完全グラフの均衡的 (C_4, C_8)-Bowtie 分解アルゴリズム

藤本　英昭　潮　和彦
近畿大学理工学部
電気電子工学科　情報学科
〒 577-8502 東大阪市小若江 3-4-1
Tel: +81-6-6721-2332 (ext. 4555(藤本)　4615(潮))
Fax: +81-6-6727-2024(藤本)　+81-6-6730-1320(潮)
E-mail: fujimoto@ele.kindai.ac.jp　ushio@is.kindai.ac.jp

アブストラクト

グラフ理論において、グラフの分解問題は主要な研究テーマである。C_4、C_8 をそれぞれ 4 点、8 点を通るサイクルとする。1 点を共有する辺素な 2 個のサイクル C_4、C_8 からなるグラフを (C_4, C_8)-bowtie という。本研究では、完全グラフ K_n を (C_4, C_8)-bowtie 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード : 均衡的 (C_4, C_8)-bowtie 分解; 完全グラフ; グラフ理論

Balanced (C_4, C_8)-Bowtie Decomposition Algorithm of Complete Graphs

Hideaki Fujimoto　Kazuhiko Ushio
Department of Electric and Electronic Engineering　Department of Informatics
Faculty of Science and Technology
Kinki University
Osaka 577-8502, JAPAN
Tel: +81-6-6721-2332 (ext. 4555(Fujimoto)　4615(Ushio))
Fax: +81-6-6727-2024(Fujimoto)　+81-6-6730-1320(Ushio)
E-mail: fujimoto@ele.kindai.ac.jp　ushio@is.kindai.ac.jp

Abstract

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced (C_4, C_8)-bowtie decomposition algorithm of the complete graph K_n.

Keywords: Balanced (C_4, C_8)-bowtie decomposition; Complete graph; Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_8 be *the* 4-*cycle* and *the* 8-*cycle*, respectively. *The* (C_4, C_8)-*bowtie* is a graph of edge-disjoint C_4 and C_8 with a common vertex and the common vertex is called *the center of the* (C_4, C_8)-*bowtie*.
When K_n is decomposed into edge-disjoint sum of (C_4, C_8)-bowties, we say that K_n *has a* (C_4, C_8)-*bowtie decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_8)-bowties, we say that K_n *has a balanced* (C_4, C_8)-*bowtie decomposition* and this number is called

the replication number.
It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system*. See Colbourn and Rosa[1] and Wallis[5, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system*.
In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced (C_4, C_8)-bowtie decomposition of K_n is $n \equiv 1 \pmod{24}$.

2. Balanced (C_4, C_8)-bowtie decomposition of K_n

We use the following notation for a (C_4, C_8)-bowtie.

Notation. We denote a (C_4, C_8)-bowtie passing through $v_1 - v_2 - v_3 - v_4 - v_1 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11})\}$.

We have the following theorem.

Theorem. K_n has a balanced (C_4, C_8)-bowtie decomposition if and only if $n \equiv 1 \pmod{24}$.

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_8)-bowtie decomposition. Let b be the number of (C_4, C_8)-bowties and r be the replication number. Then $b = n(n-1)/24$ and $r = 11(n-1)/24$. Among r (C_4, C_8)-bowties having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_8)-bowties in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/24$ and $r_2 = 10(n-1)/24$. Therefore, $n \equiv 1 \pmod{24}$ is necessary.

(Sufficiency) Put $n = 24t + 1$. Construct tn (C_4, C_8)-bowties as follows:
$B_i^{(1)} = \{(i, i+4t+1, i+21t+2, i+5t+1), (i, i+2t+1, i+13t+2, i+13t+3, i+22t+4, i+21t+3, i+11t+2, i+3t+1)\}$
$B_i^{(2)} = \{(i, i+4t+2, i+21t+4, i+5t+2), (i, i+2t+2, i+13t+4, i+13t+6, i+22t+8, i+21t+6, i+11t+4, i+3t+2)\}$
$B_i^{(3)} = \{(i, i+4t+3, i+21t+6, i+5t+3), (i, i+2t+3, i+13t+6, i+13t+9, i+22t+12, i+21t+9, i+11t+6, i+3t+3)\}$
...
$B_i^{(t)} = \{(i, i+5t, i+23t, i+6t), (i, i+3t, i+15t, i+16t, i+26t, i+24t, i+13t, i+4t)\}$
$(i = 1, 2, ..., n)$,
where the additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.
Then they comprise a balanced (C_4, C_8)-bowtie decomposition of K_n.

Note. We consider the vertex set V of K_n as $V = \{1, 2, ..., n\}$.
The additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.

Example 1. Balanced (C_4, C_8)-bowtie decomposition of K_{25}.
$B_1 = \{(1, 6, 24, 7), (1, 4, 16, 17, 2, 25, 14, 5)\}$
$B_2 = \{(2, 7, 25, 8), (2, 5, 17, 18, 3, 1, 15, 6)\}$
$B_3 = \{(3, 8, 1, 9), (3, 6, 18, 19, 4, 2, 16, 7)\}$
$B_4 = \{(4, 9, 2, 10), (4, 7, 19, 20, 5, 3, 17, 8)\}$
$B_5 = \{(5, 10, 3, 11), (5, 8, 20, 21, 6, 4, 18, 9)\}$
$B_6 = \{(6, 11, 4, 12), (6, 9, 21, 22, 7, 5, 19, 10)\}$
$B_7 = \{(7, 12, 5, 13), (7, 10, 22, 23, 8, 6, 20, 11)\}$

$B_8 = \{(8,13,6,14),(8,11,23,24,9,7,21,12)\}$
$B_9 = \{(9,14,7,15),(9,12,24,25,10,8,22,13)\}$
$B_{10} = \{(10,15,8,16),(10,13,25,1,11,9,23,14)\}$
$B_{11} = \{(11,16,9,17),(11,14,1,2,12,10,24,15)\}$
$B_{12} = \{(12,17,10,18),(12,15,2,3,13,11,25,16)\}$
$B_{13} = \{(13,18,11,19),(13,16,3,4,14,12,1,17)\}$
$B_{14} = \{(14,19,12,20),(14,17,4,5,15,13,2,18)\}$
$B_{15} = \{(15,20,13,21),(15,18,5,6,16,14,3,19)\}$
$B_{16} = \{(16,21,14,22),(16,19,6,7,17,15,4,20)\}$
$B_{17} = \{(17,22,15,23),(17,20,7,8,18,16,5,21)\}$
$B_{18} = \{(18,23,16,24),(18,21,8,9,19,17,6,22)\}$
$B_{19} = \{(19,24,17,25),(19,22,9,10,20,18,7,23)\}$
$B_{20} = \{(20,25,18,1),(20,23,10,11,21,19,8,24)\}$
$B_{21} = \{(21,1,19,2),(21,24,11,12,22,20,9,25)\}$
$B_{22} = \{(22,2,20,3),(22,25,12,13,23,21,10,1)\}$
$B_{23} = \{(23,3,21,4),(23,1,13,14,24,22,11,2)\}$
$B_{24} = \{(24,4,22,5),(24,2,14,15,25,23,12,3)\}$
$B_{25} = \{(25,5,23,6),(25,3,15,16,1,24,13,4)\}$.

This decomposition can be written as follows:
$B_i = \{(i,i+5,i+23,i+6),(i,i+3,i+15,i+16,i+1,i+24,i+13,i+4)\}$
$(i = 1,2,...,25)$.

Example 2. Balanced (C_4, C_8)-bowtie decomposition of K_{49}.
$B_i^{(1)} = \{(i,i+9,i+44,i+11),(i,i+5,i+28,i+29,i+48,i+45,i+24,i+7)\}$
$B_i^{(2)} = \{(i,i+10,i+46,i+12),(i,i+6,i+30,i+32,i+3,i+48,i+26,i+8)\}$
$(i = 1,2,...,49)$.

Example 3. Balanced (C_4, C_8)-bowtie decomposition of K_{73}.
$B_i^{(1)} = \{(i,i+13,i+65,i+16),(i,i+7,i+41,i+42,i+70,i+66,i+35,i+10)\}$
$B_i^{(2)} = \{(i,i+14,i+67,i+17),(i,i+8,i+43,i+45,i+1,i+69,i+37,i+11)\}$
$B_i^{(3)} = \{(i,i+15,i+69,i+18),(i,i+9,i+45,i+48,i+5,i+72,i+39,i+12)\}$
$(i = 1,2,...,73)$.

Example 4. Blanced (C_4, C_8)-bowtie decomposition of K_{97}.
$B_i^{(1)} = \{(i,i+17,i+86,i+21),(i,i+9,i+54,i+55,i+92,i+87,i+46,i+13)\}$
$B_i^{(2)} = \{(i,i+18,i+88,i+22),(i,i+10,i+56,i+58,i+96,i+90,i+48,i+14)\}$
$B_i^{(3)} = \{(i,i+19,i+90,i+23),(i,i+11,i+58,i+61,i+3,i+93,i+50,i+15)\}$
$B_i^{(4)} = \{(i,i+20,i+92,i+24),(i,i+12,i+60,i+64,i+7,i+96,i+52,i+16)\}$
$(i = 1,2,...,97)$.

Example 5. Balanced (C_4, C_8)-bowtie decomposition of K_{121}.
$B_i^{(1)} = \{(i,i+21,i+107,i+26),(i,i+11,i+67,i+68,i+114,i+108,i+57,i+16)\}$
$B_i^{(2)} = \{(i,i+22,i+109,i+27),(i,i+12,i+69,i+71,i+118,i+111,i+59,i+17)\}$
$B_i^{(3)} = \{(i,i+23,i+111,i+28),(i,i+13,i+71,i+74,i+1,i+114,i+61,i+18)\}$
$B_i^{(4)} = \{(i,i+24,i+113,i+29),(i,i+14,i+73,i+77,i+5,i+117,i+63,i+19)\}$
$B_i^{(5)} = \{(i,i+25,i+115,i+30),(i,i+15,i+75,i+80,i+9,i+120,i+65,i+20)\}$
$(i = 1,2,...,121)$.

References

[1] C. J. Colbourn and A. Rosa, Triple Systems. Clarendom Press, Oxford (1999).

[2] P. Horák and A. Rosa, Decomposing Steiner triple systems into small configurations, *Ars Combinatoria* 26 (1988), pp. 91–105.

[3] K. Ushio and H. Fujimoto, Balanced bowtie and trefoil decomposition of complete tripartite multigraphs, IEICE Trans. Fundamentals, Vol.E84-A, No.3, pp.839–844, March 2001.

[4] K. Ushio and H. Fujimoto, Balanced foil decomposition of complete graphs, IEICE Trans. Fundamentals, Vol.E84-A, No.12, pp.3132–3137, December 2001.

[5] W. D. Wallis, Combinatorial Designs. Marcel Dekker, New York and Basel (1988).