一様な三角形分割を与える点配置アルゴリズム

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あらまし 凸多角形 P と正整数 n が与えられたとき，ある最適化基準を満たす P の内部および境界上の n 点集合の配置，および凸多角形 P の頂点集合と配置された n 点集合の和集合に対する三角形分割を求める問題を考える。本論文では三角形分割の最適化基準として以下の 3 つを考える。(1) 最大辺と最小辺の長さの比の最小化, (2) 最大辺の長さの最小化, (3) 三角形の周長の最大値の最小化。本論文では, 逐次ボロノイ分割アルゴリズムによって得られる点配置に対するデローネ三角形分割が, 上のいずれの問題に対しても定数近似となることを示す。

An Approximate Algorithm for Finding a Uniform Triangulation for Movable Points

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Abstract Given a convex polyhedron $\boldsymbol{P}$ and a positive integer n, we consider the problem of finding a location of n points in the interior of $\boldsymbol{P}$ as well as a triangulation of the interior of $\boldsymbol{P}$ with these n points that satisfies certain optimality criterion. In this paper we consider the following three optimality criteria: (1) minimizing the ratio of the maximum edge length to the minimum one, (2) minimizing the maximum edge length, and (3) minimizing the maximum perimeter of triangles. We shall develop a heuristic called *incremental Voronoi partition algorithm* and show that it produces a constant approximation for problems under any of the above optimality criteria.

1 Introduction

Given a convex polyhedron $\boldsymbol{P}$ and a positive integer n, we consider the problem of finding a location of n points in the interior of $\boldsymbol{P}$ as well as a triangulation for the interior of $\boldsymbol{P}$ with such n points that satisfies certain optimality criteria. In particular, we shall consider the following three optimality criteria:

(1) minimizing the ratio of the maximum edge length to the minimum one,
(2) minimizing the maximum edge length, and
(3) minimizing maximum triangle perimeter.

We shall develop a heuristic called *incremental Voronoi partition algorithm* that determines a location of n points one by one in an incremental manner so that it places a point at the position which is farthest from the set of points already located as well as vertex set of $\boldsymbol{P}$. After fixing the location of n points, it constructs a Delaunay triangulation for n points and vertex set of $\boldsymbol{P}$. We shall show that the obtained triangulation produces a constant approximation for the problem under any of the above-mentioned optimality criteria. More precisely, we shall prove that the approximation factor for the problem under the criterion (1), (2) and (3) is 2, $4/\sqrt{3}$, and $2\sqrt{3}$, respectively.

Triangulating a fixed point set in the plane is one of fundamental problems in computational geometry, and has been extensively studied [1]. Triangulation of a point set has many applications such as finite element methods and computer graphics. In finite element methods, it is desirable to generate triangulations that do not have too large or too small angles. It also has an application in designing structures such as plane trusses where it is required to determine its shape from aesthetic point of view under the constraints concerning stress and nodal displacements. The plane truss can be viewed as a triangulation of points in the plane by regarding truss members and nodes as edges and points, respectively. When focusing on the shape, edge lengths should be as equal as possible from the viewpoint of design and mechanics (see [5, 6]). In such applications, the location of points are usually not fixed, but can be viewed as decision variables. In view of this, it is quite natural to consider the above criteria (1)-(3). To the best knowledge of authors, problems dealt with in this paper have not been studied in the field of computational geometry.

Finding an optimal triangulation under the above criteria seems to be difficult although minimizing the maximum edge length is known to be solvable in quadratic time for the case of a fixed point set [2]. Nooshin et al. [5] developed a potential-based heuristic method for the problem under criterion (2), but did not give any theoretical guarantee for the obtained solution.

2 Incremental Voronoi Partition

We begin with introducing several notations. Let S be a set of n points located inside or on the boundary of a given convex polyhedron $\boldsymbol{P}$. Let V be the set of vertices of $\boldsymbol{P}$. Let $\mathcal{T}(S)$ denote the set of triangulations for the interior of $\boldsymbol{P}$ using S and V. For two points u and v in the plane, let $d(u, v)$ denote the Euclidean distance between u and v. For disjoint subsets X and Y of R^2, let $d(X, Y) = \min\{d(x, y) \mid x \in X \text{ and } y \in Y\}$. For an edge e of a triangulation in the plane, let $d(e)$ denote the length of e, i.e., Euclidean distance between two end vertices of e.

We shall now describe the algorithm for determining the location of n points. Starting with an empty set S, it repeatedly places a new point at the position which is farthest from the set $V \cup S$. After determining the location of n points, we output the Delaunay triangulation for $V \cup S$. The algorithm is formally described as follows:

Algorithm INCREMENT
Input: Convex polyhedron $\boldsymbol{P}$ with vertex set V, and a positive integer n.
Output: Location of n points inside $\boldsymbol{P}$ and a triangulation of $\boldsymbol{P}$ using V and such n points.

Step 1 Set $S := \emptyset$.

Step 2 Find a Voronoi diagram Vor($V \cup S$) for $V \cup S$. For each point v of Voronoi vertices of Vor($V \cup S$), if v lies outside $\boldsymbol{P}$, let v' be the intersection of the boundary of $\boldsymbol{P}$ and the Voronoi edge one of whose endpoints is equal to v. Among Voronoi vertices inside $\boldsymbol{P}$ and those points v' on the boundary of $\boldsymbol{P}$, let v^* be the one that is farthest from $V \cup S$.

Step 3 Let $S := S \cup \{v^*\}$ and return to Step 2 if $|S| < n$.

Step 4 Output Delaunay triangulation for $V \cup S$.

Let p_k and $\hat{S}_k$ denote the point chosen in Step 2 (resp. the set obtained in Step 3) at k-th iteration of the algorithm. From the way of choosing the point p_k in Step 2, p_k is the farthest

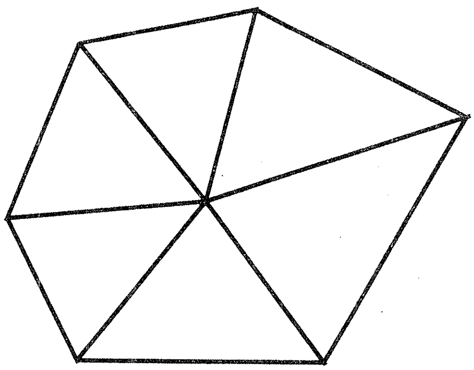

Figure 1: Illustration of Algorithm INCREMENT for $n = 1$

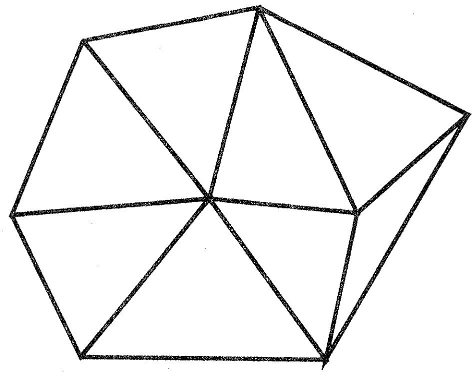

Figure 2: Illustration of Algorithm INCREMENT for $n = 2$

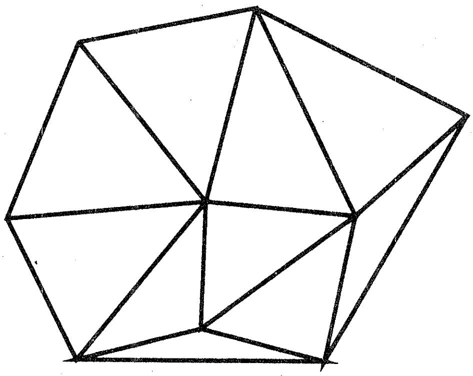

Figure 3: Illustration of Algorithm INCREMENT for $n = 3$

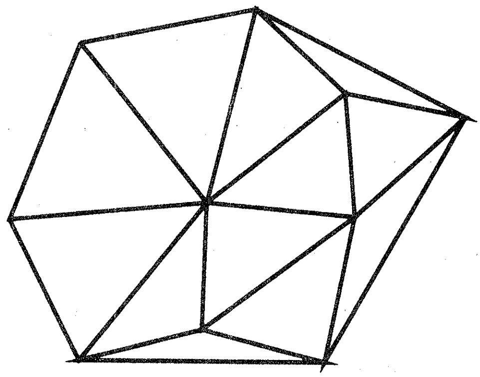

Figure 4: Illustration of Algorithm INCREMENT for $n = 4$

point in $\boldsymbol{P}$ from $\hat{S}_{k-1} \cup V$. Figures 1 through 4 illustrate how the algorithm proceeds for $n = 1, 2, 3, 4$.

3 Approximation Results

In this section we shall prove approximation results concerning the Delaunay triangulation obtained by Algorithm INCREMENT with respect to three criteria introduced in Section 1. Let us consider the following problem:

$$
\begin{aligned}
\text{Maximin}: \quad & \text{Maximize} \quad \min d(p,q) && (1)\\
& \text{subject to} \quad p, q \in V \cup S, \text{ and } S \text{ is a set of } n \text{ points} && (2)\\
& \qquad\qquad\quad \text{inside or on the boundary of } \boldsymbol{P} && (3)
\end{aligned}
$$

We call the problem *extreme packing problem*, and an optimal solution of this problem is called *extreme packing* for $\boldsymbol{P}$. Let S_n^* denote the optimal solution and let d_n denote the optimal objective value. The following theorem gives a basis for this purpose. Its proof is a simple adaptation of known results by Feder and Greene [3] and Gonzalez [4].

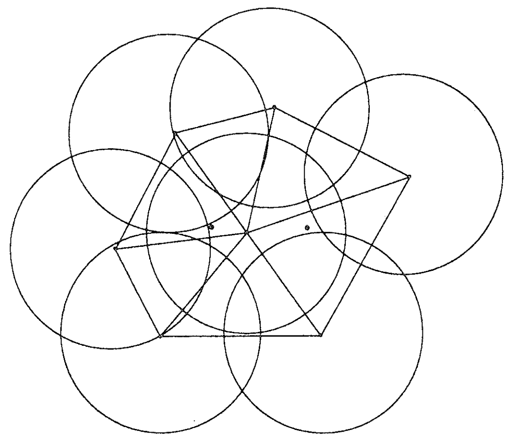

Figure 5: Illustration of circles that cover the whole area of $\boldsymbol{P}$

Theorem 1 *The solution $\hat{S}_n$ obtained by Algorithm INCREMENT is a 2-approximation for Problem* Maximin.

The proof of the theorem is done by showing the following two lemmas.

Lemma 1 *For any location of $n-1$ points, S_{n-1}, in $\boldsymbol{P}$, there exists a point q in $\boldsymbol{P}$ such that*

$$\min\ \{d(p,q) \mid p \in S'_{n-1} \cup V\} \geq d_n/2. \tag{4}$$

PROOF. Suppose that the lemma is not true. Then there exists a location of $n-1$ points, denoted by S'_{n-1}, in $\boldsymbol{P}$ such that

$$\max_{q \in \boldsymbol{P}} \min\{d(p,q) \mid p \in S'_{n-1} \cup V\} < d_n/2. \tag{5}$$

Let r be the value of left-hand side of (5). For each point $p \in S'_{n-1} \cup V$, draw a circle centered at p with radius $r+\epsilon$, where ϵ is a sufficiently small positive number that satisfies $r+\epsilon < d_n/2$. Then, the union of $n-1$ such circles must cover $\boldsymbol{P}$ (see Figure 5). Since there are n points in the set S^*_n, there exists a circle centered at some point in $S'_{n-1} \cup V$ with radius $r+\epsilon$ which contains two points of S^*_n. Since the distance between these points is less than or equal to $2(r+\epsilon)$, which is less than d_n, this contradicts the definition of d_n. □

Lemma 2 *Let $w(p_n) = d(p_n, \hat{S}_{n-1})$. Then we have*

$$w(p_n) \geq d_n/2. \tag{6}$$

PROOF. Applying Lemma 1 with $S'_{n-1} = \hat{S}_{n-1}$, it follows that there exists a point q such that $d(q, \hat{S}_{n-1}) \geq d_n/2$. Since Algorithm INCREMENT finds the farthest point in $\boldsymbol{P}$ from $\hat{S}_{n-1} \cup V$, the lemma immediately follows. □

Now we shall complete the proof of Theorem 1. Note that

$$\min\{d(p,q) \mid p,q \in \hat{S}_{n-1} \cup V\} \geq w(p_n) \tag{7}$$

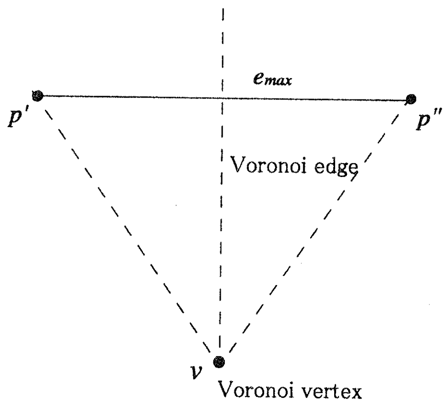

Figure 6: Illustration of edge $e_{\max}$ and the corresponding Voronoi vertex v

holds since otherwise it contradicts that the algorithm always finds at every iteration k the point which is farthest from $\hat{S}_{k-1} \cup V$. Therefore, it follows from Lemma 2 that

$$\min\{d(p,q) \mid p,q \in \hat{S}_n \cup V\} \geq d_n/2, \tag{8}$$

proving the assertion of Theorem 1.

Now we shall prove several nice properties of the triangulation obtained by Algorithm INCREMENT. We first consider the problem with optimality criterion (1) which is described as follows.

$$\text{Problem 1:} \quad \min_{S \subset \boldsymbol{P}, |S|=n} \min_{T \in \mathcal{T}} \frac{\max_{e \in T} d(e)}{\min_{e \in T} d(e)}, \tag{9}$$

where $\mathcal{T}$ denotes the set of all triangulations for $S \cup V$. Let DT denote the Dalaunay triangulation output by the algorithm, and let $e_{\max}$ and $e_{\min}$ denote the maximum and minimum length edges in DT. Let $d(e_{\max})$ and $d(e_{\min})$ denote the lengths of edges $e_{\max}$ and $e_{\min}$, respectively. The following theorem proves that DT gives a 2-approximation for Problem 1.

Theorem 2

$$\frac{d(e_{\max})}{d(e_{\min})} \leq 2. \tag{10}$$

PROOF. From (7) and (8), we have

$$w(p_n) = \min\{d(p,q) \mid p,q \in \hat{S}_n \cup V\} \;\text{ and }\; d(e_{\min}) \geq w(p_n)$$

follows. Let p' and p'' be two endpoints of $e_{\max}$. Then p' and p'' share a Voronoi edge in $\mathrm{Vor}(\hat{S}_n \cup V)$ (see Figure 6).

Let v be a Voronoi vertex on that Voronoi edge. Then we have

$$d(v,p') \leq w(p_n) \text{ and } d(v,p'') \leq w(p_n).$$

Thus,

$$d(p',p'') = d(e_{\max}) \leq 2 \cdot w(p_n) \leq 2 \cdot w(e_{\min}). \tag{11}$$

This proves the theorem. □

Now let us consider the optimality criterion (2). The problem dealt with here is described as follows.

$$\text{Problem 2:} \quad d^*_{minimax} \equiv \min_{S \subset \boldsymbol{P}, |S|=n} \min_{T \in \mathcal{T}} \max_{e \in T} d(e). \tag{12}$$

We first prove the following lemma.

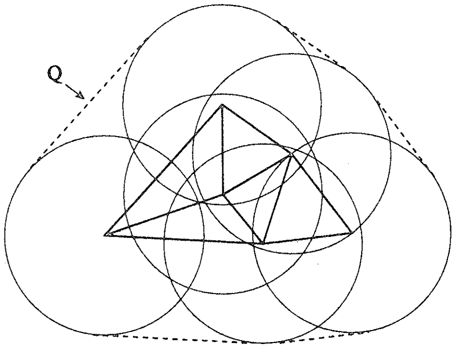

Figure 7: Illustration of circles centered at points of $\hat{S}$ and V and convex hull Q used for the proof of Lemma 3

Lemma 3

$$d^*_{minimax} \geq \frac{\sqrt{3}}{2} d_n.$$

PROOF. Let point set $\tilde{S}$ and triangulation $\tilde{T}$ for $\tilde{S}$ be the optimal solution of problem (12). Suppose the lemma is not true. Then

$$d^*_{minimax} < \frac{\sqrt{3}}{2} d_n$$

holds. For each point p in $\tilde{S} \cup V$, draw a circle centered at p with radius $\frac{1}{\sqrt{3}} d^*_{minimax}$. Let $\tilde{C}$ denote the set of such circles and let $\boldsymbol{Q}$ be the convex hull of all such circles. From the definition of the radius, for each triangle t of triangulation $\tilde{T}$, the whole area of t is covered by three such circles centered at three vertices of t (see Figure 7 for the illustration of circles and convex hull $\boldsymbol{Q}$).

Now for each point q in $S^*_n \cup V$ draw a circle centered at q with radius $\frac{1}{\sqrt{3}} d^*_{minimax}$, where S^*_n is an optimal solution to Problem Maximin. Let C^* denote the set of such circles. Circles in C^* do not either overlap or touch each other since $\frac{1}{\sqrt{3}} d^*_{minimax} < d_n/2$ holds from assumption. Let A^* be the area of $\boldsymbol{P}$ covered by circles of C^*. Then $A^* < area(\boldsymbol{P})$ holds, where $area(\boldsymbol{P})$ denotes the area of $\boldsymbol{P}$.

Now let us consider the region of $\boldsymbol{Q} - \boldsymbol{P}$, and focus on a particular edge e of $\boldsymbol{P}$ and let v and v' be two end vertices of e. Let l be the length of e. $l \geq d_n$ holds from definition of d_n. Therefore, from the assumption of $2d^*_{minimax}/\sqrt{3} < d_n \leq l$, there must have some points of $\tilde{S}$ on e such that the distance between consecutive points in $\tilde{S} \cup V$ on e is less than or equal to $d^*_{minimax}$. Now let us consider the edge e' on the boundary of $\boldsymbol{Q}$ which is parallel to e. We choose the endpoints of e' so that the convex hull of e and e' becomes a rectangle. Let $\boldsymbol{R}$ be such rectangular region. The number of circles of $\tilde{C}$ whose centers are on e is at least $\lceil l/d^*_{minimax} \rceil + 1$, and hence the area of $\boldsymbol{R}$ covered by $\tilde{C}$, denoted by $\tilde{R}$, satisfies

$$\tilde{R} > \frac{\pi (d^*_{minimax})^2}{6} (\lceil l/d^*_{minimax} \rceil + 1). \tag{13}$$

On the other hand, we claim that the number of circles of C^* that overlap e is at most $\lceil l/d^*_{minimax} \rceil + 1$. Let C' be the set of such circles and let $p_1, p_2, \ldots, p_h$ ($h = |C'|$) be the projection of centers of circles in C' onto the edge e such that $p_1(= p), p_2, \ldots, p_h(= p')$ appear on e in this order. It is then easy to see that $d(p_i, p_{i+1}) \geq d^*_{minimax}$. Therefore the above claim follows. Since circles of C^* do not overlap each other, the area in $\boldsymbol{R}$ covered by $\tilde{C}$, denoted by R^*, satisfies

$$R^* < \frac{\pi(d^*_{minimax})^2}{6}(\lceil l/d^*_{minimax}\rceil + 1). \tag{14}$$

The strict inequality holds since the contribution of each circle of C' to R^* is less than $\pi(d^*_{minimax})^2/6$ Thus, from (13) and (14), $R^* < \tilde{R}$ follows.

From the above argument, it follows that the area in $\boldsymbol{Q}$ covered by C^* is less than that by $\tilde{C}$, which is a contradiction because $|\tilde{C}| = |C^*|$ and circles of C^* do not overlap each other. □

Theorem 3

$$\frac{d(e_{\max})}{d^*_{minimax}} \leq 4/\sqrt{3}.$$

Proof. Since $d_n/2 \leq w(p_n) \leq d_n$ holds from Theorem 1, we have

$$d(e_{max}) \leq 2 \cdot w(p_n) \leq 2 \cdot d_n.$$

The first inequality was shown in the proof of Lemma 2. From Lemma 3, we have

$$\frac{d(e_{\max})}{d^*_{minimax}} \leq \frac{2 \cdot d_n}{\frac{\sqrt{3}}{2} \cdot d_n} = 4/\sqrt{3}.$$

□

Now let us consider the third optimality criterion. For a triangle t, let $length(t)$ denote the perimeter of t. For a triangulation T, we abuse the notation T to denote the set of triangles in T. The problem with optimality criterion (3) is described as follows.

$$\text{Problem 3:} \ \ l^* \equiv \min_{S \subset \boldsymbol{P}, |S|=n} \min_{T \in \mathcal{T}} \max_{t \in T} length(t). \tag{15}$$

Let

$$\tilde{l} = \max_{t \in \mathrm{DT}} length(t).$$

Theorem 4

$$\tilde{l}/l^* = 2\sqrt{3}.$$

Proof. It follows from Lemma 3 that for any S with $|S| = n$ and any triangulation T for $S \cup V$, the length of longest edge of T is at least $\frac{\sqrt{3}}{2} \cdot d_n$, we have

$$l^* \geq \sqrt{3} d_n. \tag{16}$$

For any edge e of DT,

$$l(e) \geq w(p_n) \geq d_n/2$$

holds from Lemma 2, and for $e_{\max}$ in DT, we have

$$d(e_{\max}) \leq 2w(p_n) \leq 2d_n.$$

Therefore we have

$$\tilde{l} \leq 3w(p_n) \leq 6d_n. \tag{17}$$

So, from (16) and (17),

$$\tilde{l}/l^* \leq \frac{6d_n}{\sqrt{3}d_n} = 2\sqrt{3}$$

follows. □

Finally we show that DT obtained by Algorithm INCREMENT is a 2-approximation of minimum weight triangulation for the same set of points.

Theorem 5 *Let $\mathcal{L}(T)$ denote the total edge length of triangulation T, and let $MWT(S)$ denote the minimum weight triangulation for point set S. Then*

$$\mathcal{L}(DT) \leq 2\mathcal{L}(MWT(\hat{S} \cup V))$$

holds.

PROOF. Let β be the number of edges of DT, and let $e_{\max}$ and $e_{\min}$ be those as defined above. From definition of Delaunay triangulation, the shortest line segment connecting two points in $\hat{S} \cup V$ must be in DT, we know that $d(e) \geq d(e_{\min})$ holds for any edge $e \in \mathrm{MWT}(\hat{S} \cup V)$. From Theorem 2, we have

$$\mathcal{L}(\mathrm{MWT}(\hat{S} \cup V)) \geq \beta \cdot d(e_{\min})$$

and

$$\mathcal{L}(\mathrm{DT}) \leq \beta \cdot d(e_{\max}).$$

From these two inequalities, the theorem follows. □

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