リーマン多様体でのボロノイ図に必要な点数の評価

大西 建輔 *， 伊藤 仁一 †

* 電気通信大学 大学院 情報システム学研究科， † 熊本大学 教育学部
E-mail: onishi@hol.is.uec.ac.jp, j-itoh@gpo.kumamoto-u.ac.jp

あらまし: G. Leibon と D. Letscher は, リーマン多様体上で充分密に点をとることで, その多様体上でのデローネ三角形分割やボロノイ図が存在することを示した．また, それらの構造を構築するためのアルゴリズムも提案をした.

この論文では, 多様体上でのデローネ三角形分割やボロノイ図を扱うために必要な点の数を曲率を用いて評価している．さらに, 1 つの点の回りにあり, 隣接している点の数の評価も行っている.

Estimation of the necessary number of points in Riemannian Voronoi diagram

Kensuke Onishi*, Jin-ichi Itoh†

*Graduate School of Information Systems, University of Electro-Communications,
† Faculty of Education, Kumamoto University
E-mail: onishi@hol.is.uec.ac.jp, j-itoh@gpo.kumamoto-u.ac.jp

Abstract: G. Leibon and D. Letscher showed that for *sufficiently dense point set* its Delaunay triangulation and Voronoi diagram in Riemannian manifold exist. They also proposed an algorithm to construct them for given set.

In this paper we estimate the number of points which derives the Delaunay triangulation in the manifold by its curvature. Moreover, we show that how many points exist around a point.

1 Introduction

A Riemannian manifold is a manifold M with its Riemannian metric g, denoted by (R, g). In the manifold the *canonical distance* is induced by metric g. *Voronoi diagram* in Riemannian manifold M is defined by using the distance, i.e., for given set of points the manifold M is divided into regions such that each region contains a point of the set, called *site*, and any points in the region is the nearest site rather than any other points of the set. The Voronoi diagram in Euclidean space has been investigated well and related results are collected in [3]. The diagram in special Riemannian manifold is also researched, for example, orbifold [2], Hadamard manifold [4]. G. Leibon and D. Letscher showed that the Voronoi diagram for *sufficiently dense set* in *general* Riemannian manifold exists [1]. In addition, the duality between the Voronoi diagram and the Delaunay triangulation is shown.

Their result indicates that if enough *many* points are given in the manifold, then Voronoi diagram and Delaunay triangulation can be computed. In [1] sufficiently dense point set is defined. In this paper we investigate such set more precisely and consider the following problem: How to compute sufficiently dense set for given manifold? How many points of the set need for given manifold? When Voronoi diagram for the set is constructed, how many regions adjacent with a region of a point?

In this paper, we define an ε-packing covering (ε-PC) set (Section 3.1), a similar concept is considered in [5]. This set is sufficiently dense and is easily computed for given positive constant ε. We show that the number is bounded by upper and lower curvatures of given manifold and given positive constant ε (Section 3.2). In addition, consider the Voronoi diagram for the ε PC set. the number of adjacent region of a Voronoi region is estimated (Section 3.3).

2 Preliminary

2.1 From computational geometry

In this subsection we describe the results in [1]. Their results are based on a *generic* and *sufficiently dense* set of points in a given Riemannian manifold. These concepts are defined as follows:

> A set of points is *generic* if M is d-dimensional manifold, then any $d+2$ points do not lie on the sphere in M.
>
> A finite set of points $X \subset M$ is *sufficiently dense* if for every $y \in M$ and $x \in B_{4\mathrm{rad}(y)}(y)$, the ball of radius $\mathrm{rad}(y)$ centered at z contains a point of X in its interior, where $\mathrm{rad}(x)$ is $\frac{1}{5}$ the strong convexity radius of M at x.

The former is well-known, the latter is a new concept and characterized next lemma.

> **Lemma A** If κ is a positive upper bound on the sectional curvature of M and $r = \min\left\{\frac{\mathrm{inj}(M)}{10}, \frac{\pi}{10\sqrt{\kappa}}\right\}$, then if every ball of radius r in M contains a point in X then the points are sufficiently dense.

From this lemma, sufficiently small constant r is fixed and points are distributed *well.* They propose a set of points satisfying these conditions.

Leibon and Letscher showed the following theorem:

> **Theorem B** If $X = \{x_1, \cdots, x_n\} \subset M$ is a generic, sufficiently dense set of points, then there exists a unique Delaunay triangulation with vertices $\{x_1, \cdots, x_n\}$.

By this theorem, if a generic, sufficiently dense set is constructed, then its Delaunay triangulation (Voronoi diagram) exists.

2.2 From Riemannian geometry

In this subsection some results in the Riemannian geometry are described. We state three results: Bishop's theorem, Myers' theorem and the volume of ball in the manifold with constant curvature.

Bishop's theorem is a kind of comparison theorem, which is well-known in the Riemannian geometry. The volume of ball in different manifolds is compared by Bishop's theorem. The statement is as follows:

> **Corollary C (Bishop's theorem, [5] p.155)** Let $M, \tilde{M}$ be d-dimensional complete Riemannian manifolds.
> (1) Suppose $K_\sigma \geq \tilde{K}_{\tilde{\sigma}}$ for arbitrary sectional curvatures K_σ of M and $\tilde{K}_{\tilde{\sigma}}$ of $\tilde{M}$. Let $\tilde{p} \in \tilde{M}$. For $0 < r < i_{\tilde{p}}(\tilde{M})$, take a metric ball $B_r(\tilde{p})$ in $\tilde{M}$ and a metric ball $B_r(p)$ in M. Then $\mathrm{vol}B_r(p) \leq \mathrm{vol}B_r(\tilde{p})$, and equality holds if and only if $B_r(p)$ is isometric to $B_r(\tilde{p})$.
> (2) Suppose the Ricci curvatures of M satisfy $\rho(u) \geq (d-1)\delta$ for any $u \in M$. Then for any $0 < r(\leq \pi/\sqrt{\delta})$, where $\pi/\sqrt{\delta}$ is assumed to be $+\infty$ if $\delta \leq 0$, we have $\mathrm{vol}B_r(p) \leq v_r(\delta)$. Here $v_r(\delta)$ denotes the volume of a ball of radius r in the m-dimensional complete simple connected Riemannian manifold $\tilde{M} = M^d_\delta$ and is independent of the center. If equality holds, then $B_r(p)$ is also of constant curvature. In particular, if $\delta > 0$, then $\mathrm{vol}M \leq \mathrm{vol}S^d_\delta$, where equality holds if and only if M is isometric to the sphere S^d_δ of constant curvature δ.

In the case (1) of above theorem, there is a relation about *sectional* curvature between two manifolds, then the volumes can be compared for small r less than injective radius $i_{\tilde{p}}(\tilde{M})$.

In the case (2), when *Ricci* curvature satisfies $\rho(u) \geq (d-1)\delta$, the volume can be compared. The definition of Ricci curvature is $\rho(u) := \sum_{j=2}^{d} K(u, e_j)$ where $K(u, e_j)$ is sectional curvature. If $K(u, e_j) \geq \delta$, the condition of Ricci curvature is satisfied. So, we use sectional curvature instead of Ricci. If sectional curvature $\delta > 0$, then the volume is computed by the corollary below.

Corollary D ([5] p.66) Let M be a Riemannian manifold and $p \in M$. Suppose that $\exp_p|B_r(o_p)$ is diffeomorphism. Then

$$\begin{aligned} \mathrm{vol}B_r(p) &= int_{B_r(o_p)}\sqrt{\det g_{ij} \circ \exp_p} \\ &\quad \cdot \mathrm{d}x^1 \cdots \mathrm{d}x^m \\ &= \int_{S^{d-1}} \mathrm{d}S^{d-1} \int_0^r \theta(t,u)\mathrm{d}t \end{aligned}$$

In particular, $\omega_d = \alpha_{d-1}/d$ for $(\mathbb{R}^d, g_0)$.

In the corollary above, d is the dimension of M and $\omega_d = \pi^{d/2}/\Gamma\left((d/2)+1\right)$ is the volume of the unit ball of $(\mathbb{R}^d, g_0)$ by Γ function.

Moreover, consider the manifold of constant curvature δ. Since a direction u is independent in the manifold, $\theta(t,u)$ is written by

$$s_\delta^{d-1}(t) = \begin{cases} \left\{(\sin\sqrt{\delta}t)/\sqrt{\delta}\right\}^{d-1} & \delta > 0 \\ t^{d-1} & \delta = 0 \\ \left\{(\sinh\sqrt{|\delta|}t)/\sqrt{|\delta|}\right\}^{d-1} & \delta < 0 \end{cases}.$$

In addition, the diameter of manifold M $d(M) := \sup\{d(p,q);\, p,q \in M\}$ is bounded by lowest curvature δ of M.

Corollary E (Myers' theorem, [5] p.102) Let M be a complete Riemannian manifold, and suppose that the sectional curvatures K_σ satisfy $K_\sigma \geq \sigma (>0)$ everywhere. Then M is compact, and $d(M) \leq \pi/\sqrt{\delta}$.

3 Estimation of the number of points

3.1 ε-packing-covering set

Firstly, the *packing-covering set* is described.

Definition 1 *A finite set $\mathcal{P} := \{p_1, \cdots, p_n\}$ of points in a compact Riemannian manifold M is ε-packing-covering for positive constant ε, denoted by ε-PC, if intersection of any two ε-balls with center p_i and p_j is empty and the union of 2ε-balls with center p_i is covering of M.*

The condition of ε-PC set is the following:

1. $B_\varepsilon(p_i) \cap B_\varepsilon(p_j) = \emptyset$ for any i, j.
2. $\cup_{i=1}^n B_{2\varepsilon}(p_i) = M$.

Similar set is familiar with in Riemannian geometry.

We show that ε-PC set always exists for given ε.

Lemma 1 *For any positive constant ε and for any compact Riemannian manifold, there is an ε-PC set.*

Proof: Let M be compact Riemannian manifold. It is trivial that there is a finite set of points such that all neighborhoods is covering of M because of compactness of M.

So, we show the finite set $\mathcal{P}$ satisfies the conditions above. Suppose there exist a point p in M and not in the union of 2ε neighborhoods of points in $\mathcal{P}$. The distance between p and any point in $\mathcal{P}$ is greater than 2ε. If ε-ball with center p is considered, this ε-ball does not intersect with any other ε-balls. We can add p to $\mathcal{P}$. Then we can repeat above step if such a point exists in M. Otherwise, the set is ε-PC. $\square$

An construction algorithm is considered from above lemma.

Since the manifold is compact [1], $\mathcal{P}$ becomes finite. When this algorithm is performed, suppose two oracles such that 1) a point is selected from given manifold M; 2) for any point p in M and positive number r, the ball $B_r(p)$ is computed.

3.2 For a manifold

In this subsection we consider estimation of the number of necessary points of Voronoi diagram in a given Riemannian manifold.

Let M be a complete compact Riemannian manifold. Consider an ε-PC set on M where ε is a positive constant less than or equal to $\frac{r}{2}$.

Lemma 2 *If $\varepsilon \leq \frac{r}{2}$, called ε is sufficiently small, then any ε-PC set is sufficiently dense.*

[1] a manifold is compact if the manifold are covered by neighborhoods of *finite* set

Proof: We show that every ball of radius r in M contains a point of an ε-PC set. Then we show that the ε-PC set is sufficiently dense.

Above statement is shown. Because the set is ε-PC, the manifold M is covered by 2ε-balls. In other words, for every point in M there exists a point in the ε-PC set such that the distance between two points is less than 2ε. Since $r \geq 2\varepsilon$, any ball of radius r contains one or more points in ε-PC set. $\square$

Consider ε-PC set $\mathcal{P}$ on M. From the result of [1], there exist a Delaunay triangulation and a Voronoi diagram for $\mathcal{P}$.

Corollary 1 *If a set of points $\mathcal{P}$ is a generic, ε-PC and ε is sufficiently small, then there exists a unique Delaunay triangulation (Voronoi diagram) for $\mathcal{P}$.*

So, we get an ε-PC set and its Delaunay triangulation (Voronoi diagram) exist for sufficiently small positive constant ε. Since ε gives a measure of mesh, the more ε decrease, the more the number of points increases. We prove the relation between ε and the number of points in ε-PC set by Bishop's volume comparison theorem.

Theorem 1 *Let $\mathcal{P}$ be a generic ε-PC set for sufficiently small ε on a given manifold M. Let n be the number of points in $\mathcal{P}$. The number n satisfies an inequality below:*

$$\frac{\mathrm{vol}M}{V_{2\varepsilon}} < n < \frac{\mathrm{vol}M}{v_\varepsilon},$$

where $V_{2\varepsilon}(v_\varepsilon)$ is upper(lower) bound of the volume of 2ε-ball (ε-ball) among balls with center in $\mathcal{P}$, respectively.

Proof: Suppose an ε-PC set on M. From the property of ε-PC set, all ε-balls with center point in $\mathcal{P}$ do not intersect each other. This inequality is settled:

$$\mathrm{vol}M \geq \sum_{p \in \mathcal{P}} \mathrm{vol}B_\varepsilon(p) \geq n \cdot v_\varepsilon,$$

Conversely, the union of all 2ε-balls cover M. This inequality is shown.

$$\mathrm{vol}M \leq \sum_{p \in \mathcal{P}} \mathrm{vol}B_{2\varepsilon}(p) \leq n \cdot V_{2\varepsilon},$$

$\square$

When the sectional curvature k of M is bounded we can describe the result above.

Corollary 2 *Consider the situation in Theorem 1. If the sectional curvature k of M is bounded by $(0 \leq)\kappa \leq k \leq K$, then,*

$$n \leq \frac{v_{\pi/\sqrt{\kappa}}(\kappa)}{v_\varepsilon(K)},$$

where $v_r(k)$ is the volume of r-ball on a manifold with constant sectional curvature k.

Proof: Let $d(M)$ be the diameter of the given manifold M. This diameter is estimated by Myers's theorem:

$$d(M) \leq \frac{\pi}{\sqrt{\kappa}}.$$

Since the diameter is the largest distance between two points in M, consider $d(M)$-ball with any point p in M. Then this ball contains M. So, the volume of M is bounded by

$$\mathrm{vol}M \leq B_{d(M)}(p) \leq B_{\pi/\sqrt{\kappa}}(p)$$

Apply Bishop's theorem to the volume of the ball. Suppose $r = \pi/\sqrt{\kappa}$ and a complete simple manifold $\tilde{M}$ with constant sectional curvature κ. Since for any point in M sectional curvature k is grater than or equal to κ, we can use Bishop's theorem. The volume of the ball is less than or equal to $v_{\pi/\sqrt{\kappa}}(\kappa)$. Then, we get the following inequality:

$$\mathrm{vol}M \leq v_{\pi/\sqrt{\kappa}}(\kappa).$$

The volume of v_ε is also evaluated by Bishop's theorem. In this case, a complete simple manifold $\bar{M}$ with constant sectional curvature K. In any point of M, k is less than or equal to K. ε is smaller than $\pi/\sqrt{\kappa}$. So, we get the following inequality $v_\varepsilon \geq v_\varepsilon(K)$.

Finally, the number of points is bounded by

$$\frac{v_{\pi/\sqrt{\kappa}}(\kappa)}{v_\varepsilon(K)}.$$

$\square$

In addition, this number is evaluated by computation of the volume in appendix A.

Corollary 3 *Suppose the situation in Corollary 2. The number of points n is evaluated:*

$$n \leq \begin{cases} \frac{(d-2)!!}{(d-1)!!} \cdot \frac{\pi(d+1)}{2\kappa^{d/2}K^{1/2}\varepsilon^{d+1}} & d \text{ is odd} \\ \frac{(d-2)!!}{(d-1)!!} \cdot \frac{2(d+1)}{\kappa^{d/2}K^{1/2}\varepsilon^{d+1}} & d \text{ is even.} \end{cases}$$

3.3 For a point

In previous section, the number of points for Riemannian Delaunay triangulation in manifold M is described. In this section we show that the number of points *around a points* in the ε-PC set.

Take ε-PC set of point in M for sufficiently small ε. The following lemma is shown about the distance between two adjacent points in Delaunay triangulation.

Lemma 3 *Consider the ε-PC set $\mathcal{P}$ for sufficiently small ε. Take two adjacent points in Voronoi diagram of $\mathcal{P}$. Then, the distance between the points is grater than or equal to 2ε and less than or equal to 4ε.*

Proof: Let p, p' be two adjacent points above. Suppose the distance $d(p, p') > 4\varepsilon$. The center point q between p, p' is not included in $B_{2\varepsilon}(p)$ nor $B_{2\varepsilon}(p')$. From the condition of $\mathcal{P}$, there exists a point p'' in $\mathcal{P}$ such that $B_{2\varepsilon}(p'')$ contains q. So, q is not shared point between Voronoi regions of p and of p'. Consider another point q' which is equidistant from p, p' and is not included in $B_{2\varepsilon}(p'')$. Since the center point q is nearest to $p(p')$ than any q', $d(p, q') > 2\varepsilon$ and $d(p', q') > 2\varepsilon$. Similar, there exists a point p''' such that $B_\varepsilon(p''')$ includes q'. Therefore, there is no shared point in M. This is contradiction with that p, p' are adjacent points.

Suppose the distance between p, p' is less than 2ε. $B_\varepsilon(p)$ and $B_\varepsilon(p')$ has intersection. This contradiction with the property of ε-PC set. □

We estimate the number of adjacent point around a point by this lemma.

Theorem 2 *Consider an ε-PC set $\mathcal{P}$ for sufficiently small ε on a complete compact Riemannian manifold M. Let κ and K be the lower and upper bound of sectional curvature of M. Let N be the number of adjacent region of a point p in the Voronoi diagram for $\mathcal{P}$. The following relation is settled.*

$$\frac{v_{2\varepsilon}(K)}{v_{2\varepsilon}(\kappa)} - 1 \leq N \leq \frac{v_{5\varepsilon}(\kappa)}{v_\varepsilon(K)} - 1,$$

where $v_r(k)$ is the volume of r-ball in the Riemannian manifold with constant curvature k.

Proof: Let $\mathcal{Q}$ be a set of point whose Voronoi region is adjacent with the region of p. Since the ε-balls with a point in $\mathcal{Q}$ are disjoint, the sum of volume is $\sum_{q \in \mathcal{Q}} \mathrm{vol} B_\varepsilon(q)$. Each ε-ball satisfies

$$\mathrm{vol} B_\varepsilon(q) \geq v_\varepsilon(K)$$

from Bishop's theorem. By $\|\mathcal{Q}\| = N$,

$$\sum_{q \in \mathcal{Q}} \mathrm{vol} B_\varepsilon(q) \geq N \cdot v_\varepsilon(K).$$

Let $B_{5\varepsilon}(p)$ be the 5ε-ball with center point p. Since the distance between generators is less than 4ε from above lemma, this ε-ball contains all point in $\mathcal{Q}$ and all ε-balls with center point q in $\mathcal{Q}$. So, $B_{5\varepsilon}(p)$ contains $\|\mathcal{Q} \cup q\| = N + 1$ ε-balls.

$$(N+1) v_\varepsilon(K) \leq \mathrm{vol} B_{5\varepsilon}(p) \leq v_{5\varepsilon}(\kappa).$$

The last inequality is also shown by Bishop's theorem. From these inequalities we get the following evaluation:

$$N \leq \frac{v_{5\varepsilon}(\kappa)}{v_\varepsilon(K)} - 1.$$

In the case of lower bound, take the union of 2ε-ball with center q in $\mathcal{Q} \cup \{p\}$, denoted by $\tilde{M}$. Consider a ball with center p such that the r-ball is included in $\tilde{M}$. If $r = 2\varepsilon$, then this ball is always included in $\tilde{M}$. The following relation is settled:

$$v_{2\varepsilon}(K) \leq \mathrm{vol}\tilde{M} \leq (N+1) \cdot v_{2\varepsilon}(\kappa).$$

The lower bound is proved by this inequality. □

[**Remark**] Since $\kappa \leq K$, then $v_{2\varepsilon}(\kappa) \geq v_{2\varepsilon}(K)$ is settled from the volume of 2ε-ball. So, the lower bound of the above theorem is always negative number. It gives trivial bound.

In addition, if $\kappa > 0$ then a bound by d and curvature is shown.

Corollary 4 *Suppose the situation in Theorem 2. If the lower bound κ of curvature is positive, then the number N of adjacent region is evaluated:*

$$\left(\frac{K}{\kappa}\right)^{1/2} - 1 \leq N \leq 5^{d+1}\left(\frac{\kappa}{K}\right)^{1/2} - 1.$$

4 Conclusion

We show some results about ε-PC set in a complete compact Riemannian manifold. In this section these results are adapted to a manifold with constant curvature. Consider $K = \kappa > 0$ and d is odd in Corollary 3 and Corollary 4, respectively.

$$n < \frac{(d-2)!!}{(d-1)!!} \cdot \frac{\pi(d+1)}{2\kappa^{(d+1)/2}\varepsilon^{d+1}}$$

$$0 \leq N \leq 5^{d+1} - 1$$

If $r = \pi/(10\sqrt{\kappa})$ in Lemma A, it is possible that $\varepsilon = \pi/(20\sqrt{\kappa})$ by Lemma 2. Suppose $d = 3$, then above inequalities can be computed:

$$n < \frac{20^4}{\pi^3} \sim 5160.2455, \quad N \leq 624.$$

This number is not so good, but it is possible that the evaluation is more better. Suppose ε-packing and $k\varepsilon$-covering set, called *(ε, $k\varepsilon$)-PC set* ($k > 1$). For given manifold, if (ε, $k\varepsilon$)-PC set exists, then the distance between adjacent points in the set is from 2ε to $2k\varepsilon$. This relation is applied to Theorem 2. All the points around a point are contained in $(2k+1)\varepsilon$-ball. So, the coefficient in Corollary 4 is improved:

$$N \leq (2k+1)^{d+1}\left(\frac{\kappa}{K}\right)^{1/2} - 1.$$

In actual, consider regular triangle lattice, which is the set of vertex of regular triangle and the distance between any adjacent points is 2ε in Euclidean plane. Such set of points is (ε, $\frac{2}{\sqrt{3}}$)-PC. The coefficient of above inequality becomes

$$\left(2 \cdot \frac{2}{\sqrt{3}} + 1\right)^3 \sim (3.3094011)^3 \sim 36.245009.$$

Finally, the number of points around a points is less than or equal to 35. This number is better rather than $5^3 - 1 = 124$ in case of $(\varepsilon, 2\varepsilon)$-PC set. If small positive number k is found for given manifold, this bound can be improved.

References

[1] G. Leibon and D. Letscher, "Delaunay triangulations and Voronoi diagrams for Riemannian manifolds," *Proceeding 16th Annual ACM Symposium on Computational Geometry*, 2000, pp. 341-349.

[2] M. Mazón and T. Recio, "Voronoi Diagrams on Orbifolds," *Computational Geometry : Theory and Applications*, vol. 8, No. 5, 1997, pp. 219-230.

[3] A. Okabe, B. Boots, K.Sugihara and S. N. Chiu, "Spatial Tessellations: Concepts and Applications of Voronoi Diagrams," Second Edition, John Weily, New York, 2000.

[4] Kensuke Onishi, Jin-ichi Itoh: "Voronoi Diagram in Simply Connected Complete Manifold", IEICE Transactions on Fundamentals, vol. E85-A No5, pp.944-948, 2002.

[5] T. Sakai, "Riemannian Geometry," Translation of Mathematical Monographs, vol. 149, Amer. Math. Soc., 1996.

A Estimation of Volume of r-ball with constant curvature

In this section we show computation of volume of r-ball with positive constant curvature.

$$
\begin{aligned}
v_r(\delta) &= \int_{S^{d-1}} \mathrm{d}S^{d-1} \int_0^r \left\{\sin\sqrt{\delta}t/\sqrt{\delta}\right\}^{d-1} \mathrm{d}t \\
&= \int_{S^{d-1}} \mathrm{d}S^{d-1} \int_0^{\sqrt{\delta}r} \sin^{d-1} x \mathrm{d}x \cdot \delta^{-d/2}
\end{aligned}
$$

Using this formulation, the volume is bounded.

$$
\int \sin^{2p} x\mathrm{d}x = -\cos x \left[\frac{\sin^{2p-1} x}{2p} + \frac{(2p-1)\sin^{2p-3} x}{2p(2p-2)} + \cdots + \frac{(2p-1)!!}{(2p)!!}\sin x\right] + \frac{(2p-1)!!}{(2p)!!}x
$$

$$
\int \sin^{2p+1} x\mathrm{d}x = -\cos x \left[\frac{\sin^{2p} x}{2p+1} + \frac{2p\sin^{2p-2} x}{(2p+1)(2p-1)} + \cdots + \frac{(2p)!!}{(2p+1)!!2}\sin^2 x + \frac{(2p)!!}{(2p+1)!!}\right]
$$

Firstly, $d = 2p+1$ case is considered.

$$
\begin{aligned}
\int_0^{\sqrt{\delta}r} \sin^{d-1} x\mathrm{d}x &= \left[-\cos x\left\{\frac{\sin^{2p-1} x}{2p} + \frac{(2p-1)\sin^{2p-3} x}{2p(2p-2)} + \cdots + \frac{(2p-1)!!}{(2p)!!}\sin x\right\} + \frac{(2p-1)!!}{(2p)!!}x\right]_0^{\sqrt{\delta}r} \\
&= -\cos\sqrt{\delta}r\left\{\frac{\sin^{2p-1}\sqrt{\delta}r}{2p} + \frac{(2p-1)\sin^{2p-3}\sqrt{\delta}r}{2p(2p-2)} + \cdots + \frac{(2p-1)!!}{(2p)!!}\sin\sqrt{\delta}r\right\} + \frac{(2p-1)!!}{(2p)!!}\sqrt{\delta}r
\end{aligned} \tag{1}
$$

The first term of (1) is non-positive when $0 < r \leq \frac{\pi}{\sqrt{\delta}}$. So, the volume is bounded by the second term of (1). The following bound is shown:

$$
\int_0^{\sqrt{\delta}r} \sin^{d-1} x\mathrm{d}x \leq \frac{(d-2)!!}{(d-1)!!}\sqrt{\delta}r
$$

Secondly, $d = 2p+2$ is considered.

$$
\begin{aligned}
\int_0^{\sqrt{\delta}r} \sin^{d-1} \mathrm{d}x &= \left[-\cos x\left\{\frac{\sin^{2p} x}{2p+1} + \frac{2p\sin^{2p-2} x}{(2p+1)(2p-1)} + \cdots + \frac{(2p)!!}{(2p+1)!!\cdot 2}\sin^2 x + \frac{(2p)!!}{(2p+1)!!}\right\}\right]_0^{\sqrt{\delta}r} \\
&= -\cos\sqrt{\delta}r\left\{\frac{\sin^{2p}\sqrt{\delta}r}{2p+1} + \frac{2p\sin^{2p-2}\sqrt{\delta}r}{(2p+1)(2p-1)} + \cdots + \frac{(2p)!!}{(2p+1)!!\cdot 2}\sin^2\sqrt{\delta}r + \frac{(2p)!!}{(2p+1)!!}\right\} + \frac{(2p)!!}{(2p+1)!!}
\end{aligned}
$$

The upper bound of the volume is shown:

$$
\int_0^{\sqrt{\delta}r} \sin^{d-1} \mathrm{d}x < \frac{(d-2)!!}{(d-1)!!}\left(1 - \cos\sqrt{\delta}r\right)
$$

So, the volume is bounded by

$$
\begin{aligned}
v_r(\delta) &< \int_{S^{d-1}} \mathrm{d}S^{d-1} \cdot \frac{(d-2)!!}{(d-1)!!} \cdot \frac{r\sqrt{\delta}}{2} \cdot \delta^{-d/2} \\
v_r(\delta) &< \int_{S^{d-1}} \mathrm{d}S^{d-1} \cdot \frac{(d-2)!!}{(d-1)!!} \cdot \left(1 - \cos r\sqrt{\delta}\right) \cdot \delta^{-d/2}
\end{aligned}
$$

For sufficiently small radius r, the evaluation of the volume of r-ball is possible.

$$\begin{aligned}\sin x &= x - \frac{x^3}{6} + O(x^5) \\ \sin^d x &= \left(x - \frac{x^3}{6} + O(x^5)\right)^d \\ &= x^d - \frac{d}{6}x^{d+2} + O(x^{d+4})\end{aligned}$$

Then above equation is integrated, the following is settled.

$$\begin{aligned}\int_0^{\sqrt{\delta}r} \sin^d x \mathrm{d}x &= \left[\frac{x^{d+1}}{d+1} - \frac{dx^{d+3}}{6(d+3)} + O(x^{d+5})\right]_0^{\sqrt{\delta}r} \\ &= \frac{r^{d+1}}{d+1}\delta^{(d+1)/2} - \frac{dr^{d+3}}{6(d+3)}\delta^{(d+3)/2} + O(r^{d+5})\end{aligned}$$

$$v_r(\delta) \sim \int_{S^{d-1}} \mathrm{d}S^{d-1} \cdot \delta^{-d/2} \left(\frac{r^{d+1}}{d+1}\delta^{(d+1)/2} - \frac{dr^{d+3}}{6(d+3)}\delta^{(d+3)/2} + O(r^{d+5})\right)$$