グラフの準平衡彩色 — ディスクレパンシー条件による２色彩色

ジェスパー　ジャンソン（ルンド大学）、徳山　豪（東北大学）

概要　G を n 頂点と m 辺をもつ無向グラフとする。G の準平衡彩色とは、G の最短路ハイパーグラフのディスクレパンシー条件に基づく２色彩色であり、疑平衡彩色たちは G の独立集合のあるクラスに対応する。本論文では、G の異なった準平衡彩色の数と数え上げについての結果を与える。即ち、二部グラフなら高々$n+1$, トライアングルフリーなら m, 一般に $m+1$ が準平衡彩色の数の上界である。更に、これら全ての準平衡彩色は $O(nm^2)$ 時間で列挙される。

Semi-Balanced Colorings of Graphs –
2-Colorings Based on a Relaxed Discrepancy Condition

Jesper Jansson[1] and Takeshi Tokuyama[2]

[1] Department of Computer Science, Lund University, Box 118, 221 00 Lund, Sweden,
Jesper.Jansson@cs.lth.se

[2] Graduate School of Information Sciences, Tohoku University, 980-8579 Sendai, Japan,
tokuyama@dais.is.tohoku.ac.jp

Abstract. Let G be an undirected, unweighted, connected graph with n vertices and m edges. We introduce the concept of a semi-balanced coloring of G, which is a 2-coloring of the shortest-paths hypergraph of G under a certain discrepancy condition yielding a class of independent sets of G. We prove that the number of different semi-balanced colorings of G is: (1) at most $n+1$ if G is bipartite; (2) at most m if G is non-bipartite and triangle-free; and (3) at most $m+1$ if G is non-bipartite. Based on the above combinatorial investigation, we design an algorithm that enumerates all semi-balanced colorings in $O(nm^2)$ time.

1 Introduction

Given a set V, a *coloring of* V is a mapping π from V to $\{-1, 1\}$. For a graph $G = (V, E)$, a coloring π of the vertex set V is called a *2-coloring of* G if $\pi(x) \neq \pi(y)$ for every edge $\{x, y\}$ in E. We call a vertex which has been mapped to 1 (resp. -1) a *red* (resp. *blue*) vertex. A graph has a 2-coloring if and only if it is bipartite; in fact, by symmetry, a bipartite graph always has two different 2-colorings. A natural way to extend 2-colorings is by allowing k colors to be used, where k is any positive integer. Such a coloring is called a *k-coloring of* G. The number of possible k-colorings of a graph is given by its *chromatic polynomial*, and has been studied extensively (see [10] or [12]).

Another way to generalize 2-colorings is by relaxing the restriction on two adjacent vertices never being allowed to have the same color. If we only require that no blue vertices are adjacent to each other, the problem of coloring the graph becomes equivalent to the problem of finding an independent set (often called stable set) in the graph since any set of blue vertices then forms an independent set of G. However, the number of different independent sets of G can be very large, and we usuall want a “good” one satisfying some additional restrictions: The *maximum* independent set problem and the *minimum maximal* independent set problem are famous examples, in each of which the additional restriction is basically quantative and results in an optimization problem that is hard to approximate within a factor $n^{1-\epsilon}$ under some hypothesis on computational hierarchy [4].

Discrepancy conditions

In this paper, we consider a class of independent sets with an imposed structural condition.

We can observe that the red and blue vertices along any path in a 2-colored bipartite graph are always arranged in an alternating fashion. Thus, $-1 \leq \sum_{v \in P} \pi(v) \leq 1$ must hold for the set P of vertices on any path in the graph. This can be regarded as a discrepancy condition.

Discrepancy is a popular measure of uniformity and the quality of approximations, and has been used in combinatorics, geometry, and Monte-Carlo simulations [5, 8, 9]. It is defined as follows. Let $H = (V, \mathcal{F})$ be a hypergraph, where $\mathcal{F} \subseteq 2^V$. Given a coloring π of V, let $\pi(F) = \sum_{v \in F} \pi(v)$ for every $F \in \mathcal{F}$ and let $D_c(H, \pi) = \max_{F \in \mathcal{F}} |\pi(F)|$. The *combinatorial* (or *homogeneous*) *discrepancy* $D_c(H)$ is defined as $D_c(H) = \min_\pi D_c(H, \pi)$, where we take the minimum over all possible colorings of V. In particular, if $D_c(H, \pi) \leq 1$ then π yields a coloring which is uniform in every hyperedge; this means that $-1 \leq \pi(F) \leq 1$ for every $F \in \mathcal{F}$. Such a π is called a *balanced coloring* of H. We call a coloring π *semi-balanced* if $-1 \leq \pi(F) \leq 2$ for every $F \in \mathcal{F}$.

The *shortest-paths hypergraph induced by* G is the hypergraph $\mathcal{H}(G) = (V, \mathcal{P}_G)$, where $\mathcal{P}_G$ is the set of all shortest-path vertex sets in G. A 2-coloring of G is equivalent to a balanced coloring of $\mathcal{H}(G)$; hence, we generalize 2-colorings of G by considering semi-balanced colorings of $\mathcal{H}(G)$. A balanced (semi-balanced) coloring of $\mathcal{H}(G)$ is also called a balanced (semi-balanced) coloring of G.

Naturally, the set of all blue vertices of a semi-balanced coloring is an independent set that is either maximal or submaximal. Although a semi-balanced coloring does not always exist, the semi-balancing condition defines a nonempty subpolytope of the *stable set polytope* of G [12]. Moreover, for any independent set W in G, there is a supergraph G' of G obtained by adding suitable edges such that W is the set of blue nodes in a semi-balanced coloring of G'. Thus, the set of independent sets of G corresponds to the union of sets of semi-balanced colorings of supergraphs of G, and the correspondence yields a covering structure of the set of independent sets. This motivates us to study combinatorics and algorithms for semi-balanced colorings of a graph.

Our results

The most fundamental combinatorial themes are counting and enumeration. In this paper, we show that the number of semi-balanced colorings is always polynomial in the input size. More precisely, we prove that if G is a connected graph with n vertices and m edges, the number of different semi-balanced colorings is: (1) at most $n+1$ if G is bipartite; (2) at most m if G is non-bipartite and triangle-free; and (3) at most $m+1$ if G is non-bipartite. Moreover, we can enumerate all the semi-balanced colorings of G in $O(nm^2)$ time; thus, this version of the independent set problem is polynomial time soluble.

Because of space limitation, we only deal with (1) and (2) in the present paper, and (3) will be given in our companion paper.

Relation to a rounding problem

Another motivation for studying semi-balanced colorings comes from a conjecture in [2] called the *rounding conjecture*. Given a hypergraph $H = (V, \mathcal{F})$, where $\mathcal{F} \subseteq 2^V$, along with a real-valued function $\alpha : V \to [0, 1]$, a *rounding of* α is any function from V to $\{0, 1\}$. For every rounding β of α, define the linear discrepancy $D_\ell(H, \alpha, \beta) = \max_{F \in \mathcal{F}} |\alpha(F) - \beta(F)|$, where $\alpha(F) = \sum_{v \in F} \alpha(v)$ and $\beta(F) = \sum_{v \in F} \beta(v)$. Roundings with low linear discrepancy have several applications including digital halftoning [2, 3, 1, 6, 11]. If, for a rounding β of α, it holds that $D_\ell(H, \alpha, \beta) < 1$ then β is called a *global rounding of* α *in* H. If $\mathcal{F} = \mathcal{P}_G$ for a graph G with real-valued node weights, a global rounding approximates the node weights by integral node weights such that the weight sum on each shortest path becomes either floor or ceiling of the original weight sum.

Now, the rounding conjecture states that if $G = (V, E)$ is a connected graph with n vertices and α is a function $V \to [0, 1]$ then there are at most $n+1$ global roundings of α in the shortest-paths hypergraph $\mathcal{H}(G) = (V, \mathcal{P}_G)$, regardless of α. The rounding conjecture has been proved for some special types of graphs: If G is a path then $\mathcal{P}_G$ is a set of intervals; the corresponding rounding problem was studied by Sadakane *et al.* in [11]. This is a natural extension of the fact that a single real number (i.e., the case $n = 1$) has at most two roundings (floor and ceiling). The conjecture has also been proved for cycles, meshes, trees, and trees of cycles [2]. However, it seems difficult to prove in general, and it will be helpful to investigate other special cases. One such case is when the input α is restricted to $\alpha_{U+}(v) = 1/2 + \epsilon$ for every $v \in V$, where $0 < \epsilon < 1/n$; then, the number of global roundings in $\mathcal{H}(G)$ is precisely the number of semi-balanced colorings of G[1]. Thus, although our results so far on semi-balanced colorings provide weak evidence in support of the rounding conjecture, we hope that they will give some insight. Moreover, our algorithm in Section **??** might be a useful tool when searching for a counterexample to the rounding conjecture.

2 Preliminaries

Let $H = (V, \mathcal{F})$ be a hypergraph, where $\mathcal{F} \subseteq 2^V$. A *coloring of* H is a mapping from V to $\{-1, 1\}$. For any coloring π of H and any $F \in \mathcal{F}$, let $\pi(F) = \sum_{v \in F} \pi(v)$.

Definition 2.1 *A coloring π of H is called a* balanced coloring of H *if for every $F \in \mathcal{F}$, it holds that $-1 \leq \pi(F) \leq 1$; π is called a* semi-balanced coloring of H *if for every $F \in \mathcal{F}$, it holds that $-1 \leq \pi(F) \leq 2$.*

For the rest of this paper, let $G = (V, E)$ be an undirected, unweighted, connected graph with n vertices and m edges.

Consider a path $\mathbf{p}$ in G connecting two vertices u, v. The set of all vertices on $\mathbf{p}$ (including u and v) is called the *vertex set of* $\mathbf{p}$ and is denoted by $F(\mathbf{p})$. If $\mathbf{p}$ is a shortest path between u and v, then $F(\mathbf{p})$ is a *shortest-path vertex set.* There may exist several different shortest paths between u and v, and hence each pair of vertices induces one or more shortest-path vertex sets. For any two vertices $u, v \in V$, $dist(u, v)$ denotes the length of a shortest path in G between u and v.

Given G, the *shortest-paths hypergraph induced by* G is the hypergraph $\mathcal{H}(G) = (V, \mathcal{P}_G)$, where $\mathcal{P}_G$ is the set of all shortest-path vertex sets in G. Our focus in this paper is on the semi-balanced colorings of $\mathcal{H}(G)$.

Definition 2.2 *A* coloring *is a mapping $\pi : V \to \{-1, 1\}$. A vertex v in V is said to be colored* red *if $\pi(v) = 1$, or* blue *if $\pi(v) = -1$.*

A balanced (semi-balanced) coloring of the shortest-paths hypergraph $\mathcal{H}(G)$ is also called a balanced (semi-balanced) coloring of G.

Definition 2.3 *Let π be a coloring of G and $\{u, v\} \in E$. The edge $\{u, v\}$ is called* dangerous in π *if $\pi(u) = \pi(v) = 1$, i.e., if both of u and v are colored red.*

We say that an edge is "dangerous" rather than "dangerous in π" when there is no confusion about which coloring is being referred to.

Observation. *A balanced coloring can not contain any dangerous edges. Similarly, if $\{u, v\} \in E$ then a coloring in which both u and v are colored blue can never be a semi-balanced coloring of G. Furthermore, in any semi-balanced coloring, a shortest path between two vertices cannot include two dangerous edges.*

Definition 2.4 *$\nu(G)$ is the number of different semi-balanced colorings of G.*

It is easy to calculate $\nu(G)$ for certain types of graphs. For example, $\nu(G) = n + 1$ if G is a tree since any semi-balanced coloring of a tree can have at most one dangerous edge and G has $n - 1$ edges, and there are exactly two balanced colorings of G. Also, $\nu(G) = n + 1$ if G is a complete graph because a semi-balanced coloring of a complete graph can

[1]Given a rounding β of α_{U+}, define β' as $\beta'(v) = 2\beta(v) - 1$ for every $v \in V$. Then β is a global rounding in $\mathcal{H}(G)$ if and only if β' is a semi-balanced coloring of G.

have at most one blue vertex. If G is a cycle of length n, $\nu(G) = 4$ if $n = 3$, $\nu(G) = n$ if n is odd and $n \geq 5$, $\nu(G) = n/2 + 2$ if $n \equiv 2$ (modulo 4), and $\nu(G) = 2$ if $n \equiv 0$ (modulo 4).

Not all graphs admit semi-balanced colorings. Figure 1 shows one such graph.

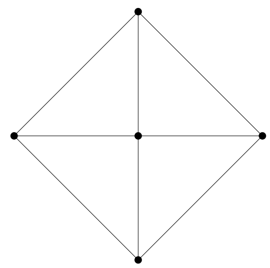

Figure 1: This graph has no semi-balanced coloring.

However, if we add an edge between the leftmost and the rightmost vertices in the graph in Figure 1, the coloring which makes the top and bottom vertices blue becomes a semi-balanced coloring. In general, we observe the following:

Proposition 2.5 *For any independent set W of $G = (V, E)$, there is a graph $G' = (V, E')$ such that $E' \supset E$ and W is the set of blue vertices in a suitable semi-balanced coloring of G'.*

Proof Let G' be the graph obtained by adding edges between all pairs of vertices in $V \setminus W$. Note that W is still an independent set in G'. Let π be the coloring of G' in which all vertices in W are colored blue, and the rest red. Consider a shortest path $\mathbf{p}$ in G' between any two vertices u and v. If u and v belong to $V \setminus W$, then $\mathbf{p}$ consists of a single dangerous edge and $\pi(\mathbf{p}) = 2$. If one of u and v belongs to W and the other to $V \setminus W$, then $\mathbf{p}$ contains one blue vertex and one or two red vertices, i.e., $\pi(\mathbf{p}) = 0$ or 1. Similarly, if both of u and v belong to W, then $\pi(\mathbf{p}) = -1$ or 0 since no path contains two consecutive blue vertices. Hence, π is a semi-balanced coloring of G'. □

A red vertex in π is called *singular* if it has no blue neighbor vertex. Two singular vertices must be adjacent, since otherwise a shortest path between them must have two dangeous edges. Hence, we have the following lemma, which implies that a semi-balanced coloring gives either a maximal independent set or a submaximal one contained in a maximal independent set given by another semi-balanced coloring.

Lemma 2.6 *The set of singular vertices (if any) of π forms a clique. Moreover, the coloring obtained by turning a singular vertex in π into blue is also semi-balanced.*

We show the following enumerative combinatorial result, and then design a polynomial time enumeration algorithm based upon it.

Theorem 2.7 *Let G be an undirected, unweighted, connected graph with n vertices and m edges. If G is bipartite, $\nu(G) \leq n + 1$. If G is not bipartite, $\nu(G) \leq m + 1$; moreover, if G is triangle-free, $\nu(G) \leq m$.*

3 The bipartite case

Proposition 3.1 *If G is a bipartite graph, then $\nu(G) \leq n + 1$.*

Proof Fix a spanning tree S of G. Any semi-balanced coloring of G is either a balanced coloring of S, or a coloring of S with one or more dangerous edges. For each edge e in S, we claim that there is at most one semi-balanced coloring of G that makes e dangerous.

Suppose $e = \{u, v\} \in S$ is dangerous in a semi-balanced coloring of G. Since G is bipartite, it contains no odd cycles. Therefore, there is no vertex whose shortest distance in G to u equals its shortest distance in G to v. Thus, we can divide the vertices into two disjoint sets V_u and V_v so that V_u contains all vertices which are closer to u than v in G, and analogously for V_v. Let T_u and T_v be two shortest path trees (in G) of V_u and V_v rooted at u and v, respectively. We claim there is no dangerous edge in $T_u \cup T_v$. Assume that an edge $\{x, y\} \in T_u$ is dangerous, where x is the father of y in T_u. Let $\mathbf{p}$ be the path from u to y in T_u. Since $dist(u, y) < dist(v, y)$ and every edge in a path contributes 1 to its length, the path appending e to $\mathbf{p}$ is a shortest path between y and v. But this path has two dangerous edges, which is a contradiction. Thus, T_u and T_v must be colored in an alternating fashion (each node in T_u is colored red or blue depending on if its distance from u is even or odd, and similarly for T_v). This shows that there is a unique (if any) semi-balanced coloring of G in which e is dangerous.

Since S has $n-1$ edges, there are at most $n-1$ semi-balanced colorings of G which make at least one edge of S dangerous. G is bipartite, so there are exactly two balanced colorings of G. Thus, we obtain the proposition. □

4 The non-bipartite, triangle-free case

In this section, we assume that G is non-bipartite and triangle-free[2]. Although the triangle-free case is a special case, we investigate it in detail since it helps the reader understand our tools and strategy.

The following dominating relation between edges is our key tool. It will be utilized later in an extended form for the case of general graphs.

Definition 4.1 (Dominating relation) *For a pair of edges $e, f \in E$, we say that e* dominates *f if we can write $e = \{u, r\}$ and $f = \{v, w\}$ so that $dist(r, v) = dist(r, w) = k$ and $dist(u, v) = dist(u, w) = k + 1$, where k is an even integer. We denote by $e > f$ that e dominates f.*

See Figure 2 for an example.

Lemma 4.2 *Let $e, f \in E$. If e is dangerous in a semi-balanced coloring π and $e > f$, then f is also dangerous in π.*

Proof Let $e = \{u, r\}$ and $f = \{v, w\}$, where r is closer than u to f. Consider a shortest path $\mathbf{p}$ from r to v. By Definition 4.1, the path appending e to $\mathbf{p}$ is a shortest path from u to v. Hence, if there

[2]Triangle-free means that if two edges $\{u, v\}$ and $\{v, w\}$ belong to G then G cannot contain the edge $\{u, w\}$.

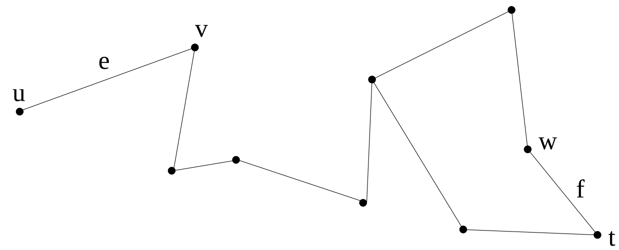

Figure 2: Edge e dominates edge f.

is a dangerous edge on $\mathbf{p}$, it contradicts the semi-balanced condition. Thus, the vertices along $\mathbf{p}$ are colored in an alternating fashion. Since $dist(r, v)$ is even, v has the same color as r, namely red. Similarly, w must be colored red, and hence f is dangerous. □

Definition 4.3 *$D(G)$, the* dominance graph *of G, is a directed graph whose vertices are in one-to-one correspondence with the edges of G. For any two edges $e, f \in E$, there is a directed edge from e to f in $D(G)$ if and only if $e > f$.*

Given a coloring π of G, a vertex of $D(G)$ is called *dangerous in* π if the corresponding edge in G is dangerous in π.

Now, consider the decomposition of $D(G)$ into strongly connected components $C_1, C_2, \ldots, C_h$.

Corollary 4.4 *If a vertex in a strongly connected component C_i is dangerous in a semi-balanced coloring π, then all vertices belonging to C_i are dangerous in π. Furthermore, all elements in its transitive closure in $D(G)$ are also dangerous.*

We remark that the dominating relation itself is not expanded to the transitive closure in our definition.

To find an upper bound on $\nu(G)$, we need one more definition.

Definition 4.5 *For an edge e of E, a* regular coloring associated with *e is a semi-balanced coloring which makes all the vertices in the strongly connected component of $D(G)$ containing e dangerous and no other vertex dominating e dangerous.*

Lemma 4.6 *If G is not bipartite then any semi-balanced coloring of G is a regular coloring associated with some edge e in E.*

Proof Let π be a semi-balanced coloring of G. π cannot be balanced since G is non-bipartite, so there is at least one dangerous edge. Let $E(\pi)$ be the set of dangerous edges. Consider the subgraph H induced by $E(\pi)$ in $D(G)$. By Corollary 4.4, H must be the union of some strongly connected components. Consider H as a directed acyclic graph on these strongly connected components, and pick a source component C. Then, for any element e in C, π is a regular coloring. □

Lemma 4.7 *For each edge $e \in E$, there is at most one regular coloring associated with e.*

Proof Let $e = \{r, p\}$ and let C be the strongly connected component of $D(G)$ containing e. We call the edges in G which are represented by vertices of C *predicted edges*. Construct a shortest path tree T of G rooted at r. In the construction of T, whenever two or more paths in G to the same vertex are of equal length, we apply a convention that a path containing a predicted edge is preferred; if two or more paths contain predicted edges, the one in which the predicted edge is nearer to r is preferred. For any vertex v, let $path(r, v)$ be the path in T from r to v.

Consider a regular coloring associated with e. By definition, e and all other predicted edges are dangerous. Below, we show that for every edge belonging to T, it is dangerous only if it is a predicted edge, implying that the set of vertices colored red is uniquely determined.

Suppose there exists a dangerous edge in T which is not predicted. Let $f = \{s, t\}$ be the nearest to r among such edges. Neither of s and t can be equal to r since otherwise f and e are adjacent dangerous edges, which is impossible in a semi-balanced coloring of a triangle-free graph. Without loss of generality, assume that s is the parent node of t in T, and let $k = dist(r, s)$. Note that k is even; otherwise, there would be another dangerous edge on $path(r, s)$, and the shortest path $path(r, t)$ would contain two dangerous edges.

Consider the path appending e to $path(r, t)$. This path has length $k + 2$, and contains two dangerous edges; hence, it cannot be a path with the shortest length. Thus, we have a path $\mathbf{p}$ between p and t whose length is less than $k + 2$. If it is less than or equal to k, the path obtained by appending e to $\mathbf{p}$ has length at most $k + 1$ from r to t. Moreover, it is preferred to the current path $path(r, t)$ in the lexicographic ordering; thus, we have a contradiction. Therefore, the path $\mathbf{p}$ has length $k + 1$. If $\mathbf{p}$ contains the edge f, then f dominates e. But because the coloring is a regular coloring associated with e, f can be dangerous only if f is in C and hence predicted.

Thus, we assume that $\mathbf{p}$ does not contain f. Since $\mathbf{p}$ has odd length, $\mathbf{p}$ has a dangerous edge $g = \{u, v\}$. We assume that u is nearer than v to p on the path. If $v = t$, we again derive a contradiction because then g and $\{s, t\}$ are adjacent dangerous edges. Hence, we assume $v \neq t$. The length ℓ of the path from p to u must be even since we cannot have a dangerous edge on $\mathbf{p}$ in the part from p to u, and both p and u are colored red. Consider the path $path(r, v)$ in T. The length of $path(r, v)$ must be $\ell + 1$; if it is less than or equal to ℓ, the path connecting $path(r, v)$ to the part from v to t of $\mathbf{p}$ has length k (or less), and contradicts that $path(r, t)$ is the shortest. If it is greater than or equal to $\ell + 2$, the path appending e to the part from p to v of $\mathbf{p}$ is shortest and has two dangerous edges. Thus, $path(r, v)$ has odd length, and hence contains a dangerous edge. If it is not predicted, it contradicts that f is the nearest edge among non-predicted dangerous edges on T. If it is predicted, the path connecting $path(r, v)$ and the part of $\mathbf{p}$ from v to t has length $k{+}1$, and it is preferred to the current path $path(r, t)$, which is a contradiction.

Thus, we have proved that all dangerous edges on T are predicted, giving a unique way (if one exists) of assigning colors to the nodes of T. □

Proposition 4.8 *If G is a non-bipartite, triangle-free graph, then $\nu(G) \leq m$.*

Proof G is non-bipartite, so any semi-balanced coloring of G must be a regular coloring associated with some edge in E by Lemma 4.6. Next, Lemma 4.7 implies that there are at most m semi-balanced colorings of G. □

5 General case and an algorithm

We give outline for the general non-bipartite case. A clique Q of G is called *maximal* if there is no other clique in G containing Q. A clique is called *submaximal* if it has at least two vertices and it is contained in a maximal clique which has one more vertex. For a clique Q in G, V_Q denotes the set of vertices of Q. The following lemma is immediate since two blue vertices can never be adjacent in a semi-balanced coloring.

Lemma 5.1 *Let Q be a maximal clique in a graph G. In any semi-balanced coloring of G, there is at most one vertex in V_Q colored blue.*

In a coloring of G, Q is called a *dangerous clique* if all vertices in V_Q are red. Lemma 5.1 implies that in any semi-balanced coloring, every maximal clique of at least three vertices is either dangerous or has a dangerous submaximal clique.

Lemma 5.2 *Let Q_1 and Q_2 be a pair of maximal cliques in G and let $W = V_{Q_1} \cap V_{Q_2}$. In any semi-balanced coloring of G, the following holds:*

(1) If $|W| \geq 2$, all vertices in W must be colored red.

(2) If $W = \{w\}$ and $|V_{Q_1}| \geq 3$ and $|V_{Q_2}| \geq 3$, the vertex w must be colored red if there is an edge between $V_{Q_1} - W$ and $V_{Q_2} - W$; otherwise, it must be colored blue.

(3) If $W = \{w\}$ and $|V_{Q_1}| \geq 3$ and $|V_{Q_2}| = 2$, the clique (indeed, the edge) Q_2 cannot be dangerous.

Because of Lemma 5.2, if two maximal cliques intersect at two or more vertices, we can fix the colors of all vertices in the intersection. Also, if two maximal cliques of size at least three intersect at one vertex, we can fix the color of that vertex.

We first remove those vertices and their incident edges from G. For any maximal clique Q, let $\tilde{Q}$ be the remaining part. Next, for each maximal clique Q of size two (i.e., edge) intersecting another clique of size greater than two, we set $\tilde{Q} = \emptyset$ and remove the corresponding edge but keep both endpoints of the edge if they have not been removed so far. Thus, we obtain a subgraph $\tilde{G}$ of G.

For a submaximal clique R in a maximal clique Q, $\tilde{R}$ denotes $R \cap \tilde{Q}$. We call $\tilde{Q}$ a *restricted clique* if Q is either maximal or submaximal.

Observe that if we give a coloring of $\tilde{Q}$ for each maximal clique Q having at least three vertices and determine the set of dangerous edges (i.e., red-colored cliques of size two), the coloring of G is uniquely determined.

We define a dominating relation among maximal and submaximal cliques which generalizes Definition 4.1. Let $\mathcal{Q}$ be the set of all cliques which are maximal or submaximal.

Definition 5.3 *Let $Q_1, Q_2 \in \mathcal{Q}$. We say that Q_1* dominates *Q_2 if there exists an even integer k such that for every $v \in V_{Q_2}$, there is a vertex $r \in V_{Q_1}$ and a vertex $u \in V_{Q_1}$ for which $dist(r, v) = k$ and $dist(u, v) = k + 1$. We write $Q_1 > Q_2$ if Q_1 dominates Q_2.*

Lemma 5.4 *If Q_1 is dangerous in a semi-balanced coloring π and $Q_1 > Q_2$, then Q_2 is dangerous in π.*

Definition 5.5 *$D(G)$, the* dominance graph *of G, is a directed graph whose vertices are in one-to-one correspondence with $\mathcal{Q}$. For any two (maximal or submaximal) cliques $Q_1, Q_2 \in \mathcal{Q}$, there is a directed edge from Q_1 to Q_2 in $D(G)$ if and only if $Q_1 > Q_2$.*

$D(\tilde{G})$, is the directed graph obtained from $D(G)$ by identifying vertices associated with Q_1 and Q_2 if $\tilde{Q_1} = \tilde{Q_2}$, and removing the vertex associated with Q if $\tilde{Q} = \emptyset$ or $Q - \tilde{Q}$ is known to contain a blue vertex in any semi-balanced coloring. We can see that if a restricted clique $\tilde{Q}$ is dangerous in a semi-balanced coloring, the restricted cliques in its transitive closure in $D(\tilde{G})$ are also dangerous.

Definition 5.6 *For a member $\tilde{Q}$ of a strongly connected component C of $D(\tilde{G})$, a* regular coloring associated with $\tilde{Q}$ *is a semi-balanced coloring which makes all restricted cliques in* C *dangerous and no other restricted clique dominating* $\tilde{Q}$ *dangerous.*

Lemma 5.7 *If* G *is not bipartite and if there is a clique* Q *of size more than two such that* $\tilde{Q} \neq \emptyset$*, then any semi-balanced coloring is a regular coloring associated with some restricted clique in* G*. On the other hand, for each restricted clique* $\tilde{Q}$*, there is at most one regular coloring associated with* $\tilde{Q}$*.*

Based on the above argument, and a counting scheme of restricted cliques (we omit it in this version, see [7]), we have the following:

Theorem 5.8 *If* G *is a non-bipartite graph, then* $\nu(G) \leq m+1$*. Moreover, all semi-balanced colorings of* G *can be enumerated in* $O(nm^2)$ *time.*

6 Concluding remarks

We have defined and studied the combinatorial concept of a semi-balanced coloring obtained by generalizing the 2-colorings of graph. Indeed, the only graph that the authors know of which satisfies $\nu(G) = m+1$ is the triangle. Incidentally, this graph also satisfies $\nu(G) = n+1$. We conjecture that for any undirected, unweighted, connected graph G with n vertices, $\nu(G) \leq n+1$. If it is true, it means that $\nu(G)$ is maximized at the two extremes: when G is a tree and when G is a complete graph. As noted in Section 1, the conjecture is a special case of the rounding conjecture. As seen in Section 2, there are graphs for which $\nu(G) = 0$. We would like to know if there is some way to characterize all graphs with $\nu(G) > 0$.

Acknowledgements

Most of this work was done during the visit of the first author to Tohoku university (Aug 1, 2001–Oct 31, 2001) supported in part by Sweden-Japan Foundation.

The authors thank Tetsuo Asano, Naoki Katoh, Kunihiko Sadakane, and Hisao Tamaki for helpful discussions and comments.

References

[1] T. Asano, N. Katoh, K. Obokata, and T. Tokuyama, Matrix Rounding under the L_p-Discrepancy Measure and Its Application to Digital Halftoning, *Proc. 13th ACM-SIAM SODA* (2002), pp. 896–904.

[2] T. Asano, N. Katoh, H. Tamaki, and T. Tokuyama, On the Number of Global Roundings of a Shortest-Path Hypergraph of a Connected Graph, in preparation.

[3] T. Asano, T. Matsui, and T. Tokuyama, Optimal Roundings of Sequences and Matrices, *Nordic Journal of Computing* **7** (2000), pp. 241–256.

[4] G. Ausiello, P. Crecenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, M. Protasi, *Complexity and Approximation*, Springer-Verlag, 1999.

[5] J. Beck and V. T. Sös, *Discrepancy Theory*, in *Handbook of Combinatorics, Vol. II* (eds. R. Graham, M. Grötschel, and L. Lovász), Elsevier (1995), pp. 1405–1446.

[6] B. Doerr, Lattice Approximation and Linear Discrepancy of Totally Unimodular Matrices, *Proc. 12th ACM-SIAM SODA* (2001), pp. 119–125.

[7] J. Jansson and T. Tokuyama, preprint (send email to tokuyama@dais.is.tohoku.ac.jp to get it).

[8] J. Matoušek, *Geometric Discrepancy*, Algorithms and Combinatorics 18, Springer Verlag (1999).

[9] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, CBMS-NSF Regional Conference Series in Applied Math., SIAM (1992).

[10] R. Read and W. Tutte, *Chromatic Polynomials*, in *Selected topics in graph theory, Vol. 3* (eds. L. Beineke and R. Wilson), Academic Press (1988), pp. 15–42.

[11] K. Sadakane, N. Takki-Chebihi, and T. Tokuyama, Combinatorics and Algorithms on Low-Discrepancy Roundings of a Real Sequence, *Proc. 28th ICALP, Springer LNCS 2076* (2001), pp. 166–177.

[12] B. Toft, *Colouring, Stable Sets and Perfect Graphs*, in *Handbook of Combinatorics, Vol. I* (eds. R. Graham, M. Grötschel, and L. Lovász), Elsevier (1995), pp. 233–288.