完全グラフの均衡的 (C_4, C_7)-Bowtie 分解アルゴリズム

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アブストラクト

グラフ理論において、グラフの分解問題は主要な研究テーマである。C_4、C_7 をそれぞれ 4 点、7 点を通るサイクルとする。1 点を共有する辺素な 2 個のサイクル C_4、C_7 からなるグラフを (C_4, C_7)-bowtie という。本研究では、完全グラフ K_n を (C_4, C_7)-bowtie 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード : 均衡的 (C_4, C_7)-bowtie 分解; 完全グラフ; グラフ理論

Balanced (C_4, C_7)-Bowtie Decomposition Algorithm of Complete Graphs

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Abstract

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced (C_4, C_7)-bowtie decomposition algorithm of the complete graph K_n.

Keywords: Balanced (C_4, C_7)-bowtie decomposition; Complete graph; Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_7 be *the* 4-*cycle* and *the* 7-*cycle*, respectively. *The* (C_4, C_7)-*bowtie* is a graph of edge-disjoint C_4 and C_7 with a common vertex and the common vertex is called *the center of the* (C_4, C_7)-*bowtie*.
When K_n is decomposed into edge-disjoint sum of (C_4, C_7)-bowties, we say that K_n *has a* (C_4, C_7)-*bowtie decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_7)-bowties, we say that K_n *has a balanced* (C_4, C_7)-*bowtie decomposition* and this number is called

the replication number.
It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[1] and Wallis[5, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system.*
In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced (C_4, C_7)-bowtie decomposition of K_n is $n \equiv 1$ (mod 22).

2. Balanced (C_4, C_7)-bowtie decomposition of K_n

We use the following notation for a (C_4, C_7)-bowtie.

Notation. We denote a (C_4, C_7)-bowtie passing through $v_1 - v_2 - v_3 - v_4 - v_1 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7, v_8, v_9, v_{10})\}$.

We have the following theorem.

Theorem. K_n has a balanced (C_4, C_7)-bowtie decomposition if and only if $n \equiv 1$ (mod 22).

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_7)-bowtie decomposition. Let b be the number of (C_4, C_7)-bowties and r be the replication number. Then $b = n(n-1)/22$ and $r = 10(n-1)/22$. Among r (C_4, C_7)-bowties having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_7)-bowties in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/22$ and $r_2 = 9(n-1)/22$. Therefore, $n \equiv 1$ (mod 22) is necessary.

(Sufficiency) Put $n = 22t + 1$. We consider 9 cases.

Case 1. $t = 1$, $n = 23$. **(Example 1.)** Construct a balanced (C_4, C_7)-bowtie decomposition of K_{23} as follows:
$B_1 = \{(1, 6, 14, 7), (1, 2, 4, 8, 11, 21, 10)\}$
$B_2 = \{(2, 7, 15, 8), (2, 3, 5, 9, 12, 22, 11)\}$
$B_3 = \{(3, 8, 16, 9), (3, 4, 6, 10, 13, 23, 12)\}$
$B_4 = \{(4, 9, 17, 10), (4, 5, 7, 11, 14, 1, 13)\}$
$B_5 = \{(5, 10, 18, 11), (5, 6, 8, 12, 15, 2, 14)\}$
$B_6 = \{(6, 11, 19, 12), (6, 7, 9, 13, 16, 3, 15)\}$
$B_7 = \{(7, 12, 20, 13), (7, 8, 10, 14, 17, 4, 16)\}$
$B_8 = \{(8, 13, 21, 14), (8, 9, 11, 15, 18, 5, 17)\}$
$B_9 = \{(9, 14, 22, 15), (9, 10, 12, 16, 19, 6, 18)\}$
$B_{10} = \{(10, 15, 23, 16), (10, 11, 13, 17, 20, 7, 19)\}$
$B_{11} = \{(11, 16, 1, 17), (11, 12, 14, 18, 21, 8, 20)\}$
$B_{12} = \{(12, 17, 2, 18), (12, 13, 15, 19, 22, 9, 21)\}$
$B_{13} = \{(13, 18, 3, 19), (13, 14, 16, 20, 23, 10, 22)\}$
$B_{14} = \{(14, 19, 4, 20), (14, 15, 17, 21, 1, 11, 23)\}$
$B_{15} = \{(15, 20, 5, 21), (15, 16, 18, 22, 2, 12, 1)\}$
$B_{16} = \{(16, 21, 6, 22), (16, 17, 19, 23, 3, 13, 2)\}$
$B_{17} = \{(17, 22, 7, 23), (17, 18, 20, 1, 4, 14, 3)\}$
$B_{18} = \{(18, 23, 8, 1), (18, 19, 21, 2, 5, 15, 4)\}$
$B_{19} = \{(19, 1, 9, 2), (19, 20, 22, 3, 6, 16, 5)\}$
$B_{20} = \{(20, 2, 10, 3), (20, 21, 23, 4, 7, 17, 6)\}$

$B_{21} = \{(21, 3, 11, 4), (21, 22, 1, 5, 8, 18, 7)\}$
$B_{22} = \{(22, 4, 12, 5), (22, 23, 2, 6, 9, 19, 8)\}$
$B_{23} = \{(23, 5, 13, 6), (23, 1, 3, 7, 10, 20, 9)\}$.

This decomposition can be written as follows:
$B_i = \{(i, i+5, i+13, i+6), (i, i+1, i+3, i+7, i+10, i+20, i+9)\}$ $(i = 1, 2, ..., 23)$,
where the additions $i + x$ are taken modulo 23 with residues $1, 2, ..., 23$.

Note. We consider the vertex set V of K_n as $V = \{1, 2, ..., n\}$.
The additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.

Case 2. $t = 2$**,** $n = 45$**. (Example 2.)** Construct a balanced (C_4, C_7)-bowtie decomposition of K_{45} as follows:
$B_i^{(1)} = \{(i, i+3, i+32, i+5), (i, i+1, i+8, i+22, i+33, i+43, i+21)\}$
$B_i^{(2)} = \{(i, i+4, i+34, i+6), (i, i+2, i+10, i+23, i+35, i+44, i+19)\}$
$(i = 1, 2, ..., 45)$.

Case 3. $t = 3$**,** $n = 67$**. (Example 3.)** Construct a balanced (C_4, C_7)-bowtie decomposition of K_{67} as follows:
$B_i^{(1)} = \{(i, i+4, i+47, i+7), (i, i+1, i+11, i+32, i+48, i+63, i+28)\}$
$B_i^{(2)} = \{(i, i+5, i+49, i+8), (i, i+2, i+13, i+33, i+50, i+64, i+30)\}$
$B_i^{(3)} = \{(i, i+6, i+51, i+9), (i, i+3, i+15, i+34, i+52, i+65, i+29)\}$
$(i = 1, 2, ..., 67)$.

Case 4. $t = 4$**,** $n = 89$**. (Example 4.)** Construct a balanced (C_4, C_7)-bowtie decomposition of K_{89} as follows:
$B_i^{(1)} = \{(i, i+5, i+62, i+9), (i, i+1, i+14, i+42, i+63, i+83, i+39)\}$
$B_i^{(2)} = \{(i, i+6, i+64, i+10), (i, i+2, i+16, i+43, i+65, i+84, i+41)\}$
$B_i^{(3)} = \{(i, i+7, i+66, i+11), (i, i+3, i+18, i+44, i+67, i+85, i+38)\}$
$B_i^{(4)} = \{(i, i+8, i+68, i+12), (i, i+4, i+20, i+45, i+69, i+86, i+37)\}$
$(i = 1, 2, ..., 89)$.

Case 5. $t = 5$**,** $n = 111$**. (Example 5.)** Construct a balanced (C_4, C_7)-bowtie decomposition of K_{111} as follows:
$B_i^{(1)} = \{(i, i+6, i+77, i+11), (i, i+1, i+17, i+52, i+78, i+103, i+51)\}$
$B_i^{(2)} = \{(i, i+7, i+79, i+12), (i, i+2, i+19, i+53, i+80, i+104, i+47)\}$
$B_i^{(3)} = \{(i, i+8, i+81, i+13), (i, i+3, i+21, i+54, i+82, i+105, i+49)\}$
$B_i^{(4)} = \{(i, i+9, i+83, i+14), (i, i+4, i+23, i+55, i+84, i+106, i+48)\}$
$B_i^{(5)} = \{(i, i+10, i+85, i+15), (i, i+5, i+25, i+56, i+86, i+107, i+46)\}$
$(i = 1, 2, ..., 111)$.

Case 6. $t \equiv 2$ **(mod** 4**),** $t \geq 6$**,** $n = 22t + 1$**.** Put $t = 4p + 2$.
Consider a sequence $S : g_1, g_2, g_3, ..., g_{4p+2}$ with $S_1 : g_1, g_3, g_5, ..., g_{2p-1}$, $S_2 : g_2, g_4, g_6, ..., g_{2p}$, $S_3 : g_{2p+1}$, $S_4 : g_{2p+2}, g_{2p+3}, g_{2p+4}, ..., g_{4p+2}$ such as
$S_1 : 10t - 2, 10t - 4, 10t - 6, ..., 10t - 2p$
$S_2 : 10t + 1, 10t - 1, 10t - 3, ..., 10t - 2p + 3$
$S_3 : 10t - 2p + 1$
$S_4 : 10t - 2p - 1, 10t - 2p - 2, 10t - 2p - 3, ..., 9t + 1$.

Construct tn (C_4, C_7)-bowties as follows:
$B_i^{(1)} = \{(i, i+t+1, i+15t+2, i+2t+1), (i, i+1, i+3t+2, i+10t+2, i+15t+3, i+20t+3, i+g_1)\}$
$B_i^{(2)} = \{(i, i+t+2, i+15t+4, i+2t+2), (i, i+2, i+3t+4, i+10t+3, i+15t+5, i+20t+4, i+g_2)\}$
$B_i^{(3)} = \{(i, i+t+3, i+15t+6, i+2t+3), (i, i+3, i+3t+6, i+10t+4, i+15t+7, i+20t+5, i+g_3)\}$
...
$B_i^{(t)} = \{(i, i+2t, i+17t, i+3t), (i, i+t, i+5t, i+11t+1, i+17t+1, i+21t+2, i+g_t)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced (C_4, C_7)-bowtie decomposition of K_n.

Case 7. $t \equiv 3$ **(mod** 4**),** $t \geq 7$**,** $n = 22t + 1$**.** Put $t = 4p + 7$.
Consider a sequence $S : g_1, g_2, g_3, ..., g_{4p+7}$ with $S_1 : g_1, g_{2p+3}, g_{4p+5}, g_{4p+6}, g_{4p+7}$, $S_2 : g_2, g_3, g_4, ..., g_{2p+2}$, $S_3 : g_{2p+4}, g_{2p+6}, g_{2p+8}, ..., g_{4p+4}$, $S_4 : g_{2p+5}, g_{2p+7}, g_{2p+9}, ..., g_{4p+3}$ such as
$S_1 : 10t + 1, 10t - 2p - 3, 9t + 5, 9t + 3, 9t + 1$
$S_2 : 10t - 1, 10t - 2, 10t - 3, ..., 10t - 2p - 1$
$S_3 : 10t - 2p - 5, 10t - 2p - 7, 10t - 2p - 9, ..., 9t + 2$
$S_4 : 10t - 2p - 2, 10t - 2p - 4, 10t - 2p - 6, ..., 9t + 7$.
Construct tn (C_4, C_7)-bowties as follows:
$B_i^{(1)} = \{(i, i+t+1, i+15t+2, i+2t+1), (i, i+1, i+3t+2, i+10t+2, i+15t+3, i+20t+3, i+g_1)\}$
$B_i^{(2)} = \{(i, i+t+2, i+15t+4, i+2t+2), (i, i+2, i+3t+4, i+10t+3, i+15t+5, i+20t+4, i+g_2)\}$
$B_i^{(3)} = \{(i, i+t+3, i+15t+6, i+2t+3), (i, i+3, i+3t+6, i+10t+4, i+15t+7, i+20t+5, i+g_3)\}$
...
$B_i^{(t)} = \{(i, i+2t, i+17t, i+3t), (i, i+t, i+5t, i+11t+1, i+17t+1, i+21t+2, i+g_t)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced (C_4, C_7)-bowtie decomposition of K_n.

Case 8. $t \equiv 0$ **(mod** 4**),** $t \geq 8$**,** $n = 22t + 1$**.** Put $t = 4p + 4$.
Consider a sequence $S : g_1, g_2, g_3, ..., g_{4p+4}$ with $S_1 : g_1, g_3, g_5, ..., g_{2p-1}$, $S_2 : g_2, g_4, g_6, ..., g_{2p+2}$, $S_3 : g_{2p+1}$, $S_4 : g_{2p+3}, g_{2p+4}, g_{2p+5}, ..., g_{4p+4}$ such as
$S_1 : 10t - 2, 10t - 4, 10t - 6, ..., 10t - 2p$
$S_2 : 10t + 1, 10t - 1, 10t - 3, ..., 10t - 2p + 1$
$S_3 : 10t - 2p - 1$
$S_4 : 10t - 2p - 2, 10t - 2p - 3, 10t - 2p - 4, ..., 9t + 1$.
Construct tn (C_4, C_7)-bowties as follows:
$B_i^{(1)} = \{(i, i+t+1, i+15t+2, i+2t+1), (i, i+1, i+3t+2, i+10t+2, i+15t+3, i+20t+3, i+g_1)\}$
$B_i^{(2)} = \{(i, i+t+2, i+15t+4, i+2t+2), (i, i+2, i+3t+4, i+10t+3, i+15t+5, i+20t+4, i+g_2)\}$
$B_i^{(3)} = \{(i, i+t+3, i+15t+6, i+2t+3), (i, i+3, i+3t+6, i+10t+4, i+15t+7, i+20t+5, i+g_3)\}$
...
$B_i^{(t)} = \{(i, i+2t, i+17t, i+3t), (i, i+t, i+5t, i+11t+1, i+17t+1, i+21t+2, i+g_t)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced (C_4, C_7)-bowtie decomposition of K_n.

Case 9. $t \equiv 1$ **(mod** 4**),** $t \geq 9$**,** $n = 22t + 1$**.** Put $t = 4p + 9$.
Consider a sequence $S : g_1, g_2, g_3, ..., g_{4p+9}$ with $S_1 : g_1, g_{2p+5}, g_{4p+7}, g_{4p+8}, g_{4p+9}$, $S_2 : g_2, g_3, g_4, ..., g_{2p+3}$, $S_3 : g_{2p+4}, g_{2p+6}, g_{2p+8}, ..., g_{4p+6}$, $S_4 : g_{2p+7}, g_{2p+9}, g_{2p+11}, ..., g_{4p+5}$ such as
$S_1 : 10t + 1, 10t - 2p - 3, 9t + 5, 9t + 3, 9t + 1$
$S_2 : 10t - 1, 10t - 2, 10t - 3, ..., 10t - 2p - 2$
$S_3 : 10t - 2p - 5, 10t - 2p - 7, 10t - 2p - 9, ..., 9t + 2$
$S_4 : 10t - 2p - 4, 10t - 2p - 6, 10t - 2p - 8, ..., 9t + 7$.
Construct tn (C_4, C_7)-bowties as follows:

$B_i^{(1)} = \{(i, i+t+1, i+15t+2, i+2t+1), (i, i+1, i+3t+2, i+10t+2, i+15t+3, i+20t+3, i+g_1)\}$
$B_i^{(2)} = \{(i, i+t+2, i+15t+4, i+2t+2), (i, i+2, i+3t+4, i+10t+3, i+15t+5, i+20t+4, i+g_2)\}$
$B_i^{(3)} = \{(i, i+t+3, i+15t+6, i+2t+3), (i, i+3, i+3t+6, i+10t+4, i+15t+7, i+20t+5, i+g_3)\}$
...
$B_i^{(t)} = \{(i, i+2t, i+17t, i+3t), (i, i+t, i+5t, i+11t+1, i+17t+1, i+21t+2, i+g_t)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced (C_4, C_7)-bowtie decomposition of K_n.

This completes the proof.

Example 6. A balanced (C_4, C_7)-bowtie decomposition of K_{133}.
$B_i^{(1)} = \{(i, i+7, i+92, i+13), (i, i+1, i+20, i+62, i+93, i+123, i+58)\}$
$B_i^{(2)} = \{(i, i+8, i+94, i+14), (i, i+2, i+22, i+63, i+95, i+124, i+61)\}$
$B_i^{(3)} = \{(i, i+9, i+96, i+15), (i, i+3, i+24, i+64, i+97, i+125, i+59)\}$
$B_i^{(4)} = \{(i, i+10, i+98, i+16), (i, i+4, i+26, i+65, i+99, i+126, i+57)\}$
$B_i^{(5)} = \{(i, i+11, i+100, i+17), (i, i+5, i+28, i+66, i+101, i+127, i+56)\}$
$B_i^{(6)} = \{(i, i+12, i+102, i+18), (i, i+6, i+30, i+67, i+103, i+128, i+55)\}$
$(i = 1, 2, ..., 133)$.

Example 7. A balanced (C_4, C_7)-bowtie decomposition of K_{155}.
$B_i^{(1)} = \{(i, i+8, i+107, i+15), (i, i+1, i+23, i+72, i+108, i+143, i+71)\}$
$B_i^{(2)} = \{(i, i+9, i+109, i+16), (i, i+2, i+25, i+73, i+110, i+144, i+69)\}$
$B_i^{(3)} = \{(i, i+10, i+111, i+17), (i, i+3, i+27, i+74, i+112, i+145, i+67)\}$
$B_i^{(4)} = \{(i, i+11, i+113, i+18), (i, i+4, i+29, i+75, i+114, i+146, i+65)\}$
$B_i^{(5)} = \{(i, i+12, i+115, i+19), (i, i+5, i+31, i+76, i+116, i+147, i+68)\}$
$B_i^{(6)} = \{(i, i+13, i+117, i+20), (i, i+6, i+33, i+77, i+118, i+148, i+66)\}$
$B_i^{(7)} = \{(i, i+14, i+119, i+21), (i, i+7, i+35, i+78, i+120, i+149, i+64)\}$
$(i = 1, 2, ..., 155)$.

Example 8. A balanced (C_4, C_7)-bowtie decomposition of K_{177}.
$B_i^{(1)} = \{(i, i+9, i+122, i+17), (i, i+1, i+26, i+82, i+123, i+163, i+78)\}$
$B_i^{(2)} = \{(i, i+10, i+124, i+18), (i, i+2, i+28, i+83, i+125, i+164, i+81)\}$
$B_i^{(3)} = \{(i, i+11, i+126, i+19), (i, i+3, i+30, i+84, i+127, i+165, i+77)\}$
$B_i^{(4)} = \{(i, i+12, i+128, i+20), (i, i+4, i+32, i+85, i+129, i+166, i+79)\}$
$B_i^{(5)} = \{(i, i+13, i+130, i+21), (i, i+5, i+34, i+86, i+131, i+167, i+76)\}$
$B_i^{(6)} = \{(i, i+14, i+132, i+22), (i, i+6, i+36, i+87, i+133, i+168, i+75)\}$
$B_i^{(7)} = \{(i, i+15, i+134, i+23), (i, i+7, i+38, i+88, i+135, i+169, i+74)\}$
$B_i^{(8)} = \{(i, i+16, i+136, i+24), (i, i+8, i+40, i+89, i+137, i+170, i+73)\}$
$(i = 1, 2, ..., 177)$.

Example 9. A balanced (C_4, C_7)-bowtie decomposition of K_{199}.
$B_i^{(1)} = \{(i, i+10, i+137, i+19), (i, i+1, i+29, i+92, i+138, i+183, i+91)\}$
$B_i^{(2)} = \{(i, i+11, i+139, i+20), (i, i+2, i+31, i+93, i+140, i+184, i+89)\}$
$B_i^{(3)} = \{(i, i+12, i+141, i+21), (i, i+3, i+33, i+94, i+142, i+185, i+88)\}$
$B_i^{(4)} = \{(i, i+13, i+143, i+22), (i, i+4, i+35, i+95, i+144, i+186, i+85)\}$
$B_i^{(5)} = \{(i, i+14, i+145, i+23), (i, i+5, i+37, i+96, i+146, i+187, i+87)\}$

$B_i^{(6)} = \{(i, i+15, i+147, i+24), (i, i+6, i+39, i+97, i+148, i+188, i+83)\}$
$B_i^{(7)} = \{(i, i+16, i+149, i+25), (i, i+7, i+41, i+98, i+150, i+189, i+86)\}$
$B_i^{(8)} = \{(i, i+17, i+151, i+26), (i, i+8, i+43, i+99, i+152, i+190, i+84)\}$
$B_i^{(9)} = \{(i, i+18, i+153, i+27), (i, i+9, i+45, i+100, i+154, i+191, i+82)\}$
$(i = 1, 2, ..., 199)$.

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