4-colorable グラフの点彩色問題に対する近似アルゴリズム

謝 旭珍、小野 孝男、平田 富夫
〒 464-8603 名古屋市千種区不老町
名古屋大学大学院工学研究科
電話: +81-52-789-3440
FAX: +81-52-789-3089
Email:
sharryx@hirata.nuee.nagoya-u.ac.jp, takao@hirata.nuee.nagoya-u.ac.jp, hirata@nuee.nagoya-u.ac.jp

概要

本論文では、グラフの点彩色問題とその近似アルゴリズムについて議論する。まず、Karger-Motwani-Sudan のアルゴリズムと Blum-Karger のアルゴリズムを組合せた 4-colorable グラフに対する $\tilde{O}(n^{7/18})$ 色のアルゴリズムを提案する。また、これを一般化して、k-colorable グラフの点彩色問題に対する $\tilde{O}(n^{1-\frac{1}{(k+1)/3-4/(11k^2-11k)}})$ 色のアルゴリズムを与える。これは、Karger らによる $\tilde{O}(n^{1-3/(k+1)})$ 色のアルゴリズムをわずかではあるか改良している。

キーワード

点彩色問題、近似アルゴリズム、NP-困難

Simple Combination of Algorithms for 4-colorable Graphs

XuZhen Xie, Takao Ono, Tomio Hirata
Graduate School of Engineering
Nagoya University
Furo-cho, Chikusa-ku
Nagoya 464–8603, Japan
Telephone: +81-52-789-3440
Fax: +81-52-789-3089
Email:
sharryx@hirata.nuee.nagoya-u.ac.jp, takao@hirata.nuee.nagoya-u.ac.jp, hirata@nuee.nagoya-u.ac.jp

Abstract

In this paper, a combination of algorithms for graph coloring is discussed. The algorithms of Karger-Motwani-Sudan and Blum-Karger can be combined in a simple way to yield a polynomial-time algorithm for an $\tilde{O}(n^{7/18})$-coloring of any n-vertex 4-colorable graph. This result can be generalized to k-colorable graphs to obtain a coloring with $\tilde{O}(n^{1-\frac{1}{(k+1)/3-4/(11k^2-11k)}})$ colors, which slightly improves the bound given by Karger et al. of $\tilde{O}(n^{1-3/(k+1)})$ colors, for any $k \geq 3$.

Keywords

Graph coloring, approximation algorithms, NP-completeness.

I. Introduction

A proper vertex coloring of a graph $G = (V, E)$ is an assignment of colors to its vertices such that no two adjacent vertices receive the same color. The **chromatic number** $\chi(G)$ is the minimum number of colors for a proper vertex coloring. It is well known [1] [2] that the problem of properly coloring a graph of chromatic number k with k colors is NP-hard, for any $k \geq 3$.

A varity of applications such as register allocation [3] [4] [5] and timetable/examination scheduling [6] [7] can be formulated as graph coloring problems. For most applications, it is sufficient to find an

approximately optimal graph coloring, that is, a coloring of the graph with a small number of colors. This along with the apparent impossibility of an exact solution has led to great interest in the problem of approximate graph coloring.

The analysis of approximation algorithms for graph coloring started with the work of Johnson [8], who showed that a version of the greedy algorithm gives an $O(n/\log n)$-approximation algorithm for k-coloring. Wigderson [9] gave a simple algorithm for coloring k-colorable graphs with $O(n^{1-1/(k-1)})$ colors. Blum [10] provided an algorithm for coloring k-colorable graphs with $\tilde{O}(n^{\alpha_k})$ colors, where the α_k satisfies a complicated recurrence relation. The first values of this sequence are $\alpha_3 = 3/8, \alpha_4 = 3/5$, and $\alpha_5 = 91/131$. Karger, Motwani and Sudan [11] showed, using semidefinite programming, that k-colorable graphs of maximum degree Δ can be colored with $\tilde{O}(\Delta^{1-2/k})$ colors, where the $\tilde{O}$ notation hides a poly-logarithmic factor. Combined with the technique of Wigderson [9], they presented a polynomial-time algorithm for coloring k-colorable graphs using $\tilde{O}(n^{1-3/(k+1)})$ colors. By combining the result of [11] with [10], Blum and Karger [12] obtained a polynomial-time algorithm that colors any 3-colorable graph with $\tilde{O}(n^{3/14})$ colors. In this paper we show how the coloring algorithms of [11] and [12] can be combined in a simple way to yield a polynomial-time algorithm for coloring any 4-colorable graph with $\tilde{O}(n^{7/18})$ colors. More generally, k-colorable graphs can be colored in polynomial time with $\tilde{O}(n^{C_k})$ colors, where $C_2 = 0$ and $C_k = 1 - \frac{1}{(k+1)/3-4/(11k^2-11k)}$, for $k \geq 3$.

Wigderson's algorithm [9], which colors any 3-colorable graph with $O(n^{1/2})$ colors, considers the graph in two cases. If there is a vertex with large degree, color its neighbors with 2 colors and set them aside. The algorithm then recurses on the remaining graph using new colors. Once the maximum degree is reduced to some small Δ, the algorithm colors the remaining graph with $(\Delta + 1)$ colors. The basic idea of our algorithm is similar. When the graph has large degree, we use the techniques of [12] to color large "3-colorable neighbors" of a vertex. When the degree is reduced to some small Δ, we use the algorithm of [11] to color the remaining graph with $\tilde{O}(\Delta^{1/2})$ colors.

Algorithms	k = 3	k = 4	k = 5	k = 6	k = 7	General
Wigderson (1983)	1/2 0.5	2/3 0.667	3/4 0.75	4/5 0.8	5/6 0.833	$1 - \frac{1}{k-1}$
Blum (1994)	3/8 0.375	3/5 0.6	91/131 0.695	105/137 0.766	5301/6581 0.806	– –
Karger, Motwani, Sudan(1998)	1/4 0.25	2/5 0.4	1/2 0.5	4/7 0.571	5/8 0.625	$1 - \frac{3}{k+1}$
Our Results ($k \geq 4, 2002$)	(3/14) (0.214)	7/18 0.389	54/109 0.495	218/383 0.569	383/614 0.624	$1 - \frac{1}{\frac{k+1}{3} - \frac{4}{11k^2-11k}}$

Table 1. The Previously Available Algorithms and Their Performance

II. Preliminaries and Definitions

Let us introduce the graph-theoretic notation that will be used throughout this paper. Given a graph G, let $V(G)$ denote the vertices of G and $E(G)$ denote the edges of G. We will use $N_G(v)$ to denote the neighborhood of a vertex v, $d_G(v)$ to denote the degree of v and $\Delta(G)$ to denote the maximum degree of the graph. That is, for $G = (V, E)$,

$$\begin{aligned} N_G(v) &= \{u|(v,u) \in E\}, \\ d_G(v) &= |N_G(v)|, \\ \Delta(G) &= \max_{v \in V}\{d_G(v)\}. \end{aligned}$$

The subgraph of G induced by $U \subseteq V$ is the graph $H(U, F)$, where

$$F = \{(u,w)|u \in U, w \in U, \text{ and } (u,w) \in E\}.$$

III. Previous Results

A. The Karger-Motwani-Sudan Algorithm

Karger, Motwani and Sudan [11] introduced the notion of vector colorings of a graph, which is closely related to Lovász's orthogonal representations and ϑ-function [13] [14] :

Definition ([11]) Given a graph $G = (V, E)$ on n vertices and a real number $k \geq 1$, a vector k-coloring of G is an assignment of n-dimensional unit vectors v_i to each vertex $i \in V$, such that for any two adjacent vertices i and j the dot product of their vectors satisfies the inequality

$$\langle v_i, v_j \rangle \square -\frac{1}{k-1}. \qquad (1)$$

Karger, Motwani and Sudan [11] obtained the following results.

Theorem 1 ([11]) Every k-colorable graph G has a vector k-coloring.

Theorem 2 ([11]) Any vector k-colorable graph on n vertices with maximum degree Δ can be colored, in probabilistic polynomial time, using $\tilde{O}(\Delta^{1-2/k})$ colors.

It is easy to see that if G is k-colorable then G also has a vector k-coloring. The semidefinite programming based coloring algorithm of Karger-Motwani-Sudan [11] can also be used to color k-colorable graphs of maximum degree Δ using $\tilde{O}(\Delta^{1-2/k})$ colors.

B. The Blum-Karger Algorithm

By combining the results of [11] with [10] , Blum and Karger [12] proved the following result.

Theorem 3 ([12]) There is a polynomial-time algorithm to color any n-vertex 3-colorable graph with $\tilde{O}(n^{3/14})$ colors.

C. The Wigderson Algorithm

Wigderson [9] presented the following simple algorithm based on the two simple facts.

Fact 4 ([9]) Any graph G can be colored in polynomial time with at most $\Delta(G) + 1$ colors.

Fact 5 ([9]) Let $G = (V, E)$ be a 3-colorable graph. Then for every vertex $v \in V$, the subgraph of G induced by $N_G(v)$ is bipartite(2-colorable) and thus can be 2-colored in polynomial time.

Wigderson's Algorithm W [9]
Input: An n-vertex 3-colorable graph $G = (V, E)$
Output: An $O(n^{1/2})$-coloring of G
1. $n \leftarrow |V|$.
2. $i \leftarrow 1$.
3. While $\Delta(G) \geq \lceil n^{1/2} \rceil$ do:
   Let v be a vertex of maximum degree in G.
   $H \leftarrow$ the subgraph of G induced by $N_G(v)$.
   Color H with colors $i, i+1$.
   Color v with color $i+2$.
   $i \leftarrow i+2$.
   $G \leftarrow$ the subgraph of G, obtained by deleting the vertices in $N_G(v) \cup \{v\}$ from G.
4. $(\Delta(G) < \lceil n^{1/2} \rceil)$
   Color G with colors $i, i+1, i+2, ..., i+\Delta(G)$ and halt.

Theorem 6 ([9]) Algorithm W colors any 3-colorable graph $G = (V, E)$ on n vertices with $O(n^{1/2})$ colors, and runs in polynomial time.

IV. Simple Combination of Karger-Motwani-Sudan and Blum-Karger

By simply combining the results of Karger-Motwani-Sudan [11] and Blum-Karger [12], we first introduce a new algorithm for coloring 4-colorable graph with $\tilde{O}(n^{7/18})$ colors.

Based on Fact 7, Theorems 2 and 3, we propose our algorithm as follows.

Fact 7 ([9]) Let $G=(V,E)$ be a k-colorable graph. Then for every vertex $v \in V$, the subgraph of G induced by $N_G(v)$ is $(k-1)$-colorable.

Algorithm-4

Input: An n-vertex 4-colorable graph $G=(V,E)$
Output: An $\tilde{O}(n^{7/18})$-coloring of G
1. $n \leftarrow |V|$.
2. While $\Delta(G) > n^{7/9}$ do:
   Let v be a vertex of maximum degree in G.
   $N'_G(v) \leftarrow$ a subset of $N_G(v)$ with $|N'_G(v)| = n^{7/9}$.
   $H \leftarrow$ the subgraph of G induced by $N'_G(v)$.
   Use the algorithm of Blum-Karger [12] to color H with $\tilde{O}(n^{1/6})$ colors.
   Remove the colors from the palette.
   $G \leftarrow$ the subgraph of G, obtained by removing from G all the vertices in $N'_G(v)$.
3. $(\Delta(G) \Box\ n^{7/9})$
   Use the algorithm of Karger-Motwani-Sudan [11], to color G with $\tilde{O}(\Delta(G)^{1/2})$=$\tilde{O}(n^{7/18})$ colors and halt.

Theorem 8 Algorithm-4 runs in probabilistic polynomial time and it colors any 4-colorable graph on n vertices with $\tilde{O}(n^{7/18})$ colors.

Proof. Based on Fact 7, the subgraph H induced by $N'_G(v)$ is 3-colorable, and can be colored by the algorithm of Blum-Karger [12] with $|N'_G(v)|^{3/14}$ colors. We observe that each time Step 2 is executed, we color $n^{7/9}$ vertices using a new set of $\tilde{O}(n^{1/6})$ colors. Note that this step can be executed at most $n^{2/9}$ times, and thus the number of colors used in this step is

$$n^{2/9}\tilde{O}(n^{1/6}) = \tilde{O}(n^{7/18}). \tag{2}$$

Step 3 is executed only once and uses $\tilde{O}(n^{7/18})$ colors by Theorem 2, so the total number of colors used in both steps is bound to $\tilde{O}(n^{7/18})$. The time bound follows from the fact that both algorithms of [12] and [11] run in probabilistic polynomial time.

How can we use the idea of Algorithm-4 to color k-colorable graphs in polynomial time for any k, and what upper bound on the number of colors can we guarantee? Still based on Fact 7, we can recursively use the algorithm for $(k-1)$-colorable graphs in the one for k-colorable graphs.

Algorithm-k

Input: An integer $k \geq 2$ and an n-vertex k-colorable graph $G=(V,E)$
Output: An $\tilde{O}(n^{C_k})$-coloring of G
1. If $k=2$, color the graph with 2 colors, in linear time.
2. If $k=3$, use the algorithm of Blum-Karger [12] to color the graph with $\tilde{O}(n^{3/14})$ colors.
3. For $k \geq 4$, let Algorithm-3 refer to the algorithm of Blum-Karger [12].
   Assume that the graph can be colored with $\tilde{O}(n^{C_k})$ colors. For $k=3$, $C_3 = 3/14$.
4. [Recursive coloring stage for $k \geq 4$]
   While $\Delta(G) > n^{\frac{C_k}{1-2/k}}$ do:
   Let v be a vertex of maximum degree in G.
   $N'_G(v) \leftarrow$ a subset of $N_G(v)$ with $|N'_G(v)| = n^{\frac{C_k}{1-2/k}}$.

$H \leftarrow$ the subgraph of G induced by $N'_G(v)$.

Use Algorithm-$(k-1)$ to color H with $\tilde{O}(n^{\frac{C_k C_{k-1}}{1-2/k}})$ colors.

Remove the colors from the palette.

$G \leftarrow$ the subgraph of G, obtained by removing from G all the vertices in $N'_G(v)$.

5. [Karger-Motwani-Sudan coloring stage for $k \geq 4$]

$(\Delta(G) \square\ n^{\frac{C_k}{1-2/k}})$

Use the algorithm of Karger-Motwani-Sudan [11] to color G with $\tilde{O}(\Delta(G)^{1-2/k})$=$\tilde{O}(n^{C_k})$ colors and halt.

Theorem 9 Algorithm-k runs in probabilistic polynomial time and it colors any k-colorable graph on n vertices with $\tilde{O}(n^{C_k})$ colors, where $C_2 = 0$ and $C_k = 1 - \frac{1}{(k+1)/3-4/(11k^2-11k)}$, for $k \geq 3$.

Proof. As in the previous proof, still based on Fact 7, the subgraph H induced by $N'_G(v)$ is $(k-1)$-colorable, thus H can be colored by Algorithm-$(k-1)$ with $|N'_G(v)|^{C_{k-1}}$ colors. It is easy to observe that each time Step 4 is executed, we color $n^{\frac{C_k}{1-2/k}}$ vertices using a new set of $\tilde{O}(n^{\frac{C_k C_{k-1}}{1-2/k}})$ colors. Note that this step can be executed at most $n^{1-\frac{C_k}{1-2/k}}$ times, and thus the number of colors used in this step is

$$n^{1-\frac{C_k}{1-2/k}} \tilde{O}(n^{\frac{C_k C_{k-1}}{1-2/k}}) = \tilde{O}(n^{1+\frac{C_k(C_{k-1}-1)}{1-2/k}}). \tag{3}$$

Step 5 is executed only once and uses $\tilde{O}(n^{C_k})$ colors by Theorem 2, so the total number of colors used in both steps is bound to $\tilde{O}(n^{C_k})$ if the following equality holds:

$$C_k = 1 + \frac{C_k(C_{k-1}-1)}{1-2/k}. \tag{4}$$

Solving this equation with respect to C_k, we obtain the recurrence relation:

$$C_k = \frac{k-2}{2k-2-kC_{k-1}}. \tag{5}$$

We can rewrite this relation as follows:

$$\frac{k(k-1)}{1-C_k} = k(k-1) + \frac{(k-1)(k-2)}{1-C_{k-1}}.$$

Let

$$\beta_k = \frac{k(k-1)}{1-C_k},$$

then

$$\beta_3 = \frac{84}{11}$$

and

$$\beta_k = k(k-1) + \beta_{k-1}.$$

We can prove that

$$\beta_k = \frac{1}{3}k(k-1)(k+1) - \frac{4}{11},$$

thus

$$C_k = 1 - \frac{1}{\frac{k+1}{3} - \frac{4}{11k^2-11k}}, \quad k \geq 3. \tag{6}$$

V. Concluding remarks

We proposed a coloring algorithm that slightly improves the result of Karger, Motwani and Sudan [11]. It is interesting to obtain further improvements. Blum [10] stated several ways of coloring techniques towards an $\tilde{O}(n^{\alpha_k})$-coloring of graphs. Finding large independent sets in a graph seems significant to reduce the number of colors. The algorithm of Karger-Motwani-Sudan [11] is used also to find large independent sets. Using the semidefinite programming technique, Alon and Kahale [15] obtained an algorithm that can be used to find large independent sets in a graph containing very large independent sets. Recently, Halperin et. al. showed that the algorithm of Alon and Kahale can be combined to obtain better approximation algorithms for graph coloring [16]. Coloring 3-colorable graphs using 4-colors is also known to be NP-hard [17] [18]. It is a challenge to find better results than $\tilde{O}(n^{3/14})$-coloring for 3-colorable graphs, which is supplied by Blum and Karger [12].

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