such that $w \in X$.

It is known that all 2-cuts in a 2-edge-connected graph can be represented by a cactus structure, which is obtained by contracting each 2-component in the graph. Each minimal 2-cut corresponds to a leaf vertex (a vertex of degree 2) in the cactus. Thus, the remaining task to find a complete $(3, s)$-feasible splitting in $G^*$ is to add a set of new edges connecting $|E_{G^*}(s)|$ pair of leaf vertices in a cactus structure of $G^*[V - s]$ to destroy all 2-cuts in $G^*$. However, the corresponding complete splitting in $G^*$ may not preserve the planarity. In this case, we can re-embed the graph $G^*[V-s]$ so that the splitting preserves the planarity in the resulting embedding. Further details are omitted (see [9] for the details). □

# 5 Augmenting Edge-Connectivity of Outerplanar Graphs

Given a multigraph $G = (V, E)$ and a positive integer $k$, the $k$-edge-connectivity (resp., $k$-vertex-connectivity) augmentation problem asks to find a minimum number of new edges to be added to $G$ such that the augmented graph becomes $k$-edge-connected (resp., $k$-vertex-connected). Watanabe and Nakamura [10] proved that the $k$-edge-connectivity augmentation problem for general $k$ is polynomially solvable. In such applications as graph drawing (see [3]), a planar graph $G$ is given, and we may want to augment its edge- (or vertex-) connectivity optimally while preserving its planarity. Kant and Boldlaender [4] proved that the planarity-preserving version of 2-vertex-connectivity augmentation problem is NP-hard. Kant [5] also showed that, if a given graph $G$ is outerplanar, then the planarity-preserving versions of both the 2-edge-connectivity and 2-vertex-connectivity can be solved in linear time. For a planar graph $G$, let $\gamma_k(G)$ (resp., $\tilde{\gamma}_k(G)$) denote the minimum number of new edges to be added to $G$ so that the resulting graph $G'$ becomes $k$-edge-connected (resp., so that the resulting graph $G''$ becomes $k$-edge-connected and remains planar). Clearly, $\gamma_k(G) \leq \tilde{\gamma}_k(G)$ for any planar graph $G$ and $k \geq 1$. From the results in the preceding sections, we can show the next result.

**Theorem 5.1** *Let $G = (V, E)$ be an outerplanar graph. If $k \geq 0$ is an even integer or $k = 3$, then $\gamma_k(G) = \tilde{\gamma}_k(G)$ and the planarity-preserving version of the k-edge-connectivity augmentation problem can be solved in $O(n^2(m+n\log n))$ time.*

**Proof:** (Sketch) Based on Theorems 4.1 and 4.2, we can apply the approach of Cai and Sun [1] (also see [2]) for solving the $k$-edge-connectivity augmentation problem by using the splitting algorithm (see [9] for details). □

Furthermore, for every odd integer $k \geq 5$, there is an outerplanar graph $G$ such that $\gamma_k(G) < \tilde{\gamma}_k(G)$. Consider the graph $G'_3$ obtained from the graph $G_3$ in Example 1(c) by deleting $s$ and the edges in $E_{G'_3}(s)$. It is easy to see that $\gamma_k(G'_3) = 2 < 3 = \tilde{\gamma}_k(G'_3)$.

**Remark:** Given an undirected outerplanar network $N = (V, L, c)$ and a real $k > 0$, we consider the $k$-edge-connectivity augmentation problem which asks how to augment $N$ by increasing link weights and by adding new links so that the resulting network $N' = (V, L \cup L', c')$ is $k$-edge-connected and remains planar while minimizing $\sum_{e\in L}(c'(e) - c(e)) + \sum_{e\in L'} c'(e)$, where $c$ and $c'$ are allowed to be nonnegative reals. It is not difficult to observe that this problem can be solved in $O(n^2(m+n\log n))$ time by the argument given so far in this paper. (It would be interesting to see whether the problem can be formulated as a linear programming or not; if the planarity is not necessarily preserved then the problem is written as a linear programming.)

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