グラフの平面への直線埋込み問題

加納幹雄（茨城大学）、金子篤司（工学院大学）、徳永伸一（東京医科歯科大学）
kano@cis.ibaraki.ac.jp kaneko@ee.kogakuin.ac.jp tokucul@tmd.ac.jp

Abstract

グラフの直線埋込みとは、グラフを平面上に各辺が直線分でかつ交差しないように描くことである。ここではグラフの点集合となる平面上の点集合が指定されており、さらにグラフのいくつかの特別な点は、その対応する点が指定されているような場合の直線埋込みを考える。具体的には、互いに素な n 個の根付き木 $T_1, T_2, \ldots, T_n$ の和グラフ $F := T_1 \cup T_2 \cup \cdots \cup T_n$ を、指定された n 個の点 $p_1, p_2, \ldots, p_n$ を含む $|F|$ 個の点からなる平面上の点集合 P 上に直線埋込みする問題を考える。このとき、各根付き木 T_i, $1 \leq i \leq n$, の根 v_i は、指定された点 p_i に対応させるものとする。またこの根の対応条件を、根の集合 $\{v_1, v_2, \ldots, v_n\}$ は全体として指定された点の集合 $\{p_1, p_2, \ldots, p_n\}$ に対応するものと条件を弱めた直線埋込みも考える。

特に、3 個の根付き木の和 $T_1 \cup T_2 \cup T_3$ には、前の意味での直線埋込みのできない点集合と根付き木が存在することを示し、かつ後の意味で直線埋込みできることをしめす。また証明から、このような埋込みが $O(N^2 \log N)$ 時間で実現できるアルゴリズムも得られる。ただし $N = |T_1 \cup T_2 \cup T_3|$ である。

Straight-line embeddings of rooted forests in the plane

Mikio Kano Ibaraki University
Atsushi Kaneko Kogakuin University
Shin-ichi Tokunaga Tokyo Medical and Dental University

Abstract

Let $F := T_1 \cup T_2 \cup \cdots \cup T_k$ be a rooted forest with roots $v_1, v_2, \ldots, v_k$ and let P be a set of $|F|$ points in the plane in general position containing n specified points $p_1, p_2, \ldots, p_k$. We say that F can be strongly (or weakly) stright-line embedded onto P if F can be embedded in the plane so that every vertex of F corresponds to a point of P, every edge corresponds to a straight-line segment, no two straight-line segments intersect except their common end-point, and so that the root v_i corresponds to p_i for every $1 \leq i \leq k$ (or the set $\{v_1, \ldots, v_k\}$ of roots corresponds to the set $\{p_1, \ldots, p_k\}$ of specified points). We give some results on stronlgy and weakly stright-line embeddeing of rooted forests.

1 Introduction

We consider finite planar graphs without loops or multiple edges. Let G be a planar graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $|G|$ the order of G, that is, $|G| = |V(G)|$. Given a planar graph G, let P be a set of $|G|$ points in the plane (2-dimentional Euclidean space) in general position (i.e., no three points of P lie on the same line). Then G is said to be *line embedded onto* P or *stright-line embedded onto* P if G can be embedded in the plane so that every vertex of G corresponds to a point of P, every edge corresponds to a straight-line segment, and no two straight-line segments intersect except their common end-point. Namely, G is line embedded onto P if there exists a bijection $\phi : V(G) \rightarrow P$ such that two points $\phi(x)$ and $\phi(y)$ are joined by a straight-line segment if and only if x and y are joined by an edge of G and all two distinct open straight-line segments have no point in common. We call such a bijection a *line embedding* or a *straight-line embedding* of G onto P.

In this paper we consider a line embedding having one more property. Let G be a planar graph with n specified viertices $v_1, v_2, \ldots, v_n$, and P a set of $|G|$ points in the plane in general position containing n specified points $p_1, p_2, \ldots, p_n$. Then we say that G is *strongly line embedded onto* P if G can be line embedded onto P so that for every $1 \leq i \leq n$, v_i corrresponds to p_i, that is, if there exists a line embedding $\phi : V(G) \rightarrow P$ such that $\phi(v_i) = p_i$ for all $1 \leq i \leq n$. The line embedding mentioned above is called a *strong line embedding* of G onto P. Similarly G is said to be *weakly line embedded onto* P if there exists a line embedding $\phi : V(G) \rightarrow P$ such that $\{\phi(v_1), \ldots, \phi(v_n)\} = \{p_1, \ldots, p_n\}$. This line embedding is called a *weak line embedding* of G onto P.

A tree with one specified vertex v is usually called a *rooted tree* with root v. Given n disjoint rooted trees T_i with root v_i, $1 \leq i \leq n$, the union $T_1 \cup T_2 \cup \cdots \cup T_n$, whose vertex set is $V(T_1) \cup V(T_2) \cup \cdots \cup V(T_n)$ and whose edge set is $E(T_1) \cup E(T_2) \cup \cdots \cup E(T_n)$, is called a *rooted forest* with roots $v_1, v_2, \ldots, v_n$, which are specified vertices of it.

We begin with the following theorem, which was conjectured by Perles [8] and partially solved by Pach and Törőcsik [4]; a simpler proof can be found in Tokunaga [10]. Another related result can be found in [2].

Theorem A (Ikebe, Perles, Tamura and Tokunaga [3]) *A rooted tree T can be strongly line embedded onto every set of $|T|$ points in the plane in general position containing a specified point.*

We obtained the following theorem in [5].

Theorem B *A rooted forest F consisting of two rooted trees can be strongly line embedded onto every set of $|F|$ points in the plane in general position containing two specified points (see Figure 1).*

Moreover, our proof of the theorem gives an $O(|F|^2 \log |F|)$ time algorithm for finding a strong line embedding. We now give an example of rooted forests consisting of four rooted trees that cannot be strongly line embedded onto certain sets of points in the plane in general position containing four specified points. However, we proposed the following conjecture.

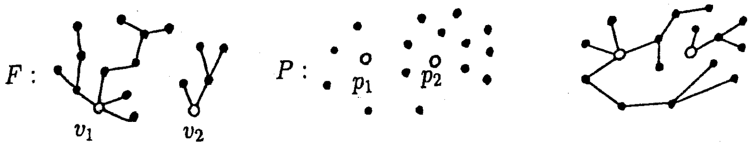


Figure 1: A rooted forest F and its strong line embedding onto P.

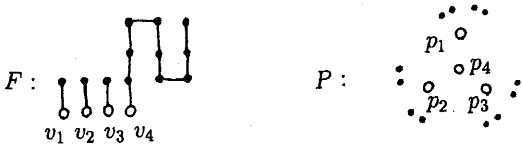


Figure 2: A rooted forest F which cannot be strongly line embedded onto P.

Conjecture C *A rooted forest F consisting of three rooted trees can be strongly line embedded onto every set of $|F|$ points in the plane in general position containing three specified points.*

In this paper we give an counterexample to the above conjecture, and give a result on weakly line-embedding of a rooted forest consisting of three rooted treees. Namely we prove the following theorems.

Theorem 1 *Let F be a rooted forest consisting of three rooted trees given in the Figure 3, and let P be the set of points given in Figure 3. Then F cannot be strongly line embedded onto P.*

Theorem 2 *Let $F := T_1 \cup T_2 \cup T_3$ be a rooted forest with roots v_1, v_2, v_3 and let P be a set of $|F|$ points in the plane in general position containing three specified points p_1, p_2, p_3. Then F can be weakly line embedded onto P. Moreover our proof gives a polynomial time ($O(n^2 \log n)$?) algorithm for finding such a weak line embedding.*

Other related results are given below.

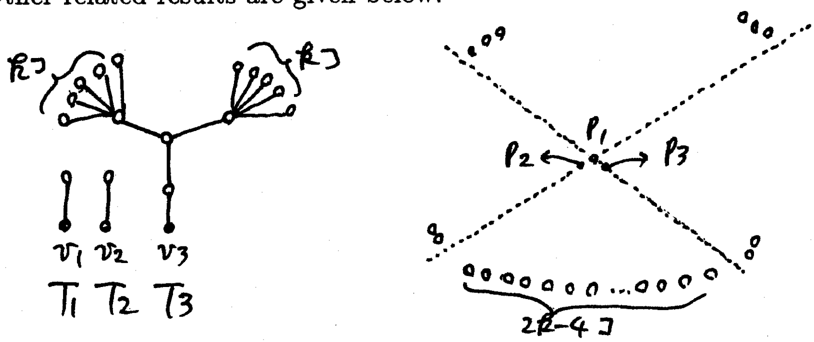


Figure 3: A rooted forest F and a set P of points.

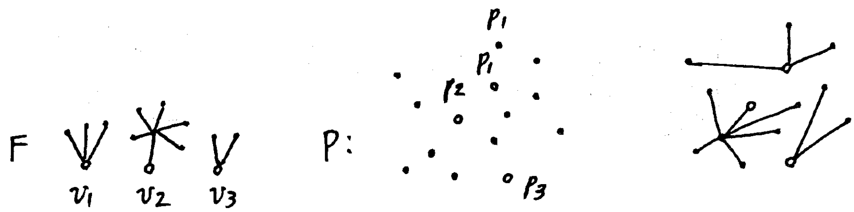


Figure 4: A strong line embedding of a forest consisting of star rooted trees

Theorem D ([7]) *Let $n \geq 3$ be an integer. Let $F := T_1 \cup T_2 \cup \cdots T_n$ be a rooted forest such that each T_i is a star with root v_i, and let P be a set of $|F|$ points in the plane in general position containing n specified points $p_1, p_2, \ldots, p_n$. Then F can be strongly line embedded onto P.*

Theorem E ([6]) *Let $k \geq 1$ and $n \geq 1$ be an integer. Let $F := T_1 \cup T_2 \cup \cdots T_n$ be a rooted forest such that each T_i is a rooted tree of order k or $k+1$ with root v_i, and let P be a set of $|F|$ points in the plane in general position containing n specified points $p_1, p_2, \ldots, p_n$. Then F can be strongly line embedded onto P.*

We can obtain polynomial time algorithms for finding strong line embeddings of Theorems D and E.

2 A sketch of proof of Theorem 2

In order to prove our theorem, we need some notation and definitions. Let X be a set of points in the plane. We denote by $\text{conv}(X)$ the convex hull of X, which is the smallest convex set containing X.

Let G be a graph. For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. For a subset $S \subseteq V(G)$, we denote by $G - S$ the graph obtained from G by deleting the vertices in S together with their incident edges.

Let P be a set of points in the plane in general position containing specified points. For convenience, we call a non-specified point of P an *ordinary point*, and denote the set of ordinary points of P by $ord(P)$.

For three non-collinear points x, y and p in the plane, the plane is partitioned into two regions by two rays emanating from p and passing through x and y, respectively. We denote by $Rgn(xpy)$ the region whose induced angle is less than ϕ. Similarly, for non-collinear point x and ray r from p, and for non-collinear rays r_1 and r_2 from p, $Rgn(xpr)$ and $Rgn(r_1pr_2)$ denote the similar internal regions (see Figure 5). If we consider a region including all its boundary, then we call it a *closed region*, and if we consider a region without its boundary, then we call it an *open region*.

Lemma 3 *Let T be a tree with two specified vertices v_1 and v_2, and P a set of $|T|$ points in the plane in general position containing two specified points p_1 and p_2. If one*

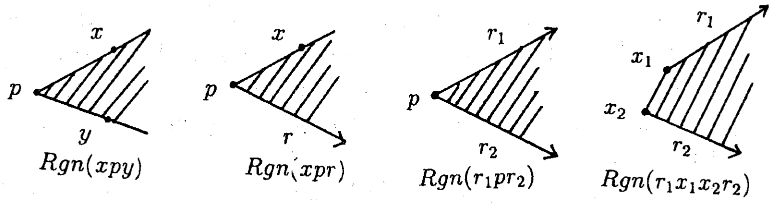


Figure 5: Regions $Rgn(xpy)$, $Rgn(xpr)$, $Rgn(r_1pr_2)$ and $Rgn(r_1x_1x_2r_2)$.

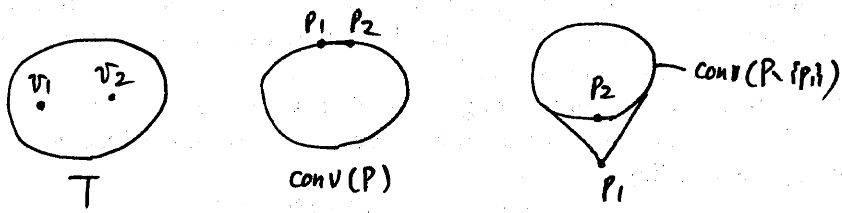


Figure 6: Conditins (i) and (ii).

of the following two conditions satisfies, then T can be line embedded onto P such that v_i corresponds to p_i for all $i = 1, 2$.

(i) p_1 and p_2 are consecutive vertices of conv(P); or

(ii) p_1 is a vetex of conv(P) and p_2 is a vertex of conv($P \setminus \{p_1\}$), and a line segment $\overline{p_1p_2}$ intersects conv($P \setminus \{p_1\}$) only at p_1 (see Figure 6).

Proof Here we prove only that if (i) is satisfied, then the lemma holds. By a suitable rotation of the plane and by the symmetry of p_1 and p_2, we may assume that p_1 lies on the bottom of conv(P) and that p_2 lies to the right of p_1. Let q be a vetex of conv(P) adjacent to p_1 and lying to the left of p_1. Suppose first $\deg_T(v_1) = 1$. Let u be the vertex of T adjacent to v_1. Then by induction, the tree $T - v_1$ with two specified vertices u and v_2 is strongly line embedded onto $P \setminus \{p_1\}$ with specified points q and p_2. By adding $\overline{p_1q}$ to this embedding, we can get the desired strong line embedding of T.

We next assume $\deg_T(v_1) \geq 2$. Let D be a component of $T - v_1$ not containing v_2. Then $1 \leq |D| \leq |T| - 2 = |P| - 2$, and so there exists a line l passing through p_1 such that the number of ordinary points of P lying on or to the left of l is equal to $|D|$. We denote the set of these ordinary points of P by Q. Then by Theorem A, the rooted tree $D \cup \{v_1\}$ with root v_1 is strongly line embedded onto $Q \cup \{p_1\}$ with specified point p_1. Furthermore, it follows from the inductive hypothesis that $T - V(D)$ with specified vertices v_1 and v_2 is strongly line embedded onto $P \setminus Q$ with specified points p_1 and p_2. By combining the above two embeddings, we can obtain the desired strong line embedding of T onto P. We can similarly prove the lemma holds under the assumption that (ii) satisfied. □

Lemma 4 *Let $k \geq 3$ be an integer. Let $T_1 \cup T_2 \cup \cdots \cup T_k$ be a rooted forest with roots $v_1, v_2, \ldots, v_k$, and P be a set of $|T_1 \cup T_2 \cup \cdots \cup T_k|$ points in the plane in general position containing k specified points $p_1, p_2, \ldots, p_k$. If $k - 2$ points $p_1, p_2, \ldots, p_{k-2}$ are verties of conv(P), then $T_1 \cup T_2 \cup \cdots \cup T_k$ can be strongly line embedded onto P.*

Let $F := T_1 \cup T_2 \cup T_3$ be a rooted forest with roots v_1, v_2 and v_3, and let P a set of $|F|$ points in the plane in general position containing three specified points p_1, p_2 and p_3. Then $ord(P) = P \setminus \{p_1, p_2, p_3\}$, which is the set of ordinary points of P. Let $\{p_i, p_j, p_k\} = \{p_1, p_2, p_3\}$, that is, p_i denotes one of the specified points of P, and p_j and p_k denote the other ones.

We may assume that $|T_1| \geq |T_2| \geq |T_3| \geq 2$ since if $|T_3| = 1$ then the theorem follows from the fact that a strong line embedding of $T_1 \cup T_2$ onto $P \setminus \{p_3\}$ is of course a weak line embedding of F onto P. For all $1 \leq i \leq 3$, put $n_i := |T_i| - 1$, which are equal to the numbers of ordinary points of P adding to p_i to construct T_i and T_2, respectively.

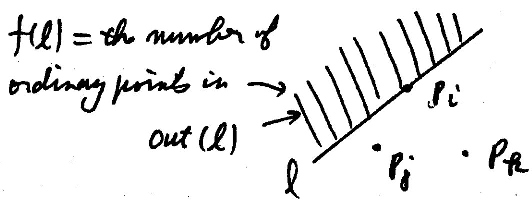
We now prove the theorem. By Lemma 4, we may assume that all the specified points p_1, p_2 and p_3 are interior points of conv(P).

Let l denote a line passing through p_i such that one of the regions determined by l contains both p_j and p_k. We denote by out(l) the region determined by l and not containing p_j, and define

$$f(l) := |\mathrm{out}(l) \cap \mathrm{ord}(P)|,$$

which is the number of ordinary points in out(l). Define the number M and N by

$$M := \max\{f(l)\} \quad \text{and} \quad N := \min\{f(l)\}.$$



Claim 1 We may assume that $f(l) < n_1$ for all line l passing through p_i such that out(l) can be defined.

Claim 2 We may assume that $f(l) > n_2 \geq n_3$ for all line l passing through p_i such that out(l) can be defined.

Let

$$T - v_1 = C_1 \cup C_2 \cup \cdots \cup C_q,$$

where each C_i is a component of $T_1 - v_1$ and $|C_1| \geq |C_2| \geq \cdots \geq |C_m|$.

Claim 3 We may assume $|C_1| > M$.

Let u denote the vertex of C_1 adjacent to v_1 in T_1. We choose a vetex w_1 of C_1 so that (i) the order of the component A_0 of $T_1 - v_1$ containing the vertex adjacent to v_1 is less than M, and (ii) the order $|A_0|$ is as large as possible subject to (i). Then we have $C_1 - w_1 = A_0 \cup A_1 \cup \cdots \cup A_r \cup A_{r+1} \cup \cdots \cup A_m$, where A_1 is one the largest components among all $A_1, \ldots, A_m$, and $A_2, \ldots, A_r$ satisfy that $|A_0 \cup A_1 \cup \cdots \cup A_r| < M$ but $|A_0 \cup A_2 \cup \cdots \cup A_r \cup A_t| \geq M$ for every $r < t \leq m$. It may happen that $w - 1 = u$ and $A_0 = \emptyset$. Let $B_1 := C_2 \cup \cdots \cup C_q$, $B_2 := A_0 \cup A_2 \cup \cdots \cup A_r$, $B_3 := A_{r+1} \cup \cdots \cup A_m$ and $B_4 := A_1$ (see Figure 7).

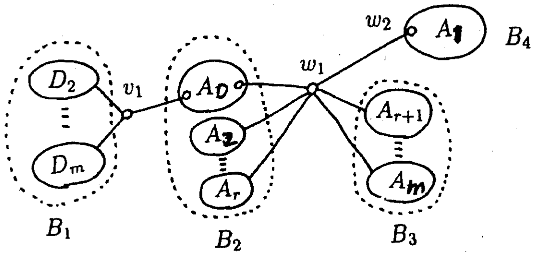


Figure 7: The rooted tree T_1

Claim 4 We may assume that $|B_2| \leq N - 2$.

Claim 5 We may assume that $B_3 \neq \emptyset$ and $|B_4| \geq 2$.

By making use of the above Claims, we can prove the theorem.

Our proof of Theorem 2 together with the following known results, we can obtan a polynomial time algorithm for finding a weak line embedding.

Theorem F (Bose, McAllister and Snoeyink [1]) *Let T be a rooted tree and P a set of $|T|$ points in the plane in general position containing a specified point. Then we can strongly embed T onto P in $ord(|T| \log |T|)$ time.*

Theorem G (Theorem 3.7 of [9]) *The convex hull of n points in the plane can be found in $O(n \log n)$ time.*

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