

## 平面グラフの内部矩形描画

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**あらまし** 平面グラフ  $G$  の各内面が長方形になるように各辺を水平線分あるいは垂直線分で描くことを、 $G$  の内部矩形描画と呼ぶ。本論文では  $G$  が内部矩形描画  $D$  を持つための必要十分条件は  $G$  から新しく作られた二部グラフが完全マッチングを持つことであることを示す。また、点数  $n$  の平面グラフ  $G$  とその外面の概形が与えられたとき、即ち外面上の全ての点が凸頂点であるか凹頂点であるかが指定されているとき、 $D$  は  $O(n^{1.5}/\log n)$  時間で求めることができることを示す。

**キーワード** グラフ描画, 矩形描画, 内部矩形描画, アルゴリズム。

## Inner Rectangular Drawings of Plane Graphs

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**Abstract** A drawing of a plane graph is called an inner rectangular drawing if every edge is drawn as a horizontal or vertical line segment so that every inner face is a rectangle. In this paper we show that a plane graph  $G$  has an inner rectangular drawing  $D$  if and only if a new bipartite graph constructed from  $G$  has a perfect matching. We also show that  $D$  can be found in time  $O(n^{1.5}/\log n)$  if  $G$  has  $n$  vertices and a sketch of the outer face is prescribed, that is, all the convex outer vertices and concave ones are prescribed.

**Key words** Graph Drawing, Rectangular Drawing, Inner Rectangular Drawing, Algorithm.

### 1. Introduction

A drawing of a plane graph  $G$  is called a *rectangular drawing* if every edge is drawn as a horizontal or vertical line segment so that every face boundary is a rectangle. Figures 1(a) and 2(d) illustrate rectangular drawings. A rectangular drawing often appears in VLSI floor-planning and architectural layout [DETT99, FW74, GT97, Len90, SY99]. Each inner face of a plane graph  $G$  represents a module of a VLSI circuit or a room of an architectural layout. Suppose that a plane graph  $G_a$  representing the requirement of adjacency among modules is given as illustrated in Fig. 2(a). Each vertex of  $G_a$  corresponds to a module of a VLSI circuit, and each edge of  $G_a$  means that the two modules corresponding to the ends are required to be adjacent in the VLSI floor-planning, that is, the two rectangular modules must share a common boundary. A conventional method obtains a floor

plan meeting the adjacency requirement represented by  $G_a$ , as follows:

- (1) add dummy edges to  $G_a$  so that every inner face of the resulting graph  $G_a'$  is a triangle, as illustrated in Fig. 2(b) where dummy edges are drawn by dotted lines;
- (2) construct a new plane graph  $G$  from a dual-like graph of  $G_a'$  by adding four vertices of degree two corresponding to the four corners of the rectangular chip, as illustrated in Fig. 2(c) where  $G_a'$  is drawn by dotted lines,  $G$  by solid lines, and the four added vertices by white circles;
- (3) find a rectangular drawing  $D$  of  $G$  as a floor plan meeting the requirement of  $G_a$ , as illustrated in Fig. 2(d).

In the plane graph  $G$  appearing in the conventional method

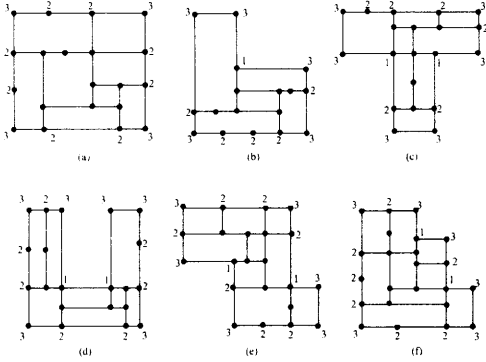


Fig. 1 Rectangular drawing (a), and inner rectangular drawings of (b) L-shape, (c) T-shape, (d) U-shape, (e) Z-shape, and (f) staircase-shape.

above, all vertices have degree three except for the four vertices of degree two corresponding to the four corners, because every inner face of  $G_a'$  is a triangle. Hence the maximum degree  $\Delta$  of  $G$  is three. However, some plane graphs with  $\Delta = 4$  may have rectangular drawing, as illustrated in Fig. 3(d). Of course,  $\Delta$  must be four or less if  $G$  has a rectangular drawing. A necessary and sufficient condition for a plane graph  $G$  with  $\Delta \leq 3$  to have a rectangular drawing is known, and linear or  $O(n^{2.5}/\log n)$  algorithms to find a rectangular drawing of  $G$  is obtained [BS88, He93, KH97, KK84, LL90, RNN02, RNN98, Tho84]. However, it has been an open problem to obtain a necessary and sufficient condition and an efficient algorithm for plane graphs  $G$  with  $\Delta \leq 4$  [MMN02, RNN98].

Let  $G_a''$  be a plane graph obtained from an adjacency requirement graph  $G_a$  by adding dummy edges to  $G_a$  so that every inner face is either a triangle or a *quadrangle*, as illustrated in Fig. 3(b). Let  $G$  be a plane graph obtained from a dual-like graph of  $G_a''$  by adding four vertices of degree two corresponding to the corners. Of course, the maximum degree  $\Delta$  of  $G$  may be four since  $G_a''$  may have a quadrangular inner face. If one can find a rectangular drawing  $D$  of  $G$  with  $\Delta \leq 4$  as illustrated in Fig. 3(d), then one can use  $D$  as a floor plan meeting the requirement of  $G_a$ .

The outer face boundary must be rectangular in a rectangular drawing, as illustrated in Fig. 1(a). However, the outer boundary of a VLSI chip or an architectural floor plan is not always rectangular, but is often a rectilinear polygon of L-shape, T-shape, U-shape or staircase-shape, as illustrated in Figs. 1(b)–(f) [FW74, Len90, MMN02, SS93, YS93]. We call such a drawing of a plane graph  $G$  an *inner rectangular drawing* if every inner face of  $G$  is a rectangle although the outer face boundary is not always a rectangle.

In the paper we show that a plane graph  $G$  has an inner rectangular drawing  $D$  if and only if a new bipartite graph

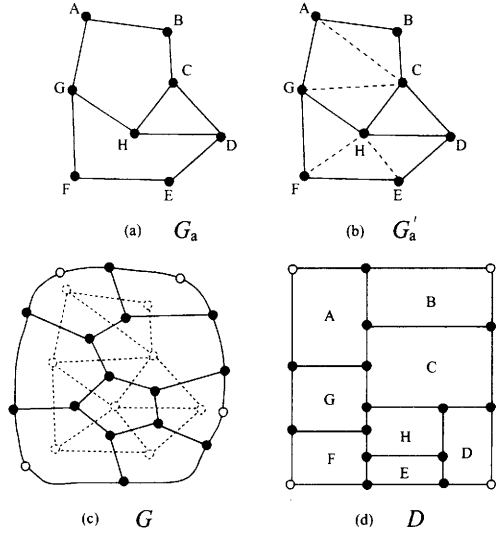


Fig. 2 (a) Adjacency requirement graph  $G_a$ , (b) inner triangulated graph  $G_a'$  augmented from  $G_a$ , (c) dual-like graph  $G$  of  $G_a'$ , and (d) rectangular drawing  $D$  of  $G$  with  $\Delta = 3$ .

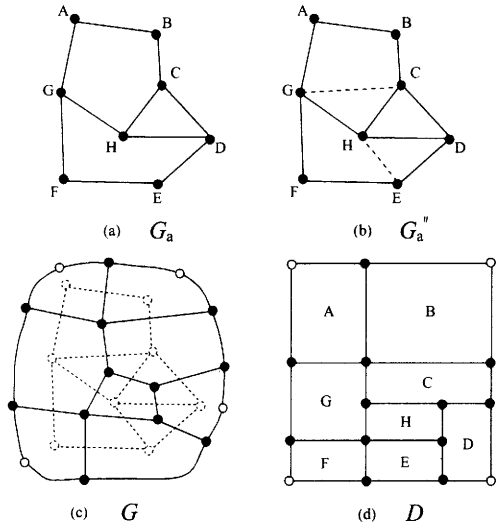


Fig. 3 (a) Adjacency requirement graph  $G_a$ , (b) inner triangulated or quadrangled graph  $G_a''$  augmented from  $G_a$ , (c) dual-like graph  $G$  of  $G_a''$ , and (d) rectangular drawing  $D$  of  $G$  with  $\Delta = 4$ .

constructed from  $G$  has a perfect matching. We also show that  $D$  can be found in time  $O(n^{1.5}/\log n)$  if a “sketch” of the outer face is prescribed, that is, all the convex outer vertices and concave ones are prescribed, where  $n$  is the number of vertices of  $G$ . We do not assume  $\Delta \leq 3$ , and an inner rectangular drawing is a rectangular drawing if the outer face

is sketched as a rectangle. Thus we solve the open problem above.

The remainder of the paper is organized as follows. In Section 2 we define some terms, describe some fundamental facts, and present our main results, Theorems 1–3. In Section 3 we prove Theorem 1 for the case where a sketch of the outer face is prescribed. In Section 4 we prove Theorem 2 for the case where the numbers of “convex” and “concave” outer vertices are prescribed. In Section 5 we prove Theorem 3 for a general case. Finally in Section 6 we conclude with some extensions of our algorithms.

## 2. Preliminaries and main results

We assume in the paper that  $G$  is a plane undirected simple graph. We denote by  $d(v)$  the degree of a vertex  $v$  in  $G$ . We denote by  $F_o$  the outer face of  $G$ . The boundary of  $F_o$  is called the *outer boundary*, and is denoted also by  $F_o$ . A vertex on  $F_o$  is called an *outer vertex*, and a vertex not on  $F_o$  is called an *inner vertex*. We may assume without loss of generality that  $G$  is 2-connected and  $\Delta \leq 4$ , and hence every vertex of  $G$  has degree 2, 3 or 4.

An angle formed by two edges  $e$  and  $e'$  incident to a vertex  $v$  in  $G$  is called an *angle of  $v$*  if  $e$  and  $e'$  appear consecutively around  $v$ . An angle of a vertex in  $G$  is called an *angle of  $G$* . An angle formed by two consecutive edges on a boundary of a face  $F$  in  $G$  is called an *angle of  $F$* . An angle of the outer face is called an *outer angle of  $G$* , while an angle of an inner face is called an *inner angle*.

In any inner rectangular drawing, every inner angle is  $\pi/2$  or  $\pi$ , and every outer angle is  $\pi/2$ ,  $\pi$  or  $3\pi/2$ . Consider a labeling  $f$  which assigns a label 1, 2, or 3 to every angle of  $G$ , as illustrated in Fig. 4(c). Labels 1, 2 and 3 correspond to angles  $\pi/2$ ,  $\pi$  and  $3\pi/2$ , respectively. We denote by  $n_{ev}$  the number of outer angles having label 3, and by  $n_{ec}$  the number of outer angles having label 1. Thus  $n_{ev}$  is the number of “convex” outer vertices, and  $n_{ec}$  is the number of “concave” outer vertices. For example  $n_{ev} = 5$  and  $n_{ec} = 1$  for the labeling  $f$  in Fig. 4(c).

We call  $f$  a *regular labeling* of  $G$  if  $f$  satisfies the following three conditions (a)–(c):

- (a) For each vertex  $v$  of  $G$ , the labels of all the angles of  $v$  total to 4;
- (b) The label of any inner angle is 1 or 2, and every inner face has exactly four angles of label 1; and
- (c)  $n_{ev} - n_{ec} = 4$ .

Figure 4(c) illustrates a regular labeling  $f$  of the plane graph in Fig. 4(a) and an inner rectangular drawing  $D$  corresponding to  $f$ .

Conditions (a) and (b) imply the following (i)–(iii):

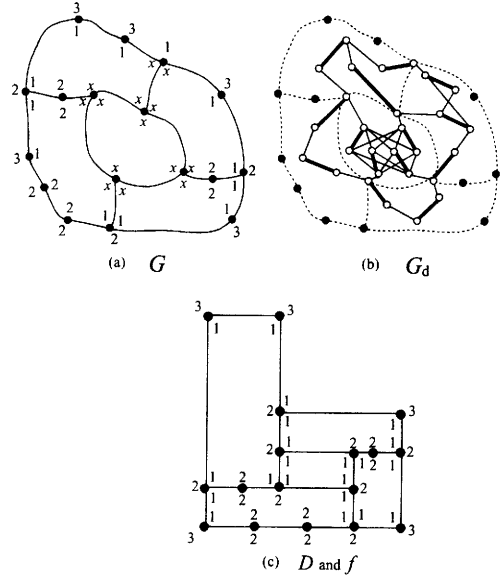


Fig. 4 (a) Plane graph  $G$ , (b) decision graph  $G_d$ , and (c) inner rectangular drawing  $D$  and regular labeling  $f$  of  $G$ .

- (i) If a vertex  $v$  has degree 2, that is,  $d(v) = 2$ , then the two labels of  $v$  are either 2 and 2 or 1 and 3. In particular, if  $v$  is an inner vertex, then the two labels are 2 and 2.
- (ii) If  $d(v) = 3$ , then exactly one of the three angles of  $v$  has label 2 and the other two have label 1.
- (iii) If  $d(v) = 4$ , then all the four angles of  $v$  have label 1.

If  $G$  has an inner rectangular drawing, then clearly  $G$  has a regular labeling. Conversely, if  $G$  has a regular labeling, then  $G$  has an inner rectangular drawing, as can be proved by means of elementary geometric considerations. We thus have the following fact.

[Fact 1] A plane graph  $G$  has an inner rectangular drawing if and only if  $G$  has a regular labeling.

A drawing of a plane graph is called an *orthogonal drawing* if each edge is drawn as an alternating sequence of horizontal and vertical line segments. A point at which an edge changes its direction is called a *bend*. An inner rectangular drawing is an orthogonal drawing with no bend such that every inner face is a rectangle. Tamassia gives an algorithm to find an orthogonal drawing of a given plane graph with the minimum number of bends in time  $O(n^2 \log n)$  by solving the “minimum cost flow problem” of a new graph constructed from  $G$  [Tam87]. Garg and Tamassia refine the algorithm so that it takes time  $O(n^{1.75} \sqrt{\log n})$  [GT97]. Tamassia presents an “orthogonal representation” of a plane graph for character-

izing an orthogonal drawing [DETT99, Tam87]. Our regular labeling can be regarded as a special case of his orthogonal representation. He gives also a linear algorithm to find an orthogonal “grid” drawing from an orthogonal representation, in which every vertex has an integer coordinate. Similarly, one can find an inner rectangular “grid” drawing from a regular labeling in linear time.

A *sketch* of the outer face  $F_o$  of a plane graph  $G$  is to assign a label 1, 2 or 3 to the outer angle  $\alpha$  of each outer vertex  $v$  of  $G$ , as illustrated in Fig. 1. If  $d(v) = 2$ , then the label of  $\alpha$  must be either 2 or 3. If  $d(v) = 3$ , then the label must be either 1 or 2. If  $d(v) = 4$ , then the label must be 1. Furthermore,  $n_{cv} - n_{cc} = 4$ . For example,  $n_{cv} = 5$  and  $n_{cc} = 1$  for the sketches in Figs. 1(b) and 4(a), and hence the sketches imply that the outer face boundary  $F_o$  must have an L-shape.

Suppose first that a sketch of  $F_o$  is prescribed. Then one can immediately determine some of the inner angles by Conditions (a) and (b) of a regular labeling, as illustrated in Fig. 4(a). The remaining undetermined inner angles are labeled with  $x$ , which means either 1 or 2. We construct from  $G$  a new graph  $G_d$ , called a *decision graph* of  $G$ , as illustrated in Fig. 4(b) where  $G_d$  is drawn by solid lines and  $G$  by dotted lines. We then have the following theorem.

[Theorem 1] Suppose that a sketch of the outer face of a plane graph  $G$  is prescribed. Then  $G$  has an inner rectangular drawing  $D$  with the sketched outer face if and only if the decision graph  $G_d$  of  $G$  has a perfect matching.  $D$  can be found in time  $O(n^{1.5}/\log n)$  whenever  $G$  has  $D$ , where  $n$  is the number of vertices in  $G$ .

The proof of Theorem 1 and together with the construction of  $G_d$  will be given in Section 3.

Suppose next that a sketch of  $F_o$  is not prescribed but the numbers  $n_{cv}$  and  $n_{cc}$  of convex and concave outer vertices are prescribed. Of course,  $n_{cv} - n_{cc} = 4$ . For example,  $n_{cv} = 4$  and  $n_{cc} = 0$  mean that  $F_o$  is rectangular as illustrated in Fig. 1(a), while  $n_{cv} = 6$  and  $n_{cc} = 2$  mean that  $F_o$  has a T-shape, U-shape, Z-shape or staircase-shape as illustrated in Figs. 1(c)–(f). Let  $G_d^*$  be a graph constructed from  $G$ , as illustrated in Fig. 6(b). Then we have the following theorem.

[Theorem 2] Suppose that a pair of non-negative integers  $n_{cv}$  and  $n_{cc}$  are prescribed. Then a plane graph  $G$  has an inner rectangular drawing  $D$  with  $n_{cv}$  convex outer vertices and  $n_{cc}$  concave ones if and only if  $G_d^*$  has a perfect matching.  $D$  can be found in time  $O(\sqrt{n}(n + n_{cv}n_o)/\log n)$  whenever  $G$  has  $D$ , where  $n_o$  is the number of outer vertices.

The proof of Theorem 2 together with the construction of  $G_d^*$  will be given in Section 4.

Consider finally a general case where neither a sketch of  $F_o$  nor a pair  $(n_{cv}, n_{cc})$  is prescribed. Let  $G_d^*$  be a graph

constructed from  $G$ . Then we have the following theorem.

[Theorem 3] A plane graph  $G$  has an inner rectangular drawing  $D$  if and only if  $G_d^*$  has a perfect matching.  $D$  can be found in time  $O(\sqrt{n}(n + (n_{o2} - n_{o4})n_o)/\log n)$  whenever  $G$  has  $D$ , where  $n_{o2}$  and  $n_{o4}$  are the numbers of outer vertices of degrees 2 and 4, respectively.

The proof of Theorem 3 together with the construction of  $G_d^*$  will be given in Section 5.

We assume that some trivial conditions for the existence of an inner rectangular drawing hold in Theorems 1, 2 and 3, as we will explain in Sections 3, 4 and 5.

### 3. Inner rectangular drawing with sketched outer face

In this section we prove Theorem 1.

Suppose that a sketch of the outer face of a plane graph  $G$  is prescribed, that is, all the outer angles of  $G$  are labeled with 1, 2 or 3, as illustrated in Figs. 1 and 4(a). Of course, the number  $n_{cv}$  of outer angles labeled with 3 and the number  $n_{cc}$  of outer angles labeled with 1 must satisfy  $n_{cv} - n_{cc} = 4$ . The outer angle of an outer vertex  $v$  must be labeled with either 2 or 3 if  $d(v) = 2$ , with either 1 or 2 if  $d(v) = 3$ , and with 1 if  $d(v) = 4$ . Then some of the inner angles of  $G$  can be immediately determined, as illustrated in Fig. 4(a). If  $v$  is an outer vertex of degree 2 and the outer angle of  $v$  is labeled with 2 and 3, then the inner angle of  $v$  must be labeled with 2 and 1, respectively. The two angles of any inner vertex of degree 2 must be labeled with 2. If  $v$  is an outer vertex of degree 3 and the outer angle of  $v$  is labeled with 2, then both of the inner angles of  $v$  must be labeled with 1. On the other hand, if  $v$  is an outer vertex of degree 3 and the outer angle of  $v$  is labeled with 1, then we label both of the inner angles of  $v$  with  $x$ , because one cannot determine their labels at this moment although one of them must be labeled with 1 and the other with 2. We also label all the three angles of an inner vertex of degree 3 with  $x$ , because one cannot determine their labels although exactly one of them must be labeled with 2 and the others with 1. We label all the four angles of each vertex of degree 4 with 1. Label  $x$  means that  $x$  is either 1 or 2, and exactly one of the labels  $x$ 's attached to the same vertex must be 2 and the others must be 1. (See Figs. 4(a) and (c).)

We now present how to construct a decision graph  $G_d$  of  $G$ . Let all vertices of  $G$  attached a label  $x$  be vertices of  $G_d$ . Thus all the inner vertices of degree 3 and all the outer vertices of degree 3 whose outer angles are labeled with 1 are vertices of  $G_d$ , and none of the other vertices of  $G$  is a vertex of  $G_d$ . We then add to  $G_d$  a complete bipartite graph inside each inner face  $F$  of  $G$ . Let  $n_x$  be the number of angles of  $F$  labeled with  $x$ . Let  $n_1$  be the number of angles of  $F$  which have been

labeled with 1. One may assume that  $n_1 \leq 4$ ; otherwise,  $G$  has no inner rectangular drawing. Exactly  $4 - n_1$  of the  $n_x$  angles of  $F$  labeled with  $x$  must be labeled with 1 by a regular labeling. We add a complete bipartite graph  $K_{(4-n_1)n_x}$  in  $F$ , and join each of the  $n_x$  vertices in the second partite set with one of the  $n_x$  vertices on  $F$  whose angles are labeled with  $x$ . Repeat the operation above for each inner face  $F$  of  $G$ . The resulting graph is a *decision graph*  $G_d$  of  $G$ . The decision graph  $G_d$  of the plane graph  $G$  in Fig. 4(a) is drawn by solid lines in Fig. 4(b), where  $G$  is drawn by dotted lines. The idea of adding a complete bipartite graph originates from Tutte's transformation for finding an " $f$ -factor" of a graph [Tut54].

A *matching* of  $G_d$  is a set of pairwise non-adjacent edges in  $G_d$ . A *maximum matching* of  $G_d$  is a matching of the maximum cardinality. A matching  $M$  of  $G_d$  is called a *perfect matching* if an edge in  $M$  is incident to each vertex of  $G_d$ . A perfect matching is drawn by thick solid lines in Fig. 4(b).

Each edge  $e$  of  $G_d$  incident to a vertex  $v$  attached a label  $x$  corresponds to an angle  $\alpha$  of  $v$  labeled with  $x$ . A fact that  $e$  is contained in a perfect matching  $M$  of  $G_d$  means that the label  $x$  of  $\alpha$  is 2. Conversely, a fact that  $e$  is not contained in  $M$  means that the label  $x$  of  $\alpha$  is 1.

We are now ready to prove Theorem 1.

**[Proof of Theorem 1]**

Suppose that  $G$  has an inner rectangular drawing with a sketched outer face. Then by Fact 1  $G$  has a regular labeling  $f$  which is an extension of the sketch, that is, a labeling of the outer angles. We include in a set  $M$  all the edges of  $G_d$  corresponding to angles of label  $x = 2$ , while we do not include in  $M$  the edges of  $G_d$  corresponding to angles of label  $x = 1$ . For each vertex  $v$  of  $G_d$  attached a label  $x$ , the labeling  $f$  assigns 2 to exactly one of the angles of  $v$  labeled with  $x$ . Therefore exactly one of the edges of  $G_d$  incident to  $v$  is contained in  $M$ . The labeling  $f$  labels exactly four of the angles of each inner face  $F$  with 1. Therefore exactly  $4 - n_1$  of the  $n_x$  angles of  $F$  labeled with  $x$  must be labeled with 1 by  $f$ , and hence all the edges of  $G_d$  corresponding to these angles are not contained in  $M$ . Including in  $M$  a number  $4 - n_1$  of edges in each complete bipartite graph, we can extend  $M$  to a perfect matching of  $G_d$ . Thus  $G_d$  has a perfect matching.

Conversely, if  $G_d$  has a perfect matching, then  $G$  has a regular labeling which is an extension of a sketch of the outer face, and hence by Fact 1  $G$  has an inner rectangular drawing with a sketched outer face.

Clearly,  $G_d$  is a bipartite graph, and  $4 - n_1 \leq 4$ . Obviously,  $n_x$  is no more than the number of edges on face  $F$ . Let  $m$  be the number of edges in  $G$ , then we have  $2m \leq 4n$  since  $\Delta \leq 4$ . Therefore the sum  $2m$  of the numbers of edges on all faces is at most  $4n$ . One can thus know that both the number  $n_d$  of vertices in  $G_d$  and the number  $m_d$  of edges in  $G_d$

are  $O(n)$ . Since  $G_d$  is a bipartite graph, a maximum matching of  $G_d$  can be found either in time  $O(\sqrt{n_d}m_d) = O(n^{1.5})$  by an ordinary bipartite matching algorithm [HK73, MV80, PS82] or in time  $O(\sqrt{n_d}m_d/\log n_d) = O(n^{1.5}/\log n)$  by a recent pseudoflow-based bipartite matching algorithm using boolean word operations on  $\log n$ -bit words [FM91, Hoc04, HC04].

This complete a proof of Theorem 1. □

Lai and Leinwand show that a plane graph  $G$  with  $\Delta \leq 3$  have a rectangular drawing if and only if a new bipartite graph constructed from  $G$  and its dual has a perfect matching [LL90]. Their bipartite graph has an  $O(n)$  number of vertices, but has an  $O(n^2)$  number of edges. Therefore their method takes time  $O(n^{2.5}/\log n)$  to find a rectangular drawing of  $G$ .

We also remark that one can enumerate all inner rectangular drawings of  $G$  by enumerating all perfect matchings of  $G_d$ .

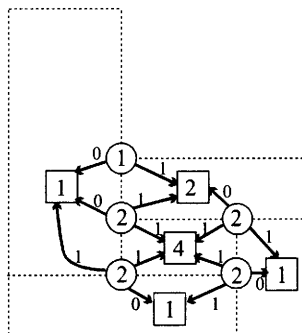


Fig. 5 Network  $N$ .

Clearly the bipartite matching problem can be reduced to the maximum flow problem [AMO93, PS82]. Conversely, modifying Tamassia's formulation of the bend-minimum orthogonal drawing problem [Tam87], one can directly reduce the inner rectangular drawing problem to a flow problem on a new planar bipartite network  $N$  with multiple sources and sinks. A network  $N$  for the plane graph  $G$  in Fig. 4(a) is illustrated in Fig. 5;  $N$  is drawn by solid lines, a flow value is attached to each arc, and an inner rectangular drawing of  $G$  corresponding to the flow is drawn by dotted lines. Every arc of  $N$  has a capacity 1. A node of  $N$  corresponding to a vertex of  $G$  is a source, the supply of which is written in a circle in Fig. 5. A node of  $N$  corresponding to an inner face of  $G$  is a sink, the demand of which is written in a square in Fig. 5. An inner angle of  $\pi/2$  is represented by an arc of flow 1, while an inner angle of  $\pi$  is represented by an arc of flow 0. One can observe that  $G$  has an inner rectangular drawing with the sketched outer face if and only if  $N$  has a feasible

(single commodity) flow satisfying all the demands. A feasible flow in such a planar network or a bipartite network can be found either in time  $O(n^{1.5})$  by a planar flow algorithm [MN95] or a bipartite flow algorithm [AMO93, ET75] or in time  $O(n^{1.5}/\log n)$  by the pseudoflow-based bipartite flow algorithm [Hoc04, HC04].

Thus both our matching approach and the flow approach solve the inner rectangular drawing problem in the same time complexity. However, the bipartite matching algorithm in [HK73] can be quite easily implemented in comparison with the flow algorithms in [Hoc04, HC04, MN95].

#### 4. Inner rectangular drawing with prescribed numbers $n_{cv}$ and $n_{cc}$

In this section we prove Theorem 2.

Suppose that an outer face of a plane graph  $G$  is not sketched but a pair  $(n_{cv}, n_{cc})$  is prescribed, where  $n_{cv}$  is the number of convex outer vertices and  $n_{cc}$  is the number of concave outer vertices. If  $(n_{cv}, n_{cc})$  is prescribed as  $n_{cv} = 5$  and  $n_{cc} = 1$  like in Fig. 6(a), then the outer boundary must have an L-shape as illustrated in Fig. 6(c), but it has not been prescribed which outer vertices are convex and which outer vertices are concave. We label all the four angles of each vertex of degree 4 in  $G$  with 1, and label both of the angles of each inner vertex of degree 2 with 2. The labels of all the other angles of  $G$  are not determined at this moment, and we label them with  $x$  or  $y$  as follows:

- (1) label all the three angles of any vertex  $v$  of degree 3 with  $x$ ; and
- (2) label the inner angle of any outer vertex  $v$  of degree 2 with  $x$ , and label the outer angle of  $v$  with  $y$ .

The labeling of the same plane graph as in Fig. 4(a) is depicted in Fig. 6(a). Label  $x$  means that  $x$  is either 1 or 2, similarly as in Section 3. On the other hand, label  $y$  means that  $y$  is either 2 or 3. Each outer vertex of degree 2 is attached two labels  $x$  and  $y$ , and if  $x = 1$  then  $y = 3$  and if  $x = 2$  then  $y = 2$ .

We now present how to construct a decision graph  $G_d^*$  of  $G$ . The construction is similar to that of  $G_d$ . Let all the vertices of  $G$  attached label  $x$  or  $y$  be vertices of  $G_d^*$  as illustrated in Fig. 6(b). Thus all the outer vertices of degree 2 in  $G$  and all the vertices of degree 3 in  $G$  are vertices of  $G_d^*$ . All the other vertices of  $G$  are not vertices of  $G_d^*$ : all the vertices of degree 4 and all the inner vertices of degree 2 are not vertices of  $G_d^*$ .

For each inner face  $F$  of  $G$ , we add a complete bipartite graph  $K_{(4-n_1)n_x}$  inside  $F$ , where  $n_x$  is the number of angles of  $F$  labeled with  $x$  and  $n_1$  is the number of angles of  $F$  labeled with 1. Of course, one may assume that  $n_1 \leq 4$ .

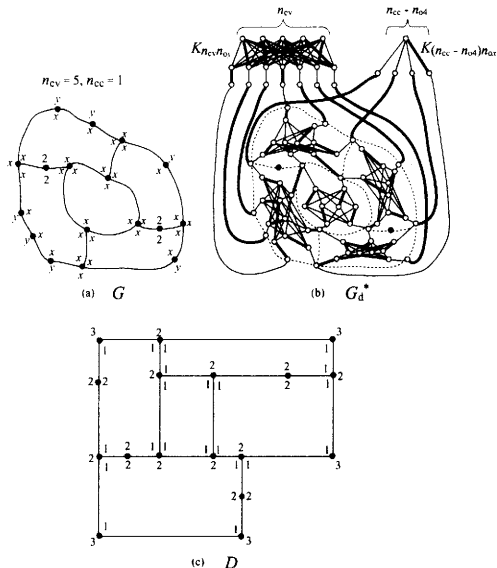


Fig. 6 (a) Plane graph  $G$  and pair  $(n_{cv}, n_{cc})$ , (b) decision graph  $G_d^*$ , and (c) inner rectangular drawing  $D$  and regular labeling  $f$  of  $G$ .

We then add two complete bipartite graphs inside the outer face  $F_o$ , as follows. Let  $n_{ox}$  be the number of outer angles of  $G$  labeled with  $x$ , and let  $n_{oy}$  be the number of outer angles labeled with  $y$ . For the example in Fig. 6(a)  $n_{ox} = 4$  and  $n_{oy} = 7$ . Let  $n_{o4}$  be the number of outer vertices  $v$  of degree 4. For the example in Fig. 6(a)  $n_{o4} = 0$ . The outer angle of  $v$  must be labeled with 1, and  $v$  must be a concave outer vertex. One may assume without loss of generality that  $n_{cc} \geq n_{o4}$ ; otherwise,  $G$  has no inner rectangular drawing with  $n_{cc}$  concave outer vertices. Exactly  $n_{cc} - n_{o4}$  of the  $n_{ox}$  outer angles of label  $x$  must be labeled with 1. We add a complete bipartite graph  $K_{(n_{cc} - n_{o4})n_{ox}}$  in  $F_o$ , and joint each of the  $n_{ox}$  vertices in the second partite set with one of the  $n_{ox}$  outer vertices of degree 3. Exactly  $n_{cv}$  of the  $n_{oy}$  outer angles of label  $y$  must be labeled with 3. We add a complete bipartite graph  $K_{n_{cv}n_{oy}}$  inside  $F_o$ , and connect each of the  $n_{oy}$  vertices in the second partite set with one of the  $n_{oy}$  outer vertices of degree 2 via a path of length 2.

This completes a construction of  $G_d^*$ . In Fig. 6(b)  $G_d$  is drawn by solid lines, and  $G$  by dotted lines. A perfect matching of  $G_d^*$  is drawn by thick solid lines in Fig. 6(b).

Let  $\alpha$  be an angle of a vertex  $v$  of  $G$  labeled with  $x$  or  $y$ , and let  $e$  be the edge of  $G_d^*$  which is incident to  $v$  and corresponds to  $\alpha$ . A fact that  $e$  is contained in a perfect matching  $M$  of  $G_d^*$  means that the label of  $\alpha$  is 2 if it is  $x$  and the label is 3 if it is  $y$ . Conversely, a fact that  $e$  is not contained in  $M$  means that the label of  $\alpha$  is 1 if it is  $x$  and the label is

2 if it is  $y$ . Similarly as for Theorem 1, one can easily prove that  $G$  has an inner rectangular drawing with  $n_{cv}$  convex outer vertices and  $n_{cc}$  concave ones if and only if  $G_d^*$  has a perfect matching. Clearly  $G_d^*$  is a bipartite graph. Since  $n_o \geq n_{cv} > n_{cc}$ ,  $G_d^*$  has an  $O(n)$  number of vertices and an  $O(n + n_{cv}n_o)$  number of edges, where  $n_o$  is the number of outer vertices of  $G$ . Thus a maximum matching of  $G_d^*$  can be found in time  $O(\sqrt{n}(n + n_{cv}n_o)/\log n)$ .

This completes a proof of Theorem 2.

## 5. Inner rectangular drawing

In this section we prove Theorem 3.

Suppose that neither a sketch of the outer face  $F_o$  nor a pair of integers  $(n_{cv}, n_{cc})$  is prescribed for a plane graph  $G$ . Let  $n_{o2}, n_{o3}$  and  $n_{o4}$  be the numbers of outer vertices having degrees 2, 3 and 4, respectively, then  $n_o = n_{o2} + n_{o3} + n_{o4}$ . Since  $n_{cv} \leq n_{o2}$  and  $n_{cv} - n_{cc} = 4$ , there are at most a number  $n_{o2}$  of pairs which are possible as  $(n_{cv}, n_{cc})$ . Examining all these pairs, one can know whether  $G$  has an inner rectangular drawing for some pair. Such a straightforward method would take time  $O(n_{o2}\sqrt{n}(n + n_{o2}n_o)/\log n)$ . However, we can show as in Theorem 3 that  $G$  has an inner rectangular drawing  $D$  for some pair if and only if a new graph  $G_d^*$  constructed from  $G$  has a perfect matching, and that  $D$  can be found in time  $O(\sqrt{n}(n + (n_{o2} - n_{o4})n_o)/\log n)$  whenever  $G$  has  $D$ .

We label each angle of  $G$  with 1, 2,  $x$  or  $y$  in the same way as in Section 4, as illustrated in Fig. 6(a). There are  $n_{o4}$  outer angles labeled with 1,  $n_{o3}$  with  $x$ , and  $n_{o2}$  with  $y$ . One may assume without loss of generality that  $n_{o2} \geq n_{o4} + 4$ ; otherwise,  $G$  has no inner rectangular drawing. The construction of a new graph  $G_d^*$  is the same as  $G_d^*$  except for the outer face  $F_o$ . We add a complete bipartite graph  $B = K_{(n_{o2}-n_{o4}-4)(n_{o2}+n_{o3})}$  in  $F_o$ , and join each of the  $n_{o2} + n_{o3}$  vertices in the second partite set of  $B$  with one of the  $n_{o2} + n_{o3}$  outer vertices of degree 2 or 3.

Suppose that  $G_d^*$  has a perfect matching  $M$ . Let  $a$  be the number of edges of  $G_d^*$  which correspond to outer angles of outer vertices of degree 2 and are contained in  $M$ . Let  $b$  be the number of edges of  $G_d^*$  which correspond to outer angles of outer vertices of degree 3 and are contained in  $M$ . Since  $M$  covers all the  $n_{o2} - n_{o4} - 4$  vertices in the first partite set of  $B$ , we have  $(n_{o2} - a) + (n_{o3} - b) = n_{o2} - n_{o4} - 4$ , and hence  $a + b = n_{o3} + n_{o4} + 4$ . We assign 2 to the  $b$  outer angles of outer vertices which have label  $x$  and are ends of edges of  $M$  in  $F_o$ , and assign 1 to the remaining  $(n_{o3} - b)$  outer angles of label  $x$ . We assign 3 to the  $a$  outer angles of outer vertices which have label  $y$  and are ends of edges of  $M$  in  $F_o$ , and assign 2 to the remaining  $(n_{o2} - a)$  outer angles of label  $y$ . Then we have  $n_{cv} = a$  and  $n_{cc} = (n_{o3} - b) + n_{o4}$ , and hence

$n_{cv} - n_{cc} = a - (n_{o3} - b + n_{o4}) = 4$ . One can thus know that  $G$  has a regular labeling if  $G_d^*$  has a perfect matching.

Conversely  $G_d^*$  has a perfect matching if  $G$  has an inner rectangular drawing.

This completes a proof of Theorem 3.

## 6. Conclusions

In this paper, we reduced the problem of finding an inner rectangular drawing of a plane graph to the perfect matching problem. More precisely, we showed that a plane graph  $G$  has an inner rectangular drawing  $D$  if and only if a new bipartite graph constructed from  $G$  has a perfect matching. We also showed that  $D$  can be found in time  $O(n^{1.5}/\log n)$  if a sketch of the outer face is prescribed, in time  $O(\sqrt{n}(n + n_{cv}n_o)/\log n)$  if a pair  $(n_{cv}, n_{cc})$  is prescribed, and in time  $O(\sqrt{n}(n + (n_{o2} - n_{o4})n_o)/\log n)$  for a general case where neither a sketch nor a pair is prescribed, where  $n_o$  is the number of outer vertices and  $n_{o2}$  and  $n_{o4}$  are the numbers of outer vertices of degrees 2 and 4, respectively.

If a sketch of several faces of  $G$  including the outer face is prescribed, then one can examine whether  $G$  has a drawing  $D$  such that each of the other faces is a rectangle, and can find  $D$  whenever  $G$  has  $D$ , by finding a maximum matching in a bipartite graph constructed from  $G$ .

If faces  $F_o, F_1, \dots, F_i, i \geq 1$ , of  $G$  are pairwise vertex-disjoint and the numbers of convex and concave vertices are prescribed for each of these faces, then one can examine whether  $G$  has a drawing  $D$  such that each of the faces  $F_o, F_1, \dots, F_i$  has the prescribed numbers of convex and concave vertices and each of the other faces is a rectangle, and can find  $D$  whenever  $G$  has  $D$ , by finding a maximum matching of a bipartite graph constructed from  $G$ .

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