グラフを k-辺連結かつ 3-点連結に最適増大させる問題

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あらまし 辺連結度, 点連結度を同時に最適増大させる問題とは, 入力として, 無向多重グラフ $G=(V,E)$ と, 2 つの正整数 k,ℓ が与えられたとき, 最小本数の辺を G に加えることで, グラフの辺連結度および点連結度をそれぞれ k以上, かつ ℓ以上にする問題である. 本研究では, kが任意に固定された正整数で $\ell=3$ である場合, この問題が, 任意の入力グラフに対して, 多項式時間で解けることを示す.

和文キーワード: 無向多重グラフ, 連結度増加問題, 辺連結度, 点連結度, 辺分離.

k-Edge and 3-Vertex Connectivity Augmentation in an Arbitrary Multigraph

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Abstract Given an undirected multigraph $G=(V,E)$ and two positive integers k and ℓ, the edge-and-vertex connectivity augmentation problem asks to augment G by the smallest number of new edges so that the resulting multigraph becomes k-edge-connected and ℓ-vertex-connected. In this paper, we show that the problem with a fixed k and $\ell=3$ can be solved in polynomial time for an arbitrary multigraph G.

英文 key words: undirected multigraph, connectivity augmentation problem, edge-connectivity, vertex-connectivity, edge-splitting.

1 Introduction

The problem of augmenting a graph by adding the smallest number of new edges to meet edge-connectivity or vertex-connectivity requirement has been extensively studied as an important subject in network design, and many efficient algorithms have been developed so far. However, it was only very recent to have algorithms for augmenting both edge-connectivity and vertex-connectivity simultaneously (see [6, 7, 8] for those results).

Let $G = (V, E)$ stand for an undirected multigraph with a set V of vertices and a set E of edges. We denote the number of vertices by n, and the number of pairs of adjacent vertices by m. The local edge-connectivity $\lambda_G(x, y)$ (resp., the local vertex-connectivity $\kappa_G(x, y)$) is defined to be the maximum k (resp., ℓ) such that there are k edge disjoint (resp., ℓ internally vertex disjoint) paths between x and y in G (where at most one edge between x and y is allowed in the set of internally vertex disjoint paths). The *edge-connectivity* and *vertex-connectivity* of G are defined by $\lambda(G) = \min\{\lambda_G(x, y) \mid x, y \in V, x \neq y\}$ and $\kappa(G) = \min\{\kappa_G(x, y) \mid x, y \in V, x \neq y\}$. Let r be a function: $\binom{V}{2} \rightarrow Z^+$, where $\binom{V}{2}$ denotes the set of unordered pairs of $x, y \in V$ and Z^+ denotes the set of nonnegative integers. We call a multigraph G *r_λ-edge-connected* if $\lambda_G(x, y) \geq r_\lambda(x, y)$ for all $x, y \in V$. Analogously, G is called *r_κ-vertex-connected* if $\kappa_G(x, y) \geq r_\kappa(x, y)$ for all $x, y \in V$. The *edge-connectivity augmentation problem* (resp., the *vertex-connectivity augmentation problem*) asks to augment G by the smallest number of new edges so that the resulting multigraph G' becomes r_λ-edge-connected (resp., r_κ-vertex-connected).

As to the edge-connectivity augmentation problem, Watanabe and Nakamura [10] first proved that the problem with $r_\lambda(x, y) = k$ for all $x, y \in V$ can be solved in polynomial time for any given integer k. For a general requirement function r_λ, Frank [2] showed by using Mader's edge-splitting theorem that the problem can be solved in polynomial time.

As to the vertex-connectivity augmentation problem, the problem of making a $(\ell-1)$-vertex-connected multigraph ℓ-vertex-connected was shown to be polynomially solvable for $\ell = 2$ [1] and for $\ell = 3$ [11]. It was later found out that, for $\ell \in \{2, 3, 4\}$, the vertex-connectivity augmentation problem can be solved in polynomial time in [1, 4] (for $\ell = 2$), [3, 11] (for $\ell = 3$), and [5] (for $\ell = 4$), even if the input multigraph G is not necessarily $(\ell - 1)$-vertex-connected. However, whether there is a polynomial time algorithm for an arbitrary constant ℓ was still an open question (even if G is $(\ell - 1)$-vertex-connected).

Hsu and Kao [6] first treated the problem of augmenting edge-connectivity and vertex-connectivity simultaneously, and presented a linear time algorithm for augmenting $G = (V, E)$ with two specified subsets X and Y of V by adding the smallest number of edges so that the resulting multigraph G' satisfies $\lambda_{G'}(x, x') \geq 2$ for all $x, x' \in X$ and $\kappa_{G'}(y, y') \geq 2$ for all $y, y' \in Y$. The connectivity augmentation problem in the general setting was first studied in [7, 8]. For two given functions $r_\lambda, r_\kappa : \binom{V}{2} \rightarrow Z^+$, we say that G is *(r_λ, r_κ)-connected* if G is r_λ-edge-connected and r_κ-vertex-connected. The *edge-and-vertex-connectivity augmentation problem*, denoted by EVAP(r_λ, r_κ), asks to augment G by adding the smallest number of new edges so that the resulting multigraph G' becomes (r_λ, r_κ)-connected, where $r_\lambda(x, y) \geq r_\kappa(x, y)$ is assumed for all $x, y \in V$ without loss of generality. When a requirement function r_κ satisfies $r_\kappa(x, y) = \ell \in Z^+$ for all $x, y \in V$, this problem is also denoted as EVAP(r_λ, ℓ). The authors presented algorithms EV-AUGMENT [7] and EV-AUGMENT3 [8]. The first algorithm solves EVAP$(r_\lambda, 2)$ in $O(n^3 m \log(n^2/m))$ time, and the second solves EVAP$(k, 3)$ in $O(n^4)$ time, under the assumption that k is a fixed constant and the input multigraph is already 2-vertex-connected. However, it is left open whether EVAP$(k, 3)$ with a fixed k can be solved in polynomial time for an arbitrary input multigraph G.

In this paper, we consider EVAP$(k, 3)$ for an arbitrary input multigraph $G = (V, E)$, which is not necessarily 2-vertex-connected. It seems difficult to extend directly the above EV-AUGMENT3 to this case, since many properties used in EV-AUGMENT3 heavily depend on the 2-vertex-connectivity of the input multigraph. Alternatively, one may first apply EV-AUGMENT to an input multigraph G to obtain a $(k, 2)$-connected multigraph G', and then apply EV-AUGMENT3 to G' to obtain a $(k, 3)$-connected multigraph G''. However, in this case, the resulting multigraph G'' may not be optimally augmented from the original multigraph G.

In this paper, we first derive two lower bounds $\lceil \alpha(G)/2 \rceil$ and $\beta(G)$ on $opt(G)$, where $opt(G)$ is the optimal value of EVAP$(k, 3)$. We next obtain a $(k, 2)$-connected multigraph $G_2 = (V, E \cup F)$ with $|F| = \lceil \alpha(G)/2 \rceil$. We show that such G_2 can be computed by applying EV-AUGMENT. We then apply several procedures used in EV-AUGMENT3 in order to replace some edges in F with the same number of edges to attain the 3-vertex-connectivity while preserving its $(k, 2)$-connectivity. However, those procedures cannot be directly used unless G is 2-vertex-connected. To remedy this, we show that there exist some edges in F for which the procedures can be applied. As a result of applying these procedures, we can show that either $opt(G) = \max\{\lceil\alpha(G)/2\rceil, \beta(G)\}$ or $opt(G) \leq 2k-3$ holds and that a set of the smallest number of new edges can then be constructed in $O((2k-3)n^{4k-3}+n^2m+n^3\log n)$ time. Furthermore, we can show that if $\delta(G) \geq k$ or $opt(G) \geq 2k-2$ holds, then it can be found in $O(n^2m+n^3\log n)$ time, and if $opt(G) \leq 2k-3$ holds, then a feasible solution F' with $|F'| \leq \min\{2opt(G)-1, 2k-3, opt(G)+(k+1)/2\}$, can be found in $O(n^2m+n^3\log n)$ time. The entire algorithm is called EV-AUGMENT3*.

In Section 2, after introducing basic definitions and the concept of edge-splitting, we derive a lower bound on the optimal value of EVAP$(k,3)$ for an arbitrary multigraph G. In Sections 3, 4 and 5, we give an outline of EV-AUGMENT3*.

2 Preliminaries

2.1 Definitions

For a multigraph $G=(V,E)$, an edge with end vertices u and v is denoted by (u,v). Given two disjoint subsets of vertices $X,\ Y \subset V$, we denote by $E_G(X,Y)$ the set of edges connecting a vertex in X and a vertex in Y, and denote $c_G(X,Y)=|E_G(X,Y)|$. A singleton set $\{x\}$ is also denoted x. In particular, $E_G(u,v)$ is the set of multiple edges with end vertices u and v and $c_G(u,v)=|E_G(u,v)|$ denotes its multiplicity. For a subset $V' \subseteq V$ (resp., $E' \subseteq E$) in G, $G[V']$ (resp., $G[E']$) denotes the subgraph induced by V' (resp., E'), and we denote $G[V-V']$ (resp., $G[E-E']$) simply by $G-V'$ (resp., $G-E'$). For an edge set F, we denote by $V[F]$ the set of end vertices of edges in F. If F satisfies $F \cap E = \emptyset$, we denote $G=(V, E\cup F)$ by $G+F$. A *partition* $X_1,\cdots,X_t$ of a vertex set V is a family of nonempty disjoint subsets X_i of V whose union is V, and a *subpartition* of V is a partition of a subset of V. A *cut* is defined to be a subset X of V with $\emptyset \neq X \neq V$, and the *size* of a cut X is defined by $c_G(X, V-X)$, which may also be written as $c_G(X)$. In particular, $c_G(v)$ for $v \in V$ denotes the *degree* of v. Let $\delta(G)$ denote the minimum degree of G. We say that a cut X *intersects* another cut Y if none of subsets $X\cap Y$, $X-Y$ and $Y-X$ is empty, and X *crosses* Y if in addition $V-(X\cup Y)\neq\emptyset$. A family $\mathcal{X}$ of subsets $X_1,\cdots,X_u$ is called *laminar* if no two subsets in $\mathcal{X}$ intersect each other. A multigraph G is called k-edge-connected if $\lambda(G)\geq k$. For a subset X of V, a vertex $v\in V-X$ is called a *neighbor* of X if it is adjacent to some vertex $u\in X$, and the set of all neighbors of X is denoted by $\Gamma_G(X)$. A maximal connected subgraph G' in a multigraph G is called a *component* of G, and the number of components in G is denoted by $p(G)$. A *disconnecting set* of G is defined as a cut S of V such that $p(G-S)>p(G)$ holds and no $S'\subset S$ has this property. Let $\bar{G}$ denote the simple graph obtained from G by replacing multiple edges in $E_G(u,v)$ by a single edge (u,v) for all $u,v\in V$. A component G' of G with $|V(G')|\geq 3$ always has a disconnecting set unless $\bar{G}'$ is a complete graph. If G is connected and contains a disconnecting set, then a disconnecting set of the minimum size is called a *minimum disconnecting set*, whose size is equal to $\kappa(G)$. A cut $T\subset V$ is called *tight* if $\Gamma_G(T)$ is a minimum disconnecting set in G. A tight set T is called *minimal* if no proper subset T' of T is tight (hence, the induced subgraph $G[T]$ is connected). A disconnecting set S is called a *disconnecting vertex* (resp., *disconnecting pair*) if $|S|=1$ (resp., $|S|=2$). We say that a disconnecting set $S\subset V$ *disconnects* two disjoint subsets Y and Y' of $V-S$ if no two vertices $x\in Y$ and $y\in Y'$ are connected in $G-S$. For a disconnecting set S, there is a unique component X of G such that $X\supseteq S$, and we call the components in $G[X]-S$ *the S-components*.

2.2 Edge-Splitting

Given a multigraph $H=(V\cup\{s\},E)$, a designated vertex s, vertices $u,v\in\Gamma_H(s)$ (possibly $u=v$) and a nonnegative integer $\delta\leq\min\{c_H(s,u),c_H(s,v)\}$, we construct multigraph $H'=(V\cup\{s\},E')$ from H by deleting δ edges from $E_H(s,u)$ and $E_H(s,v)$, respectively, and adding new δ edges to $E_H(u,v)$. We say that H' is obtained from H by *splitting* δ pair of edges (s,u) and (s,v). A sequence of splittings is *complete* if the resulting multigraph H' has no neighbor of s. The following theorem is due to Lovász [9].

Theorem 2.1 *Let $H=(V\cup\{s\},E)$ be a multigraph with a designated vertex s and an integer $k\geq 2$ such that $c_H(s)$ is an even integer and $\lambda_H(x,y)\geq k$ for all pairs $x,y\in V$. Then, for any neighbor u of s, there is a neighbor v (possibly $v=u$) such that $\lambda_{H'}(x,y)\geq k$ for all $x,y\in V-s$ in the multigraph H' resulting from H by splitting one pair of edges (s,u) and (s,v).* □

By applying this repeatedly, we see that there always exists a complete splitting at s such that the resulting multigraph H' satisfies $\Gamma_{H'}(s)=\emptyset$ and $\lambda_{H'}(x,y)\geq k$ for all $x,y\in V$.

2.3 Lower Bound on the number of new edges

In the subsequent discussion, we consider EVAP(k, 3) for an arbitrary multigraph G, and assume $k \geq 4$ (since the problem is equivalent to the 3-vertex-connectivity augmentation problem if $k = 3$). In this section, we derive two types of lower bounds $\alpha(G)$ and $\beta(G)$ on the optimal value $opt(G)$ to EVAP(k,3).

Let X be a cut in G. To make G $(k, 3)$-connected, it is necessary to add at least $\max\{k - c_G(X), 0\}$ edges between X and $V - X$, or at least $\max\{3 - |\Gamma_G(X)|, 0\}$ edges between X and $V - X - \Gamma_G(X)$ if $V - X - \Gamma_G(X) \neq \emptyset$. Given a subpartition $\mathcal{X} = \{X_1, \cdots, X_{q_1}, X_{q_1+1}, \cdots, X_{q_2}\}$ of V, where $V - X_i - \Gamma_G(X_i) \neq \emptyset$ holds for $i = q_1 + 1, \cdots, q_2$, we can sum up "deficiency" $\max\{k - c_G(X_i), 0\}$, $i = 1, \cdots, q_1$, and $\max\{3 - |\Gamma_G(X_i)|, 0\}$, $i = q_1 + 1, \cdots, q_2$. Adding one edge to G contributes to the deficiency of at most two cuts in $\mathcal{X}$. Hence, to make G $(k, 3)$-connected, we need at least $\lceil \alpha(G)/2 \rceil$ new edges, where

$$\alpha(G) = \max_{\text{all subpartitions } \mathcal{X}} \left\{ \sum_{i=1}^{q_1} (k - c_G(X_i)) + \sum_{i=q_1+1}^{q_2} (3 - |\Gamma_G(X_i)|) \right\}, \tag{2.1}$$

and the maximum is taken over all subpartitions $\mathcal{X} = \{X_1, \cdots, X_{q_1}, X_{q_1+1}, \cdots, X_{q_2}\}$ of V with $V - X_i - \Gamma_G(X_i) \neq \emptyset$, $i = q_1 + 1, \cdots, q_2$.

We now consider another case in which different type of new edges become necessary. For a pair of two vertices $S = \{v, v'\}$ of G, let $T_1, \cdots, T_r$ denote all the components in $G - S$, where $r = p(G - S)$ (note that S may not be a disconnecting pair in G). To make G 3-vertex-connected, a new edge set F must be added to G so that all T_i form a single connected component in $(G + F) - S$. For this, it is necessary to add

(i) at least $p(G - S) - 1$ edges to connect all components in $G - S$.

Moreover, if $k > c_G(u)$ holds for a $u \in S$, then it is necessary to add at least $k - c_G(u)$ edges in order to make G k-edge-connected. Since adding an edge between v and v' contribute to the requirement of both v and v', we require

(ii) at least $\max\{k - c_G(v), k - c_G(v'), 0\}$ edges.

In the above, no edge in (i) is incident to v or v', while all edges in (ii) are incident to v or v'; hence there is no edge that belongs to both (i) and (ii). Therefore, it is necessary to add

(iii) at least $p(G - S) - 1 + \max\{k - c_G(v), k - c_G(v'), 0\}$ edges for $S = \{v, v'\}$.

This means that the following number of new edges are necessary to make G $(k, 3)$-connected.

$$\beta(G) = \max_{\substack{\text{all vertex pairs} \\ S = \{v, v'\} \text{ in } G}} \left[p(G - S) - 1 + \max\{k - c_G(v), k - c_G(v'), 0\} \right]. \tag{2.2}$$

Lemma 2.1 (Lower Bound) $\gamma(G) \leq opt(G)$, *where* $\gamma(G) = \max\{\lceil \alpha(G)/2 \rceil, \beta(G)\}$. □

Based on this, we shall prove the next result in this paper.

Theorem 2.2 *Let G be an arbitrary multigraph with n vertices and m adjacent vertex pairs.*
(1) For any integer $k \geq 4$, $\gamma(G) \leq opt(G) \leq \max\{\gamma(G), 2k - 3\}$ holds and an optimal solution of EVAP(k, 3) *can be found in $O((2k - 3)n^{4k-3} + n^2 m + n^3 \log n)$ time.*
(2) If $\delta(G) \geq k$ or $\gamma(G) \geq 2k - 2$, then $opt(G) = \gamma(G)$ holds and an optimal solution F can be found in $O(n^2 m + n^3 \log n)$ time.
(3) If $\gamma(G) \leq 2k - 3$, then a feasible solution F' of EVAP(k,3) *such that $|F'| \leq \min\{2opt(G) - 1, 2k - 3, opt(G) + (k + 1)/2\}$, can be found in $O(n^2 m + n^3 \log n)$ time.* □

3 Algorithm for EVAP(k,3)

Given a multigraph $G = (V, E)$, let $\mathcal{P}_3(G)$ denote the set of unordered pairs $\{x, y\}$ of vertices $x, y \in V$ with $\kappa_G(x, y) \geq 3$. Thus $\mathcal{P}_3(G) = \binom{V}{2}$ if $\kappa(G) \geq 3$. For a subset $F \subseteq E$ in G, an operation of removing a subset F' from F followed by adding a set F'' of new edges with $|F''| = |F'|$ to F is called a *shifting* in F, and denoted by F''/F'. In particular, a shifting F''/F' in F is called a *switching* if it does not change the degree $c_G(v)$ of any vertex v in G. Given an k-edge-connected multigraph G, a sequence of switchings or shiftings of edges is called *feasible* to G if the resulting multigraph G' remains k-edge-connected and $\mathcal{P}_3(G') \supseteq \mathcal{P}_3(G)$ holds.

We now present a polynomial time algorithm EV-AUGMENT3* for solving EVAP(k,3). The proofs of the properties and the analysis of the time complexity are omitted due to space limitation. An example of computational process of EV-AUGMENT3* is shown in Fig. 1.

Algorithm EV-AUGMENT3*

Input: An undirected multigraph $G = (V, E)$ ($|V| \geq 4$) and an integer $k \geq 4$.

Output: A set of new edges F with $|F| = opt(G)$ such that $G + F$ is $(k, 3)$-connected.

Step I (Addition of vertex s and associated edges): Add a new vertex s together with a set F_1 of new edges between s and V such that the resulting $G_1 = (V \cup \{s\}, E \cup F_1)$ satisfies

$$c_{G_1}(X) \geq k \quad \text{for all cuts } X \subset V, \tag{3.1}$$

$$|\Gamma_G(X)| + |\Gamma_{G_1}(s) \cap X| \geq 3 \quad \text{for all cuts } X \subset V \text{ with } V - X - \Gamma_G(X) \neq \emptyset \text{ and } |\Gamma_G(X)| + |X| \geq 3, \tag{3.2}$$

$$|\Gamma_G(x)| + c_{G_1}(s, x) \geq 3 \quad \text{for all } x \in V, \tag{3.3}$$

and $|F_1|$ is *minimum* subject to (3.1) – (3.3). We describe in Section 4 how to find such F_1.

Property 3.1 $|F_1| = \alpha(G)$. □

If $c_{G_1}(s)$ is odd, then we add to F_1 a new edge $\hat{e} = (s, w)$ for an arbitrary vertex $w \in V$ so that $c_{G_1}(s)$ becomes even. Call this edge $\hat{e} = (s, w)$ *extra.*

Step II (Edge-splitting): Here we show the following property which is stronger than Theorem 2.1.

Property 3.2 *There is a complete splitting at s in G_1 such that the resulting multigraph $G_2 = (V, E \cup F_2)$ is $(k, 2)$-connected.* □

Now if $\kappa(G_2) \geq 3$ holds, then we are done, because $|F_2| = |F_1|/2 = \lceil \alpha(G)/2 \rceil$ attains the lower bound of Lemma 2.1. Otherwise ($\kappa(G_2) = 2$), we can observe from (3.2) and (3.3) that the following holds for the family of minimal tight sets in G_2, denoted by $\mathcal{T}(G_2)$.

$$\begin{aligned} &T \cap V[F_2] \neq \emptyset \text{ for all } T \in \mathcal{T}(G_2), \\ &|E_{G_2}(x) \cap F_2| \geq 2 \text{ for all } T = \{x\} \in \mathcal{T}(G_2) \text{ with } |\Gamma_G(x)| = 1, \\ &|E_{G_2}(x) \cap F_2| \geq 3 \text{ for all } T = \{x\} \in \mathcal{T}(G_2) \text{ with } |\Gamma_G(x)| = 0. \end{aligned} \tag{3.4}$$

Step III (Switching edges): Now the current $G_2 = (V, E \cup F_2)$ is $(k, 2)$-connected, and satisfies (3.4). During this step, we try to make G_2 3-vertex-connected by switching some edges in F_2 while preserving the k-edge-connectivity. In [8, Property 3.2], some sufficient conditions for two edges $e_1, e_2 \in F_2$ that are disconnected by some disconnecting pair and satisfies $\kappa(G_2 - \{e_1, e_2\}) \geq 2$, are given to admit a feasible switching such that at least one new pair of vertices in G_2 becomes 3-vertex-connected. However, if an input multigraph G is not 2-vertex-connected (a situation not assumed in [8]), there may be an edge $e \in F_2$ such that $G_2 - e$ is not 2-vertex-connected. Let $F^*(G_2) \subset F_2$ denotes the set of edges in F_2 such that the removal of any edge $e \in F^*(G_2)$ violates 2-vertex-connectivity of G_2.

Property 3.3 *For each tight set T in G_2, $G_2[T \cup \Gamma_{G_2}(T)]$ contains at least one edge $e \in F_2 - F^*(G_2)$ with $T \cap V[e] \neq \emptyset$.* □

Property 3.4 *Let S be a disconnecting pair in G_2. If two edges $e_1, e_2 \in F_2 - F^*(G_2)$ satisfy $T_i \cap V[e_i] \neq \emptyset$, $i = 1, 2$, for two distinct S-components T_1 and T_2 in G_2, then $G_2 - \{e_1, e_2\}$ is 2-vertex-connected.* □

From these properties, every tight set contains an edge in $F_2 - F^*(G_2)$ and the two edges $e_1, e_2 \in F_2 - F^*(G_2)$ satisfying one of the conditions in [8, Property 3.2] always satisfy $\kappa(G_2 - \{e_1, e_2\}) \geq 2$. Therefore we can repeat executing a feasible switching of pairs of edges in $F_2 - F^*(G_2)$ until none of the conditions in [8, Property 3.2] holds in G_2. Then it is not difficult to see that all disconnecting pairs in G_2 contain one common vertex, say v^*.

In [8, Property 3.3], another case in which a feasible switching can be performed is given. In this paper, we show a generalization of [8, Property 3.3] as it is now proved including the case of $\kappa(G_2-\{e_1,e_2,e_3\}) \leq 1$.

If none of the conditions in [8, Properties 3.2, 3.3] for a feasible switching holds any longer, let $G_3 = (V, E \cup F_3)$ denote the resulting multigraph, where $F_3 = F_2$. If $\kappa(G_3) \geq 3$, we are done since $|F_3| = \lceil \alpha(G)/2 \rceil$ attains the lower bound $\gamma(G)$. Otherwise go to Step IV.

Step IV (Shifting edges):

Property 3.5 *Let $S_i = \{v^*, v_i\}$, $i = 1, \cdots, q$, denote all disconnecting pairs in G_3. Then every $e \in F^*(G_3)$ satisfies $e = (v_i, v_j)$ for some $i \neq j$, or $e = (t, v_i)$ with $\Gamma_{G_3}(t) = \{v^*, v_i\}$ for some i.* □

In [8, Property 3.4], some conditions are given to admit a feasible shifting of an edge $e_1 \in F_3 - F^*(G_3)$ incident to the common vertex v^* (and another edge $e_2 \in F_3 - F^*(G_3)$ such that $\kappa(G_3 - \{e_1, e_2\}) \geq 2$ holds and e_1 and e_2 are disconnected by some disconnecting pair in G_3, if necessary) that decreases the degree of v^* by one. From Property 3.5, $E_{G_3}(v^*) \cap F_3 \subseteq F_3 - F^*(G_3)$ follows. Furthermore, from Property 3.4, every two edges $e_1, e_2 \in F_3 - F^*(G_3)$ that are disconnected by some disconnecting pair satisfy $\kappa(G_3 - \{e_1, e_2\}) \geq 2$. Therefore we can apply [8, Property 3.4] to G_3, and repeat a feasible shifting and switching edges in $F_3 - F^*(G_3)$ until $|F_3 \cap E_{G_3}(v^*)| = 0$ or $c_{G_3}(v^*) = k$ holds.

Property 3.6 *If $F^*(G_3) \neq \emptyset$ holds, we can execute a feasible switching or shifting of edges in $F_3 - F^*(G_3)$ by applying [8, Properties 3.2, 3,3, and 3.4].* □

Let $G_4 = (V, E \cup F_4)$ denote the resulting multigraph $G_3 = (V, E \cup F_3)$ obtained by repeating a feasible switching or shifting of edges in F_3 until none of conditions in [8, Properties 3.2, 3,3, and 3.4] holds in G_3. It holds $F^*(G_4) = \emptyset$ from Property 3.6. In this case, we observe that we can apply the latter part of Step IV and Step V of EV-AUGMENT3 [8] to G_4 (while maintaining condition $F^*(G_4) = \emptyset$). Then as observed in [8, Property 3.8], we see that an input multigraph G becomes $(k,3)$-connected by adding $\min\{\lceil \alpha(G)/2 \rceil, \beta(G)\}$ new edges or by adding at most $2k - 3$ new edges. This establishes Theorem 2.2. □

4 Algorithm for Step I

We consider how to find a set F_1 such that $G_1 = (V \cup s, E \cup F_1)$ satisfies (3.1) – (3.3), and has a subpartition $\mathcal{X} = \{X_1, \cdots, X_{q_1}, X_{q_1+1}, \cdots, X_{q_2}\}$ of V, satisfying

$$\begin{array}{l} c_{G_1}(s, X_i) = k - c_G(X_i) \text{ for } i = 1, \cdots, q_1, \\ c_{G_1}(s, X_i) = 3 - |\Gamma_G(X_i)| \text{ and } V - X_i - \Gamma_G(X_i) \neq \emptyset \text{ for } i = q_1+1, \cdots, q_2, \\ \Gamma_{G_1}(s) \subseteq \cup_{X \in \mathcal{X}} X. \end{array} \quad (4.1)$$

Note that such an edge set F_1 attains $|F_1| = \alpha(G)$.

Algorithm ADD-EDGE*

1. After adding a sufficiently large number (say, k) of edges between s and each vertex $v \in V$, discard them one by one as long as (3.1), (3.2) or (3.3) is not violated. Let $G'_1 = (V \cup s, E \cup F'_1)$ be the resulting multigraph where $F'_1 = E_{G'_1}(s, V)$. Unless this G'_1 has a subpartition of V satisfying (4.1), we continue shifting or removing edges in $E_{G'_1}(s, V)$, while preserving (3.1) – (3.3).

Property 4.1 *For each edge $e = (s,t) \in E_{G'_1}(s, V)$ such that $G'_1 - e$ violates (3.2) or (3.3), G'_1 has a cut $T \subset V$ with $t \in T$ and $V - T - \Gamma_G(T) \neq \emptyset$, satisfying*

(I) $|\Gamma_G(T)| = 2$ *and* $E_{G'_1}(s, T) = \{(s,t)\}$, *or*
(II) $|\Gamma_G(T)| = 1$ *and* $c_{G'_1}(s,t) = c_{G'_1}(s,u) = 1$ *hold for* $t \neq u \in T$ *if* $|T| \geq 2$,
or $|\Gamma_G(t)| = 1$ *and* $c_{G'_1}(s,t) = 2$ *if* $T = \{t\}$. □

If $T \subseteq V$ satisfies (I) or (II) in Property 4.1, T is called *κ-critical.* Let $\mathcal{T}_1$ (resp., $\mathcal{T}_2$) denote the family of all κ-critical cuts T of type (I) (resp., (II)) such that no $T' \subset T$ is of type (I) (resp., (II)). For each edge $e = (s, u) \in F'_1$ such that $G'_1 - e$ violates (3.1), there is a unique cut $X_u \subset V$, called *λ-critical,* such that $u \in X_u$, $c_{G'_1}(X_u) = k$ and $c_{G'_1}(Y) > k$ holds for all cuts Y with $u \in Y \subset X_u$. Let $\mathcal{X}_1$ denotes the family all λ-critical cuts X_u.

Property 4.2 *Let $\mathcal{T}_1' := \mathcal{T}_1 - \{T_i, T_j \in \mathcal{T}_1 \mid T_i$ and T_j cross each other in G_1' $\} - \{T_i \in \mathcal{T}_1 \mid T_i \subseteq T_j$ for some $T_j \in \mathcal{T}_2\}$. Then $\mathcal{X}_1 \cup \mathcal{T}_1' \cup \mathcal{T}_2$ covers $\Gamma_{G_1'}(s)$, and every two cuts in $\mathcal{T}_1' \cup \mathcal{T}_2$ are pairwise disjoint.* □

From this property, if G_1' does not have a subpartition of V satisfying (4.1), then there are two cuts $X \in \mathcal{X}_1$ and $T \in \mathcal{T}_1' \cup \mathcal{T}_2$ satisfying the following:

$$\begin{array}{l}\text{(i) } c_{G_1'}(s,T) = 1 \text{ if } T \in \mathcal{T}_1', \text{ and } c_{G_1'}(s,T) = 2 \text{ if } T \in \mathcal{T}_2. \\ \text{(ii) } T \cap X \neq \emptyset \text{ and } (T - \sum_{X_i \in \mathcal{X}_1} X_i) \cap \Gamma_{G_1'}(s) \neq \emptyset.\end{array} \tag{4.2}$$

2. Let H denote an arbitrary 2-vertex-connected multigraph. Let us regard $\mathcal{T}_1'$ as the family $\mathcal{T}(H)$ of all minimal tight sets in H, since every two cuts in $\mathcal{T}_1'$ are pairwise disjoint and every cut $T \in \mathcal{T}_1'$ satisfies $|\Gamma_G(T)| = 2$ and $V - T - \Gamma_{G_1'}(T) \neq \emptyset$. It was shown in the algorithm ADD-EDGE in [8] that the number of two cuts $X \in \mathcal{X}_1$ and $T \in \mathcal{T}(H)$ satisfying (4.2) in H can be decreased to 0. Hence, by applying the same algorithm, the number of pairs of cuts $X \in \mathcal{X}_1$ and $T \in \mathcal{T}_1'$ satisfying (4.2) can be decreased to 0. Moreover, we can see that the resulting multigraph G_1'' has a subpartition of V satisfying (4.1) and hence let $G_1 := G_1''$. □

5 Algorithm for Step II

Algorithm SPLIT

1. Let $F_1' \subseteq F_1$ be a set of edges such that $G_1' = (V \cup s, E \cup F_1')$ satisfies (3.1) and the following (5.1), but no $(V \cup s, E \cup F')$ with $F' \subset F_1'$ satisfies them.

$$\begin{array}{rl}|\Gamma_G(X)| + |\Gamma_{G_1'}(s) \cap X| \geq 2 & \text{for all cuts } X \subset V \text{ with } V - X - \Gamma_G(X) \neq \emptyset \\ & \text{and } |\Gamma_G(X)| + |X| \geq 2, \\ c_{G_1'}(s,x) \geq 2 & \text{for all } X = \{x\} \text{ with } \Gamma_G(X) = \emptyset.\end{array} \tag{5.1}$$

If $c_{G_1'}(s)$ is odd, then we choose one edge $e^* \in F_1 - F_1'$, and update $G_1' := G_1' + e^*$ and $F_1' := F_1' + e^*$. Then it is shown in [7] that by a complete splitting at s in G_1', which does not create a self-loop, we can obtain a multigraph $G_2' = (V \cup \{s\}, E \cup (F_1 - F_1') \cup F_2')$ satisfying one of the following conditions, where F_2' is the set of edges obtained from splitting edges in F_1'.

(i) $G_2'[V]$ is $(k,2)$-connected.
(ii) $G_2'[V]$ is $(k,1)$-connected and has exactly one disconnecting vertex v. At most one edge in F_2' is incident to v, and each edge $e \in F_2'$ which is not incident to v satisfies $p(G_2'[V] - v) = p((G_2'[V] - e) - v) - 1$.

If G_2' satisfies (ii), go to step 2; otherwise go to step 3.

2. Let $T_i, i = 1, \cdots, r'$, denote all v-components in $G_2'[V]$. If $G_2'[V]$ has the edge $e = (v, t_1) \in F_2'$ incident to v, then assume $t_1 \in T_1$ without loss of generality. Here we can prove from the above property of edges in F_2' and properties (3.2) and (3.3) in G_1 that $c_{G_2'}(s, T_1) \geq 1$ and $c_{G_2'}(s, T_i) \geq 2$ hold for $i = 2, \cdots, r'$. Let $G_2'' = (V \cup \{s\}, E \cup (F - F_1' - F_1'') \cup F_2' \cup F_2'')$ be the multigraph resulting from a sequence of splitting pairs of edges (s, a_i) and (s, b_{i+1}) to (a_i, b_{i+1}), $i = 1, \cdots, r' - 1$, where $a_1 \in T_1 \cap \Gamma_{G_2'}(s)$, $a_i, b_i \in T_i \cap \Gamma_{G_2'}(s)$ for $i = 2, \cdots, r'$, $F_1'' = \{\{(s, a_i), (s, b_{i+1})\} \mid i = 1, \cdots, r' - 1\}$, and $F_2'' = \{(a_i, b_{i+1}) \mid i = 1, \cdots, r' - 1\}$. This $G_2''[V]$ is now $(k,2)$-connected. Let $G_2' := G_2''$, $F_1' := F_1' \cup F_1''$, and $F_2' := F_2' \cup F_2''$, and go to 3.

3. Now $G_2'[V]$ is $(k,2)$-connected and $c_{G_2'}(s)$ is even. Here it is not difficult to see that G_2' has a complete splitting at s which creates no self-loop (if necessary, the extra edge $\hat{e}$ chosen in Step I will be rechosen). Let $G_2 := G_2'' - s$, where G_2'' denotes the multigraph resulting from such a complete splitting in G_2'. □

6 Concluding Remarks

In this paper, we combined the algorithms EV-AUGMENT [7] and EV-AUGMENT3 [8], and gave an algorithm for augmenting a given arbitrary graph G to an k-edge-connected and 3-vertex-connected graph by adding the smallest number of new edges. However, our lower bound on the optimal value does not always give the exact optimal value. So, it is desired to find a new and stronger lower bound on the optimal value.

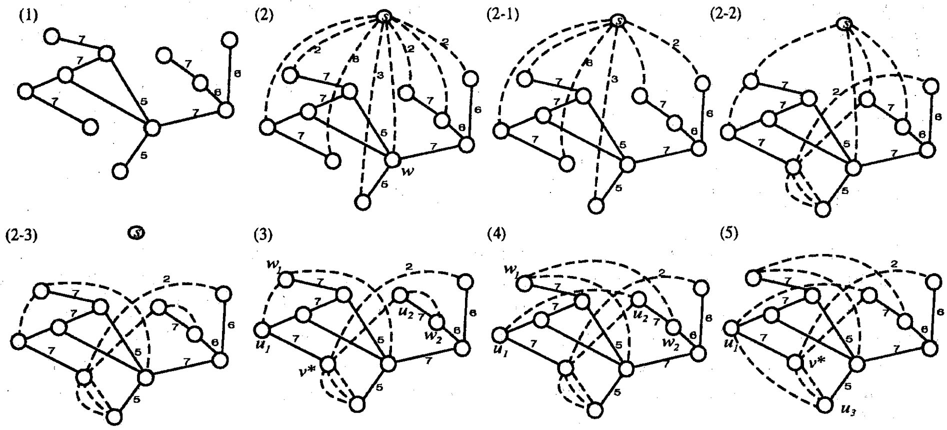

Figure 1: Computational process of algorithm EV-AUGMENT3* for $k = 8$. (1) An input multigraph $G = (V, E)$ with $\lambda(G) = \kappa(G) = 1$, where the number beside each edge is the multiplicity of the edge (the numbers for multiplicity 1 are omitted). The two lower bounds in Section 2 are $\lceil \frac{\alpha(G)}{2} \rceil = \frac{18}{2} = 9$ and $\beta(G) = 3$. (2) $G_1 = (V \cup \{s\}, E \cup F_1)$ obtained by Step I. Edges in F_1 are drawn as broken lines. Now G_1 satisfies (3.1) for $k = 8$, (3.2) and (3.3), and the edge $\hat{e} = (s, w)$ is extra. (2-1) $G_1' = (V \cup \{s\}, E \cup F_1')$ with $F_1' \subseteq F_1$ in Step II, satisfying (3.1) and (5.1). (2-2) $G_2' = (V \cup \{s\}, E \cup (F_1 - F_1') \cup F_2')$ in Step II, where F_2' is a set of edges obtained from splitting edges in F_1'. Now $G_2'[V]$ is $(8, 2)$-connected and every edge in $E_{G_2'}(s, V) - \hat{e}$ is κ-critical. (2-3) $G_2'' = (V \cup \{s\}, E \cup F_2)$ in Step II obtained from splitting edges in $F_1 - F_1'$ in G_2', which creates no self-loop. (3) $G_2 = (V, E \cup F_2)$ obtained by Step II. The G_2 satisfies $\lambda(G_2) \geq 8$ but has a disconnecting pair $S = \{v^*, v\}$. (4) $G_2^{(1)} = (V, E \cup F_2^{(1)})$ obtained from G_2 by a feasible switching $\{(u_1, u_2), (w_1, w_2)\}/\{(u_1, w_1), (u_2, w_2)\}$ in Step III. Moreover, any switching is no longer feasible in $G_2^{(1)}$. (5) $G_4 = (V, E \cup F_4)$ obtained by shifting $\{(u_1, u_3)\}/\{(v^*, u_3)\}$ in Step IV. This G_4 is $(8, 3)$-connected. □

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