パラメトリックなポリマトロイドとその幾何学的応用

加藤 直樹（京都大学），玉木 久夫（明治大学），徳山 豪（日本 IBM）

要旨

本論文では、整数値ポリマトロイドに関するパラメトリックな最適基底の数を、重みがパラメターに関して線形である場合に解析し、漸近的に最適な理論的上界を与えた。この結果は、計算幾何学における最近の注目すべき結果である Tamal Dey による k セットの数及びアレンジメントの k レベル (図 1) の複雑度の $O(k^{1/3}n)$ の上界、David Eppstein による線形パラメトリックマトロイドの基底の数の最適下界、アレンジメントの at-most-k レベルに対する古典的な$\Theta(kn)$ の最適下界の３つを統一し、一般化したものになっている。更に、幾何学的応用として、 Welzl によるアレンジメントの多重レベルの複雑度の上界を改良し、さらに平面点集合の平行線による等分の個数の上界を与えている。

Parametric Polymatroid Optimization and Its Geometric Applications

Naoki Katoh (Kyoto University) , Hisao Tamaki (Meiji University) ,
Takeshi Tokuyama (IBM Tokyo Research Laboratory)

Abstract

We give an optimal bound on the number of transitions of the minimum weight base of an integer valued parametric polymatroid. This generalizes and unifies Tamal Dey's $O(k^{1/3}n)$ upper bound on the number of k-set (and complexity of k-level of an arrangement), David Eppstein's lower bound on the transition of the minimum weight base of a parametric matroid, and also the $\Theta(kn)$ bound for the complexity of at-most-k level (union of i-levels for $i = 1, 2, .., k$) of the arrangement. As applications, we improve Welzl's upper bound of the sum of complexities of multiple levels, and considered the number of different equal-sized-bucketing of a planar point set with parallel partition lines.

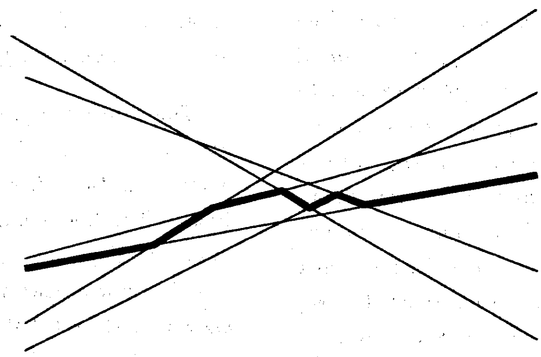

図 1: k-level of an arrangement of lines ($k = 3$)

1 Introduction

A *polymatroid* (to be precise, integer valued polymatroid) is a pair (E,r) of a set $E = \{1,2,..,n\}$ and the *rank function* r, which assigns nonnegative integers to $X \in 2^E$ and satisfies the following three conditions:
(a) $r(\emptyset) = 0$;
(b) Monotonicity: $X \subset Y$ implies $r(X) \le r(Y)$; and
(c) Submodularity:
$f(X) + f(Y) \ge f(X \cup Y) + f(X \cap Y)$ for every pair $X, Y \in 2^E$

$r(X)$ is called the rank of X, and $r(E)$ is called the rank of the polymatroid. A nonnegative integral vector $v = (v_1, ..., v_n)$ is called an independent vector of the polymatroid (E,r) if $\sum_{j\in X} v_j \le r(X)$ for every $X \in 2^E$. We often say that v_i is the *multiplicity* of i in the vector. A *base* of the polymatroid is an independent vector satisfying $\sum_{j\in E} v_j = r(E)$. Let $\mathcal{B}$ be the set of bases. Let $U = U(r) = \max_{i\in E} r(\{v_i\})$, which is the largest multiplicity in the polymatroid.

For each $i \in E$, we define a weight w_i, and consider the base $\mathbf{v} \in \mathcal{B}$ which minimizes $\sum_{i=1}^{n} w_i v_i$. This base is denoted by $B_{r,min}$, and called the *minimum weight base*. We also consider the maximum weight base $B_{r,max}$.

For example, (E,r) where $r(X) = \min\{k, |X|\}$ for a fixed $k < n$ is a polymatroid, and its base must be a 0-1 vector with k nonzero entries. The minimum weight base is the support vector of the set of k smallest elements (with respect to weights) of E; Here, the support vector of a set $X \subset E$ is a 0-1 vector where multiplicity of i is 1 if and only if $i \in X$. In this case, the set of subsets of cardinality at most k of E forms a *uniform matroid* of rank k, and r coincides with its rank function in the terminology of matroids. Indeed, for any matroid $M \subset 2^E$, its rank function r_M (as a matroid) is a rank function of a polymatroid, where the independent vectors are support vectors of elements of M. In other words, a matroid can be considered as a polymatroid where $U = 1$. See Fujishige [9] and Murota [14, 15] for the theory of matroids and polymatroids.

Gusfield [10] considered the change of the minimum (or maximum) weight base when the weight changes as a linear function with respect to a parameter t. The weight function for i is denoted by $w_i(t)$. The minimum weight base and maximum weight base at a fixed t is denoted by $B_{r,min}(t)$ and $B_{r,max}(t)$, respectively. Since the argument for the maximum weight base is analogous, we concentrate on $B_{r,min}(t)$. The trajectory of the map $t \to B_{r,min}(t)$ from $\mathcal{R}$ to $\mathcal{B}$ is called the *parametric minimum weight base*. We consider the number of transitions in this trajectory.

If we consider the set of weight functions as a set of lines, the problem can be considered as a combinatorial problem in an arrangement of n lines. In particular, if we consider the rank function of the uniform matroid of maximum rank k, the number of transitions is asymptotically same as the complexity of k-level (Figure 1) of the arrangement, which is the trajectory of the k-th lowest line at t in the arrangement.

A classical result by Lovász [13] for k-sets (dual concept of k-level) and the result by Gusfield [10] for the graphic matroid can be extended to the following upper bound:

Theorem 1.1 *The number of transitions on* $B_{r,min}(t)$ *is* $O(n(r(E))^{1/2})$.

The above bound can be improved for some special r. Indeed, for the rank function r_M of a matroid, Dey [7] showed an $O(n(r_M(E))^{1/3})$ bound. This bound matches $\Omega(n(r_M(E))^{1/3})$ lower bound of Eppstein [8], and hence tight for the rank function of matroids. Unfortunately, we can show that $\Omega(n(r(E))^{1/2})$ transitions can occur if we consider the polymatroid associated with *at-most-k level* of an arrangement for $k = U$, where

$r(E) = U^2$. Here, at-most-k level is the union of i-levels for $i = 1, 2, .., k$.

We give the following combinatorial bound which is sensitive to both $r(E)$ and U:

Theorem 1.2 *The number of transitions on* $B_{r,min}(t)$ *is* $O(n(Ur(E))^{1/3})$. *Also, the bound is tight if* $r(E) = \Omega(U^2)$.

The above theorem unifies Dey's upper bound and Eppstein's lower bound on parametric matroid, and Alon-Györi's tight upper bound [3] on the at-most-k level of an arrangement.

We apply this to give the $O(n(s\sum_{i=1}^{s} k_i)^{1/3})$ upper bound for the sum of complexities of k_i-level for $i = 1, 2, .., s$ in an arrangement of n lines. This improves Welzl's $O(n(\sum_{i=1}^{s} k_i)^{1/2})$ bound [16], since $s^2 \leq \sum_{i=1}^{s} k_i$. For a special case where $k_i = k_1 + i - 1$ (i.e., consecutive levels), Day gave a $O(nk^{1/3}s^{2/3})$ result, which matches our bound.

If we consider the dual of these levels, and set $k_i = \lfloor in/s \rfloor$, this gives the transition on the equal-sized bucketing of n points in the plane with s parallel lines if we rotate the angle of parallel lines. Hence, we can show that The number of different equal-sized bucketings of n points in a plane with s parallel lines is $O(n^{4/3}s^{2/3})$. This can be utilized to find the optimal angle for the orthogonal grid bucketing generated by an pair of equal-sized bucketings to have the most uniform distribution.

2 Upper bounds for the number of transitions

When we consider the parametric polymatroid, each element e of E has a weight $w_e(t)$ which is linear with respect to t. Since $y = w_e(t)$ can be considered as a line in the (t, y)-plane, we regard E as a set of line from now. Theorem 1.1 can be proven analogously to Dey's proof for the $O(k^{1/3}n)$ bound for the k-level of an line arrangement.

An important fact is the following simultaneous exchange property [15] (see [11] for a proof):

Lemma 2.1 *If there are two bases* $B = (b_1, ..., b_n)$ *and* $B' = (b'_1, ..., b'_n)$, *there are two indices* i *and* j *satisfying* $b_i > b'_i$, $b_j < b'_j$, *such that both of* $B + \vec{e_j} - \vec{e_i}$ *and* $B' - \vec{e_j} + \vec{e_i}$ *are bases, where* e_i *is the* i*-th characteristic vector.*

Let us consider the arrangement of lines corresponding to $w_i(t)$ ($i = 1, 2, .., n$). Then, each transition corresponding to a vertex of this arrangement. The following lemma is straightforward from Lemma 2.1 :

Lemma 2.2 *There exists a family* Y *of* $r(E)$ *concave chains in the arrangement, satisfying that:*

(1) Each edge of the arrangement is contained in at most U *concave chains, and*

(2) transitions of the minimum base occurs at vertices of the concave chains.

Consider a graph G which has n vertices each corresponding to a line in the arrangement. There is an edge between two vertices in G if and only if the intersection of the corresponding two lines in the arrangement corresponds to a transition. It suffices to estimate the number m of edges in G.

We can draw G in a plane so that each vertex is placed to the dual point of the corresponding line in the arrangement. We draw each edge as the straight-line segment between vertices. Let $Cr(G)$ be the number of crossing of edges in this drawing. The following is a very famous lemma:

Lemma 2.3 *[2]*

There is a constant c *such that*
$Cr(G) > cm^3/n^2 - O(n)$.

Dey [7] gave the following observation:

Lemma 2.4 $Cr(G)$ *is at most the number of common tangents of the family* Y *of concave chains.*

For each common tangent, we charge it to the intersection between the concave chains lying below the tangent. The only difference from Dey's proof for the matroid is that more than two concave chains may pass an intersection. For a vertex p in the arrangement, let $mult(p)$ be the number of concave chains in Y passing p. Thus, the charge at an intersection is $O(mult(p)^2)$.

Lemma 2.5 *Let I be the set of intersections between concave chains in Y. Then,*
$\sum_{p\in I} mult(p) = O(r(E)n)$.

Proof: A concave chain intersects at most twice with a line. We count these intersections for each pair of concave chain and line. Then, each vertex p is counted $mult(p)$ times. Hence, we obtain the lemma. ∎

Thus, we have $Cr(G) = O(r(E)nU)$, and hence, $r(E)nU > cm^3/n^2$. Thus, $m = O(n(r(E)U)^{1/3}$, and our upper bound is obtained.

3 Lower bound

First, we demonstrate that $O((r(E))^{1/2}n)$ bound of Lovasz is tight for some polymatroid function r. We define a function f_k on 2^E by $f_k(A) = k + k - 1 + ... + (k - |A| + 1)$ if $|A| < k$, and otherwise $k(k+1)/2$. It is easy to see that (f_k, E) is a polymatroid, and $f_k(E) = k(k+1)/2$.

We can observe that $B_{f_k,min}(t)$ correspond to the sorted list of the smallest k elements in $\{w_i(t) : i = 1, 2, .., n\}$. Hence, a transition on $B_{f_k,min}(t)$ occur at each vertex in the "at-most-k-level" of the arrangement, for which an $\Omega(kn)$ bound is known. Hence, the following bound is obtained:

Proposition 3.1 *For any $k < n$, there exists a polymatroid function r satisfying that $r(E) = k(k+1)/2$ and the number of transitions on $B_{r,min}(t)$ is $\Omega(r(E)^{1/2}n)$.*

In the above lower bound construction, $U(f_k) = k$, and indeed, we can make U larger without decreasing the number of transitions. If $U = \Omega(r(E)^{1/2})$, the $O(\min\{n^2, r(E)^{1/2}n\})$ bound is tight, and also better than $O((r(E)U)^{1/3}n)$.

However, most of interesting cases satisfy that $U < r(E)^{1/2}$. We show that our $O((r(E)U)^{1/3}n)$ bound is tight for this case.

Theorem 3.1 *For any positive integers U and R satisfying that $U^2 \leq R$, there exists a polymatroid function r such that $r(E) \leq R$ and $U(r) \leq U$ on a set E of n elements satisfying that the number of transitions on $B_{r,min}(t)$ is $\Omega(n(RU)^{1/3})$.*

We devote the rest of this section for proving the above lower bound. Since $r(E) \leq nU$ always holds, we can assume that $U \leq n$.

Let $k = \lfloor R/U^2 \rfloor$. We use the Eppstein's construction [8] of concave chains $C_1, ..., C_k$ in an arrangement of $m = n/U$ lines where each concave chain has $\Omega(mk^{-2/3})$ vertices (without loss of generality, we assume that n/U is an integer). In this construction, no pair of different concave chains shares an edge in the arrangement. Also, we can make that each line in the arrangement has a positive slope.

We replace each line l in the arrangement by U copies $l^{(1)}, ...l^{(U)}$, where $l^{(j)}$ is obtained by horizontally translating l by $j\epsilon$ to the right, where ϵ is an infinitesimally small positive real number. E is the set of all these copied lines; hence E has n lines. Let S_i be the set of translated copies of lines of those contributing C_i. We considered the transversal polymatroid associated with the set $\{S_i : 1 \leq i \leq k\}$ of subsets of E.

We consider the function r_P on 2^E as follows: For a subset X of E, we subdivide X into $X_1, ..., X_k$ such that $X_i \subset S_i$. We define $g()$ such that for $Y \subset E$ $g(Y) = U(U+1)/2$ if $|Y| \geq U$; otherwise, $g(Y) = \sum_{j=1}^{|Y|}(U - j)$. We define $r_P(X)$ to be the maximum of $\sum_{i=1}^{N} g(X_i)$ over all possible subdivision of X.

Lemma 3.1 *The function $r_P(X)$ is a polymatroid function.*

Proof: For subsets X and Y of E, let $(X \cup Y)_i$ and $(X \cap Y)_i$ be the partitions of $X \cup Y$ and $X \cap Y$ assigned to the index i, respectively. Note that $(X \cup Y)_i$ need not include $(X \cap Y)_i$. The key observation is that we can find subsets X_i and Y_i of $X \cap S_i$ and $Y \cap S_i$, so that $X_i \cup Y_i = (X \cup Y)_i \cup (X \cap Y)_i$, $X_i \cap Y_i = (X \cup Y)_i \cap (X \cap Y)_i$, $\min\{|X_i|, |Y_i|\} \geq \min\{|(X \cap Y)_i|, |(X \cup Y)_i|\}$, and $X_i (i = 1, 2, .., k)$ and $Y_i (i = 1, 2, .., k)$ are partitions of X and Y, respectively. Thus, $g(X_i) + g(Y_i) \geq g((X \cup Y)_i) + g((X \cap Y)_i)$, and r_P is a polymatroid function. ∎

Note that $r_P(E) = k(U(U+1)/2)$. Next, let us consider the minimum base at $t = t_0$ where there is no vertex of the chains in the horizontal interval $[t_0 - U\epsilon, t_0 + U\epsilon]$. Let $l(i; t_0)$ be the line forming the edge of C_i at $t = t_0$. Then, the minimum weight base at t_0 has the following multiplicity vector: $l(i; t_0)^{(j)}$ has multiplicity j for $j = 1, 2, .., U$ and $i = 1, 2, .., k$; Other lines have multiplicity 0.

Let us consider the number of transitions near a vertex of C_i. In our construction, C_i is transformed into U copies, and a vertex of C_i causes U^2 intersections between copied chains. Among these intersections, the $U(U+1)/2$ intersections circled in Figure 2 are associated with transitions on the parametric minimum base.

Since there are $\Omega(mk^{1/3}) = \Omega(nU^{-1}(R/U^2)^{1/3})$ vertices in the concave chains $C_1, ..., C_k$, the total number of transitions of the minimum base of our polymatroid is $\Omega(n(UR)^{1/3})$. $r(E) = U(U+1)/2 \times k \approx R$, and obviously, $U(r) = U$. Thus, we have the lower bound.

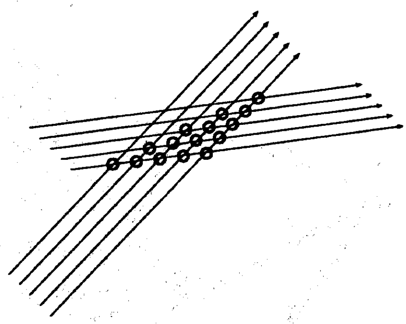

図 2: $U(U+1)/2$ intersections

4 Applications

4.1 Network flow

In this subsection, we consider a polymatroid associated with maximum flows (equivalently, minimum cuts) of a network. We later give geometric applications using this polymatroid in the next subsection.

Let $G = (V, F)$ be a directed graph (e.g., Figure 3). We fix a source node s and a set $T = \{1, 2, .., n\}$ of sink nodes of V. Each edge e of F has its capacity $c(e)$, which is a positive integer.

Suppose that we want to distribute marchandize shipped from s, which will be sold at each vertex $i \in T$ with w_i dollars per unit. What transportation flow gives the maximum profit? We define a function r on 2^T so that $r(X)$ is the size of maximum flow from s to X. Then, it is well-known that r is submodular, and becomes the rank function of a polymatroid (r, T). A vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ is a base of this polymatroid if and only if there is a maximum flow from s to T such that the node i receives a flow of size x_i. In this polymatroid, $U(r)$ is $\max_i Cut(s, i)$, where $Cut(s, i)$ is the size of minimum cut separating s and i. $r(T)$ is the size of the minimum cut separating s and T. Each node i of T has

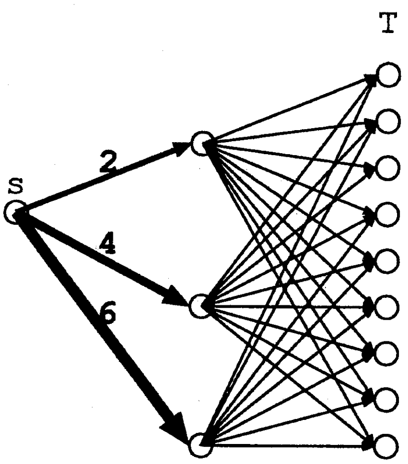

図 3: Graph G (capacity of an edge without a number is one)

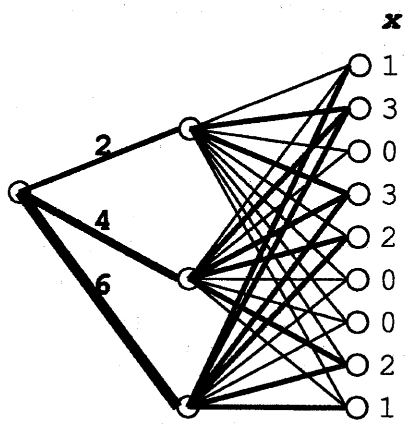

図 4: Maximum base $\mathbf{x}$ and its associated flow

a weight w_i. The weight of the flow associated with $\mathbf{x}$ is $\sum_{i=1}^{n} x_i w_i$, and the base maximizing the weight naturally gives the maximum profit. Since this is a polymatroid problem, the maximum weight base can be computed by using a greedy algorithm. For example, if the weight vector $\mathbf{w} = (w_1, w_2, .., w_9)$ is $(2,4,1,5,3,0,1,3,2)$ in the network of Figure 3, the maximum weight base $\mathbf{x}$ is $(1,3,0,3,2,0,0,2,1)$, and flow is the one shown in Figure 4.

Now, we replace w_i by $w_i(t)$, and consider the parametric version. For example, suppose t is the discount parameter on which the cost of manufacturing a unit of the marchandize also depends. We control t, and a transition of the maximum weight base of the polymatroid gives a transition of the maximum flow to give the maximum profit. Note that we do not give weights to edges, and the weight only depends on the node weight at sink nodes.

Theorem 4.1 *There are $O(n(r(T)U(r))^{1/3})$ transitions of the maximum weight max flow of the above network if $w_i(t)$ are linear. The same bound holds for the minimum weight max flow. Moreover, the above bound is tight if $U(r)^2 \leq r(T)$.*

Proof: The argument for the minimum weight base can be applied similarly to the maximum weight base; hence, the upper bound is immediate. The lower bound is obtained since the polymatroid in the lower bound construction in the previous section can be realized as the above polymatroid associated with a network flow. We omit details. ∎

Usually, edge weights are also considered in a transportation problem. If the graph is a tree, we can admit edge weights, since there is a unique path from s to each sink node, and we can accumulate the sum of the edge weights on the path to consider it as the node weight of the sink. Unfortunately, this argument fails for a general graph.

4.2 Geometric applications

Parametric matroids are useful in computational geometry [7, 12], and so are parametric polymatroids. Let $0 = k_0 < k_1 < k_2 < ... < k_s < n$ be an increasing sequence of integers. Consider a network $G = (V, F)$, where $V = \{s\} \cup Y \cup T$,

$Y = \{y_1, ..., y_s\}$, and $T = \{1, 2, .., n\}$ From s to y_i, an edge with capacity k_i is given. Between Y and T, we have the complete directed bipartite graph (direction of each edge is from Y to T) $K_{s,n}$ in which each edge has capacity one. Consider the polymatroid (r, T) defined in the previous subsection. It is easy to see that $r(T) = \sum_{i=1}^{s} k_i$ and $U(r) = s$. Indeed, the network in Figure 3 is this network for $(k_1, k_2, k_3) = (2, 4, 6)$.

The minimum weight base is obained as follows: First, we sort $\{w_i : i = 1, 2, .., n\}$, and let $h(i)$ be the sorting rank (in non-increasing fashion) of w_i in the sorted list. The base is defined by $x_i = s-j$ if $k_j < h(i) \leq k_{j+1}$, where we artificially set $k_{s+1} = n$. Note that $x_i \geq x_j$ if $w_i < w_j$. In other words, if we re-order the nodes of T so that $w_1 < w_2 < ... < w_n$, the base is $(s, s, .., s, (s-1), ..., (s-1), ..., 2, 2, 2, ..2, 1, 1, 1..1, 0, ..., 0)$, where j occurs $k_j - k_{j-1}$ times as entries, and hence it is the "conjugate" (see [4]) of the nonincreasing sequence $(k_s, k_{s-1}, ..., k_1, 0)$.

If we consider the parametric version, each weight $w_i(t)$ changes linearly on t, and the transitions on $B_{r,min}(t)$ occurs at $t = t_0$ if and only if two indices i and j satisfies $x_i(t_0) = x_j(t_0) + 1$ (before the transition) and $w_i(t_0) = w_j(t_0)$. After the transition, $x_i(t)$ is increased by one, and $x_j(t)$ is decreased by one.

If $x_i(t_0) = j$, the transition must be a transition of k_j-th level of the arrangement of lines $\{l_1, .., l_n\}$, where l_i is defined by $y = w_i(t)$. Hence, the transitions on $B_{r,min}$ are corresponded to the transitions on the union of k_i-th levels for $i = 1, 2, .., s$ in the arrangement of the lines in an one-to-one fashion. Hence, we have the following:

Theorem 4.2 *The number of transitions on* $k_1, ..., k_s$*-th levels is* $O(n(s \sum_{i=1}^{s} k_i)^{1/3})$.

The dual of these levels gives the partition of a set of n points in a plane with s parallel lines with the slope θ into $s + 1$ subsets with k_1, $k_2 - k_1$,...,$k_s - k_{s-1}$, and $n - k_s$ points. The angle θ is the counterpart of the parameter t. In the dual, the transitions on the levels corresponds to the change of the partition when we rotate the angle θ of parallel lines from 0 to 2π.

For the special case where $k_i = ni/s$, the partition is called *equal-sized bucketing* of the point set with the projection angle θ (Figure 5).

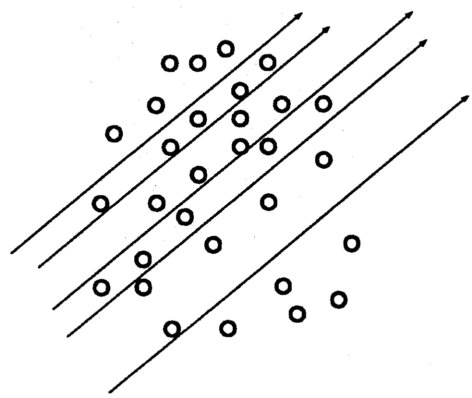

図 5: Equal-sized bucketing with five parallel lines

We can have different equal-sized bucketing if we change θ. From Theorem 4.2, we have the following:

Corollary 4.1 *There are* $O(s^{2/3}n^{4/3})$ *equal-sized bucketings.*

If we use the direct product of a pair of equal-sized bucketings with projection angles θ and $\theta + \pi/2$, we have a partition of the point set into a grid with $(s+1)^2$ rectangular buckets. (Figure 6). This is similar to the data partition considered in [5] (in [5], equal-width bucketings were used instead of equal-sized bucketings). The distribution of this data partition depends on the rotation angle θ. Since there are only $O(s^{2/3}n^{4/3})$ transitions on equal-sized bucketing when we rotate θ from 0 to 2π, the number of different partitions into a grid constructed as above is also $O(s^{2/3}n^{4/3})$. Moreover, we can compute the angle θ generating the most uniform partition (for example, minimizing the valuation of the numbers of points in rectangles) in $O(s^{2/3}n^{4/3}\log^2 n)$ time by using the algorithm of Cole et al. [6] for constructing levels of an arrangement.

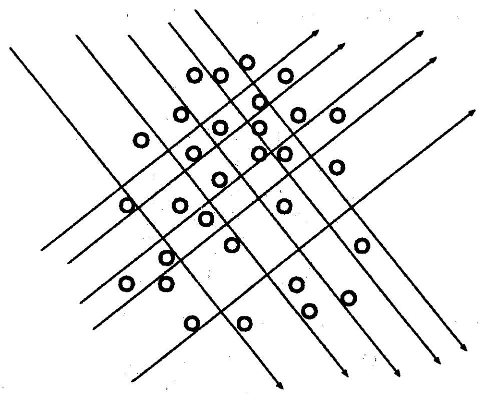

図 6: Grid generated by an orthogonal pair of equal-sized bucketings

5 Conclusion

We have considered the transition of the minimum weight base of parametric polymatroid. Although we only consider the base minimizing the sum of the weights (with multiplicities), we can similarly handle the lexicographic minimum weight base. There are many applications of polymatroids in combinatorial optimization [9]. By applying our theorem on several parametric polymatroids (other than the one in the previous section), we can obtain some results on graphs, linear algebra, and combinatorial systems, although we do not know whether they have practical or theoretical impact. Extensions of the results to the nonlinear weight functions or bivariate weight functions will have several impacts to computational geometry and control theory.

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