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重み付き独立集合問題に対する近似アルゴリズム

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概 要

重みなし最大独立集合問題の近似アルゴリズムに対する近似率は, 頂点数, 最大次数, 平均次数などをパラメータとして解析されている. しかし、重み付きの問題では平均次数をパラメータとした近似率の解析はなされていない. 本論文では, 平均次数を拡張した重み付き平均次数と重み付き inductiveness というパラメータを導入し, それを用いて近似率を解析する.

Approximation Algorithms for the Weighted Independent Set Problem

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Abstract

In unweighted case, approximation ratio for the maximum independet set problem has been analyzed in terms of the graph parameters, such as the number of vertices, maximum degree, and average degree. In weighted case, no corresponding results are given for average degree. In this paper, we analyze approximation ratio in terms of the “weighted” average degree and “weighted” inductiveness.

1 Introduction

An independent set in a graph is a set of vertices in which no two vertices are adjacent. The (weighted) independent set problem is that of finding a maximum (weight) independent set. There have been proposed and analyzed numerous approximation algorithms for this problem. In unweighted case, an algorithm with approximation ratio $\Delta/6+O(1)$ was proposed by Halldórsson and Radhakrishnan [6] for the graphs with the maximum degree Δ. Vishwanathan proposed the SDP-based algorithm whose approximation ratio is $O(\Delta \log\log\Delta/\log\Delta)$ [3]. For the graphs with the average degree $\overline{d}$, Hochbaum [7] proved that a version of Greedy algorithm has approximation ratio $(\overline{d}+1)/2$. Halldórsson and Radhakrishnan [5] improved this approximation ratio to $(2\overline{d}+3)/5$. Moreover, an algorithm with approximation ratio $O(\overline{d}\log\log\overline{d}/\log\overline{d})$ was proposed by Halldórsson [2]. In weighted case, Halldórsson and Lau [4] gave an algorithm with approximation ratio $(\Delta+2)/3$. For the δ-inductive graphs approximation ratio $(\delta+1)/2$ is known due to Hochbaum [7], and Halldórsson [2] proposed an algorithm with approximation ratio $O(\delta\log\log\delta/\log\delta)$. Note that $\delta \leq \Delta$ for any graph.

In this paper, we extend the approximation algorithms of [2, 7] to the weighted case. Since inserting the vertices with small weight decreases $\overline{d}$ arbitrarily without significantly changing approximation ratio, we introduce the *weighted average degree* $\overline{d}_w$ and analyze the approximation ratio. For weighted graphs, there exist approximation algorithms whose approximation ratio is analyzed in terms of inductiveness. We extend inductiveness to weighted version and introduce the *weighted inductiveness* δ_w.

The rest of this paper is organized as follows. In Section 2 we define the weighted average degree and the weighted inductiveness. We also show the relationship between these degrees. In Section 3 we propose a greedy algorithm whose lower bound is $\max(W/(\overline{d}_w+1), W/(\delta_w+1))$, where W is the total weight. In Section 4 we prove that the approximation ratio of $\min((\overline{d}_w+1)/2, (\delta_w+1)/2)$ can be achieved. Finally we will prove that the approximation ratios of $O(\overline{d}_w\log\log\overline{d}_w/\log\overline{d}_w)$ and $O(\delta_w\log\log\delta_w/\log\delta_w)$ can be achieved in Section 5. We will assume that the input graphs have no isolated vertices, because isolated vertices may always be included in the independent set.

2 Preliminaries

2.1 Definitions

Let G be an undirected graph where each vertex v has positive weight w_v. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively, as usual. Let $W(G)$ be the sum of the weights of all vertices. $n(G)$ is the number of vertices in G. Let $\Delta(G)$ and $\overline{d}(G)$ denote the maximum and the average degree of G, respectively. $d(v,G)$ is the degree of vertex v in G. The inductiveness $\delta(G)$ of a graph G is given by

$$\delta(G) = \max_{H\subseteq G}\min_{v} d(v,H), \tag{2.1}$$

where $H \subseteq G$ denotes that H is a subgraph of G. Let π be an ordering of vertices in V, that is, π is a one to one map $\pi : V \to \{1,2,\ldots,n\}(n=|V|)$. We define the right degree of a vertex v in G with respect to π as follows:

$$d^{\pi}(v,G) = |\{u \in V | (u,v) \in E, \pi(u) > \pi(v)\}|. \tag{2.2}$$

The right degree of a vertex v is the number of adjacent vertices to the right when we arrange vertices from left to right according to π. If there exists π such that $m \geq \max_v d^{\pi}(v,G)$, we call G an m-inductive graph.

For a vertex set X, let $w(X)$ denote the sum of the weights of the vertices in X. Let $N_G(v)$ denote the set of vertices adjacent to vertex v in G. For a vertex v, we define the weighted degree $d_w(v,G)$ in G as follows:

$$d_w(v,G) = \frac{w(N_G(v))}{w_v}. \tag{2.3}$$

$\Delta_w(G) = \max_v d_w(v,G)$ is the maximum weighted degree of G. We will omit G if it is clear from the context. We define the weighted average degree $\overline{d}_w(G)$ of graph G as follows:

$$\overline{d}_w(G) = \frac{\sum_{v\in V} w_v d_w(v)}{W}. \tag{2.4}$$

In fact, we can represent the weighted average degree in the following form:

$$\begin{aligned}\overline{d}_w(G) &= \frac{\sum_{v\in V} w(N(v))}{W} && (2.5)\\ &= \frac{\sum_{v\in V} w_v d(v)}{W}. && (2.6)\end{aligned}$$

The weighted inductiveness $\delta_w(G)$ of a graph G is given by

$$\delta_w(G) = \max_{H\subseteq G}\min_{v} d_w(v,H). \tag{2.7}$$

We define the right weighted degree of a vertex v for an ordering π in G as follows:

$$d_w^{\pi}(v,G) = \frac{w(\{u \in V | (u,v) \in E, \pi(u) > \pi(v)\})}{w_v}.$$

If there exists π such that $m \geq \max_v d_w^\pi(v,G)$, we call G a weighted m-inductive graph.

We denote $\alpha_w(G)$ as the weight of the optimal solution of the weighted independent set problem on G. For an algorithm A, $A(G)$ denotes the weight of the independent set obtained by A on G. Then the approximation ratio of A is defined by

$$\sup_G \frac{\alpha_w(G)}{A(G)}.$$

We will consider unweighted graphs as weighted ones where each vertex has unit weight. $\alpha(G)$ denotes the size of a maximum independent set on G.

2.2 Weighted inductiveness

Let π be an ordering of the vertices of G and v_i be a vertex with $\pi(v_i) = i$. We define $V_i^\pi = \{v_j | j \geq i\}$. Let G_i^π be the induced subgraph of G by V_i^π. Smallest-first ordering π is an ordering such that the weighted degree of v_i is minimum in G_i^π for all i $(1 \leq i \leq n)$. We can find a smallest-first ordering in polynomial time. We can prove the following theorem by the same method as in the case of unweighted inductiveness [8].

Proposition 2.1 *For any ordering π, the inequality*

$$\delta_w(G) \leq \max_v d_w^\pi(v,G)$$

holds.

Proof: Let H^* be a subgraph of G with $\min_v d_w(v, H^*) = \delta_w(G)$. Let j be the largest index such that H^* is the subgraph of G_j^π. Then the following inequalities hold:

$$\delta_w(G) = \min_v d_w(v,H^*) \leq d_w(v_j,H^*) \leq d_w(v_j,G_j^\pi).$$

Thus, $\max_v d_w^\pi(v,G) = \max_i d_w(v_i,G_i^\pi) \geq \delta_w(G)$. Hence, the given inequality holds. □

Theorem 2.2 *If π is a smallest-first ordering, then the equality*

$$\delta_w(G) = \max_v d_w^\pi(v,G)$$

holds.

Proof: The following inequalities hold:

$$\begin{aligned}
\max_v d_w^\pi(v,G) &= \max_i d_w(v_i,G_i^\pi) \\
&= \max_i \min_{v\in V_i^\pi} d_w(v,G_i^\pi) \\
&\leq \max_{H\subseteq G} \min_{v\in V(H)} d_w(v,H) \\
&= \delta_w(G).
\end{aligned}$$

Thus, $\delta_w(G) \geq \max_v d_w^\pi(v,G)$. From this inequality and Proposition 2.1, this theorem holds. □

Corollary 2.3 *A smallest-first ordering π minimizes* $\max_v d_w^\pi(v,G)$.

2.3 Relationship between weighted and unweighted degrees

Theorem 2.4 *The following relationships hold for all graphs G and all weight functions w:*

$$\begin{aligned}
\delta &\leq \Delta_w &(2.8)\\
\delta_w &\leq \Delta &(2.9)\\
\overline{d} &\leq \Delta_w &(2.10)\\
\overline{d}_w &\leq \Delta &(2.11)\\
\delta_w &\leq \Delta_w &(2.12)\\
\overline{d}_w &\leq \Delta_w. &(2.13)
\end{aligned}$$

Proof: Let π_1 be an ordering of the vertices in non-decreasing ordering of weight. Then the inequalities

$$\delta \leq \max_v d^{\pi_1}(v,G) \leq \max_v d_w^{\pi_1}(v,G) \leq \Delta_w$$

hold. Let π_2 be an ordering of the vertices in non-increasing ordering of weight. Then the following inequalities hold:

$$\delta_w \leq \max_v d_w^{\pi_2}(v,G) \leq \max_v d^{\pi_2}(v,G) \leq \Delta.$$

(2.11), (2.12) and (2.13) follow immediately from the definition of measures. Finally, we prove inequality (2.10). We can get the following inequalities:

$$\begin{aligned}
\sum_{v\in V} d_w(v) &= \sum_{v\in V}\sum_{u:(u,v)\in E} \frac{w_u}{w_v} \\
&= \sum_{(u,v)\in E} \left[\frac{w_u}{w_v} + \frac{w_v}{w_u}\right] \\
&\geq 2|E| \\
&= n\overline{d}.
\end{aligned}$$

Thus,

$$\Delta_w = \max_{v\in V} d_w(v) \geq \frac{1}{n}\sum_{v\in V} d_w(v) \geq \overline{d}.$$

Hence, this theorem holds. □

3 Greedy algorithm

3.1 Previous results

For unweighted graphs, the greedy algorithm can be written as follows. We select a minimum degree vertex as a vertex in the independent set I, and delete this vertex and all of its neighbors from the graph. We repeat this process for the remaining subgraph. When the induced subgraph becomes empty, we terminate the algorithm. This algorithm attains the Turán bound [5, 7];

$$|I| \geq \frac{n}{\overline{d}+1}. \tag{3.1}$$

For weighted graphs, there exists an algorithm which attains the following lower bound [2, 8]

$$w(I) \geq \frac{W}{\delta+1}. \tag{3.2}$$

3.2 Algorithm for the weighted graphs

Our greedy algorithm for the weighted graphs is almost the same as the unweighted greedy algorithm. The difference is that, instead of selecting a minimum degree vertex, our algorithm selects a minimum weighted degree vertex. We call this algorithm WG.

3.3 Lower bound

We use the following proposition.

Proposition 3.1 *Assume that $a_i > 0$, $b_i > 0$ for all $1 \leq i \leq n$. Then the inequality*

$$\sum_i \frac{b_i^2}{a_i} \geq \frac{\left(\sum_i b_i\right)^2}{\sum_i a_i}$$

holds.

Proof: The inequality is equivalent to

$$\sum_i a_i \sum_i \frac{b_i^2}{a_i} \geq \left(\sum_i b_i\right)^2.$$

This inequality comes from the Cauchy-Schwarz inequality

$$\left(\sum_i x_i^2\right)\left(\sum_i y_i^2\right) \geq \left(\sum_i x_i y_i\right)^2,$$

by assigning $x_i = \sqrt{a_i}$ and $y_i = b_i/\sqrt{a_i}$. □

Let I be the independent set obtained by WG. Let v_i be the i-th vertex selected into the independent set I. Let G_i be the subgraph induced by the remaining vertices at the beginning of the i-th iteration.

Theorem 3.2 WG *produces the independent set satisfying the inequality*

$$\mathsf{WG}(G) \geq \frac{W}{\overline{d}_w+1}. \tag{3.3}$$

Proof: We first argue a lower bound of $\overline{d}_w W$ as follows:

$$\begin{aligned}
\overline{d}_w W &= \sum_{v \in V(G)} w_v d_w(v, G) \\
&= \sum_i \sum_{v \in N_{G_i}(v_i) \cup \{v_i\}} w_v d_w(v, G_i) \\
&\geq \sum_i \sum_{v \in N_{G_i}(v_i) \cup \{v_i\}} w_v d_w(v_i, G_i) \\
&= \sum_i \left(w(N_{G_i}(v_i)) + w_{v_i}\right) d_w(v_i, G_i).
\end{aligned}$$

The inequality follows from the property of the greedy algorithm, that is, $d_w(v_i, G_i) \leq d_w(v, G_i)$, and the last equality comes from the equality $\sum_{v \in N_{G_i}(v_i) \cup \{v_i\}} w_v = w(N_{G_i}(v_i)) + w_{v_i}$. We note that $d_w(v_i, G_i) = w(N_{G_i}(v_i))/w_{v_i}$ because v_i is the i-th selected vertex. Adding $W = \sum_i \left(w(N_{G_i}(v_i)) + w_{v_i}\right)$, we can deduce the inequality

$$\left(\overline{d}_w + 1\right) W \geq \sum_i \frac{\left(w(N_{G_i}(v_i)) + w_{v_i}\right)^2}{w_{v_i}}.$$

Finally we apply Proposition 3.1 with $a_i = w_{v_i}$, $b_i = w(N_{G_i}(v_i)) + w_{v_i}$. The inequality

$$\left(\overline{d}_w + 1\right) W \geq \frac{W^2}{\mathsf{WG}(G)}$$

holds, which implies the theorem. □

Note that WG can find an independent set with the following lower bound [9]:

$$\mathsf{WG}(G) \geq \sum_{v \in V} \frac{w_v^2}{w(N(v)) + w_v}.$$

This lower bound also leads to Theorem 3.2.

Theorem 3.3 WG *produces the independent set satisfying the inequality*

$$\mathsf{WG}(G) \geq \frac{W}{\delta_w + 1}. \tag{3.4}$$

Proof: Because $\delta_w \geq d_w(v_i, G_i)$ for all i and $W = \sum_i \left(w(N_{G_i}(v_i)) + w_{v_i}\right)$, the inequality

$$W\delta_w \geq \sum_i \left(w(N_{G_i}(v_i)) + w_{v_i}\right) d_w(v_i, G_i)$$

holds. With this inequality, we can prove this theorem in the same way as Theorem 3.2. □

Corollary 3.4 WG *produces the independent set satisfying the inequality*

$$\mathsf{WG}(G) \geq \max\left(\frac{W}{\overline{d}_w + 1}, \frac{W}{\delta_w + 1}\right). \tag{3.5}$$

Proposition 3.5 *The lower bounds of Theorems 3.2 and 3.3 are tight.*

Proof: Let G be a star graph with n vertices. We assign weight w to the central vertex and $w/\sqrt{n-1}$ to the other vertices. It is easy to see that the weighted degree of each vertex is $\sqrt{n-1}$. Thus, the weighted average degree of G is $\sqrt{n-1}$. It is obvious that the weighted inductiveness is also $\sqrt{n-1}$. The sum of the weights assigned to all vertices is clearly $(\sqrt{n-1}+1)w$. WG may select the central vertex as a vertex in the independent set I, and in this case the weight of the independent set $\mathsf{WG}(G) = w$. Thus the inequalities in Theorems 3.2 and 3.3 hold with equality, which means that these theorems give the tight lower bounds. □

4 Linear programming algorithm

4.1 Unweighted results

We will consider the combination of linear programming and the greedy algorithm. With the lower bound (3.1), Hochbaum [7] proved that this combination achieves the approximation ratio $(\overline{d}+1)/2$. In this section we extend Hochbaum's algorithm to the weighted case and prove that the proposed algorithm has the approximation ratios $(\overline{d}_w + 1)/2$ and $(\delta_w + 1)/2$.

4.2 LP relaxation for the weighted independent set problem

The weighted independent set problem can be formulated in the integer programming as follows:

$$\begin{array}{rll} \text{maximize} & \sum_{i \in V} w_i x_i, & \\ \text{subject to} & x_i + x_j \leq 1 & \text{for all } (i,j) \in E, \\ & x_i \in \{0,1\} & \text{for all } i \in V. \end{array} \tag{4.1}$$

Relaxing the integral constraint, we can deduce the following linear programming:

$$\begin{array}{rll} \text{maximize} & \sum_{i \in V} w_i x_i, & \\ \text{subject to} & x_i + x_j \leq 1 & \text{for all } (i,j) \in E, \\ & 0 \leq x_i \leq 1 & \text{for all } i \in V. \end{array} \tag{4.2}$$

We can obtain the optimal solution to this LP each of whose elements is 0, 1/2, or 1 [10]. We classify the vertices into three sets according to the value of x_i:

$$\begin{array}{ll} i \in S_1 & \text{if } x_i = 1, \\ i \in S_{\frac{1}{2}} & \text{if } x_i = \dfrac{1}{2}, \\ i \in S_0 & \text{if } x_i = 0. \end{array}$$

Note that S_1 is an independent set of G and no vertex in $S_{\frac{1}{2}}$ has a neighbor in S_1. We also note that $S_{\frac{1}{2}}$ induces the subgraph with no isolated vertices.

4.3 Algorithm

We first solve the LP relaxation to divide the vertex set V into three subsets S_1, $S_{\frac{1}{2}}$, and S_0 as above. We then apply WG to the subgraph H induced by the vertices in $S_{\frac{1}{2}}$ to obtain an independent set I_H of H. Finally, we output the independent set $I = S_1 \cup I_H$. We call this algorithm WGL.

4.4 Approximation ratio

Theorem 4.1 *Approximation ratio of* WGL *is* $(\overline{d}_w + 1)/2$.

Proof: By the definition (2.6) of $\overline{d}_w$ and assumption that the graph has no isolated vertices, we can show that the following inequality is satisfied:

$$\overline{d}_w(G) \geq \frac{w(S_{\frac{1}{2}})\overline{d}_w(H) + w(S_1) + w(S_0)}{w(S_{\frac{1}{2}}) + w(S_1) + w(S_0)}.$$

The inequality $\alpha_w \leq w(S_1) + \frac{1}{2}w(S_{\frac{1}{2}})$ holds because the optimal value of LP (4.2) is larger than that of IP (4.1). The lower bound of WG implies that $\mathsf{WG}(H) \geq w(S_{\frac{1}{2}})/(\overline{d}_w(H)+1)$. Thus the independent set I satisfies the inequality $\mathsf{WGL}(G) = w(S_1)+\mathsf{WG}(H) \geq w(S_1)+w(S_{\frac{1}{2}})/(\overline{d}_w(H)+1)$. We claim that

$$\frac{w(S_1)+\frac{1}{2}w(S_{\frac{1}{2}})}{w(S_1)+\frac{w(S_{\frac{1}{2}})}{\overline{d}_w(H)+1}} \leq \frac{1}{2}\left(\frac{w(S_{\frac{1}{2}})\overline{d}_w(H)+w(S_1)+w(S_0)}{w(S_{\frac{1}{2}})+w(S_1)+w(S_0)}+1\right),$$

which completes the theorem as follows:

$$\begin{aligned}
&\frac{\alpha_w(G)}{\mathsf{WGL}(G)} \\
&\leq \frac{w(S_1)+\frac{1}{2}w(S_{\frac{1}{2}})}{w(S_1)+\frac{w(S_{\frac{1}{2}})}{\overline{d}_w(H)+1}} \\
&\leq \frac{1}{2}\left(\frac{w(S_{\frac{1}{2}})\overline{d}_w(H)+w(S_1)+w(S_0)}{w(S_{\frac{1}{2}})+w(S_1)+w(S_0)}+1\right) \\
&\leq \frac{\overline{d}_w+1}{2}.
\end{aligned}$$

Now we prove that our claim holds. The claim is equivalent to

$$\begin{aligned}
&(w(S_{\frac{1}{2}})+w(S_1)+w(S_0)) \\
&\quad\times(2w(S_1)+w(S_{\frac{1}{2}}))(\overline{d}_w(H)+1) \\
&\leq [w(S_{\frac{1}{2}})(\overline{d}_w(H)+1)+2w(S_1)+2w(S_0)] \\
&\quad\times[w(S_1)(\overline{d}_w(H)+1)+w(S_{\frac{1}{2}})].
\end{aligned}$$

Rearranging this inequality, we have to prove the following inequality:

$$w(S_{\frac{1}{2}})(\overline{d}_w(H)-1)(\overline{d}_w(H)w(S_1)-w(S_0)) \geq 0. \quad (4.3)$$

$\overline{d}_w(H)$ is at least 1 because H has no isolated vertices. Moreover, we can show that $w(S_1)$ is no less than $w(S_0)$ as follows: We assume in contrast that $w(S_1) < w(S_0)$. In this case, the objective function becomes larger if we assign 1/2 to all variables corresponding to the vertices in S_1 and S_0, which is contradiction. Thus the inequality (4.3) is proved. □

Proposition 4.2 *The approximation ratio of Theorem 4.1 is tight.*

Proof: We consider the split graph $G = (V,E)$, where $V = \{u_1, u_2, \ldots, u_t, v_1, v_2, \ldots, v_{2t-1}\}$ and $E = \{(u_i, v_j) | 1 \leq i \leq t, 1 \leq j \leq 2t-1\} \cup \{(u_i, u_j) | 1 \leq i < j \leq t\}$. The induced subgraph by u_i is a clique and induced subgraph by v_i is an independent set. We give each vertex u_i weight $w/t+\epsilon$, each vertex v_i weight $w/(2t-1)$, where ϵ is positive and small enough. Then, the weighted average degree $\overline{d}_w$ is as follows:

$$\overline{d}_w = 2t-1+\frac{3t^2-2t}{2w}\epsilon.$$

In the optimal solution for LP (4.2), each value of x_i is 1/2. Thus, $S_{\frac{1}{2}} = V(G)$. $\mathsf{WGL}(G) = w/t+\epsilon$ because the weighted degree of a vertex u_i is smaller than that of a vertex v_i. The weight of the optimal solution is clearly w. So, the ratio of the weight of the approximate solution to the weight of the optimal solution is as follows:

$$\begin{aligned}
\frac{\alpha_w(G)}{\mathsf{WGL}(G)} &= \frac{w}{w/t+\epsilon} \\
&= t-\frac{\epsilon t^2}{w+\epsilon t} \\
&= \frac{\overline{d}_w+1}{2}-\left(\frac{t^2}{w+\epsilon t}-\frac{3t^2-2t}{4w}\right)\epsilon.
\end{aligned}$$

Hence, Theorem 4.1 is tight because ϵ can be arbitrarily small. □

Theorem 4.3 *Approximation ratio of* WGL *is* $(\delta_w+1)/2$.

Proof: From Theorem 3.3,

$$\begin{aligned}
\frac{\alpha_w(G)}{\mathsf{WGL}(G)} &\leq \frac{w(S_1)+\frac{1}{2}w(S_{\frac{1}{2}})}{w(S_1)+\frac{w(S_{\frac{1}{2}})}{\delta_w(H)+1}} \\
&\leq \frac{\delta_w(H)+1}{2} \\
&\leq \frac{\delta_w+1}{2}.
\end{aligned}$$

Thus, this theorem holds. □

Corollary 4.4 *Approximation ratio of* WGL *is* $\min((\overline{d}_w+1)/2, (\delta_w+1)/2)$.

5 Semi-definite programming

5.1 Previous result

The following theorem was proved in [2]:

Theorem 5.1 *For any fixed real number k, if $\vartheta_w(G) \geq 2W/k$, then we can construct an independent set in G whose weight is $\Omega(W/(k\delta^{1-1/(2k)}))$.*

The function $\vartheta_w(G)$, defined in [1], is the weighted version of Lovász's ϑ-function. This function can be computed using a semi-definite programming (SDP) in polynomial time, and has the property $\alpha_w(G) \leq \vartheta_w(G)$.

This theorem suggests the following algorithm [2]. We arbitrarily select a unit vector $d \in \mathbf{R}^n$. We can find unit vectors $\{b_v \in \mathbf{R}^n | v \in V(G)\}$ such that $\vartheta_w(G) = \sum_{v\in V(G)}(d \cdot b_v)^2 w_v$ and $b_i \cdot b_j = 0$ for any two adjacent vertices i and j. We remove every vertex v with $(d \cdot b_v)^2 < 1/k$. Let H be the subgraph induced by the remaining vertices. We then find unit vectors $\{u_v \in \mathbf{R}^n | v \in V(H)\}$ such that for any two adjacent vertices i and j the corresponding vectors u_i and u_j satisfy $u_i \cdot u_j = -1/k$. Using the method of "rounding by hyperplanes", we can obtain an independent set satisfying Theorem 5.1.

For the unweighted graphs, the combination of this algorithm and the greedy algorithm yields the algorithm with approximation ratio $O(\overline{d} \log\log \overline{d} / \log \overline{d})$.

5.2 Approximation ratio for the weighted graphs

We will prove the following result for the weighted version of the algorithm.

Theorem 5.2 *For any fixed real number t, if $t \geq W(G)/\alpha_w(G)$, we can approximate the weighted independent set problem within $O(t^2 \overline{d}_w^{1-1/(8t)})$.*

Proof: Assume that $t \geq W(G)/\alpha_w(G)$ is fixed. Let K be the subgraph induced by the vertices whose degrees in G are less than $2t\overline{d}_w$. Then we can estimate the value $\overline{d}_w W(G)$ as follows:

$$\begin{aligned}
\overline{d}_w W(G) &= \sum_{v\in V(G)} w_v d(v) \\
&\geq \sum_{v\in V(G)\setminus V(K)} w_v d(v) \\
&\geq 2t\overline{d}_w \sum_{v\in V(G)\setminus V(K)} w_v.
\end{aligned}$$

Thus the following inequality holds:

$$\sum_{v\in V(G)\setminus V(K)} w_v \leq \frac{W(G)}{2t}.$$

From the assumption $t \geq W(G)/\alpha_w(G)$,

$$\sum_{v\in V(G)\setminus V(K)} w_v \leq \frac{\alpha_w(G)}{2}.$$

Thus, we have the inequality

$$\alpha_w(K) \geq \alpha_w(G) - \sum_{v\in V(G)\setminus V(K)} w_v \geq \frac{\alpha_w(G)}{2}. \quad (5.1)$$

Using inequalities $t \geq W(G)/\alpha_w(G)$ and $W(G) \geq W(K)$ along with (5.1), we can prove that

$$\vartheta_w(K) \geq \alpha_w(K) \geq \frac{\alpha_w(G)}{2} \geq \frac{W(G)}{2t} \geq \frac{W(K)}{2t}.$$

We can obtain the independent set I whose weight is $\Omega(W(K)/(t\delta(K)^{1-1/(8t)}))$ by applying Theorem 5.1 with $k = 4t$. With the inequalities $\delta(K) \leq 2t\overline{d}_w$ and $W(K) \geq \alpha_w(K) \geq \alpha_w(G)/2$, the lower bound of the weight of I is $\Omega(W(K)/(t\delta(K)^{1-1/(8t)})) = \Omega(\alpha_w(G)/(t^2\overline{d}_w^{1-1/(8t)}))$. Thus when $t \geq W(G)/\alpha_w(G)$, approximation ratio becomes $O(t^2\overline{d}_w^{1-1/(8t)})$. □

Theorem 5.3 *For any fixed real number t, if $t \geq W(G)/\alpha_w(G)$, we can approximate the weighted independent set problem within $O(t^2\delta_w^{1-1/(8t)})$.*

Proof: Let π be an ordering of vertices in G with which the value of $\max_v d_w^\pi(v)$ is equal to δ_w. Let π' be the reverse ordering of π. Assume that $t \geq W(G)/\alpha_w(G)$ is fixed. Let K be the subgraph induced by the vertices whose right degrees $d^{\pi'}(v, G)$ are less than $2t\delta_w$. Thus K is a $2t\delta_w$-inductive graph. Then the following inequalities hold:

$$\begin{aligned}
W\delta_w &\geq \sum_{v\in V(G)} w_v d_w^\pi(v) \\
&= \sum_{v\in V(G)} w_v d^{\pi'}(v) \\
&\geq \sum_{v\in V(G)\setminus V(K)} w_v d^{\pi'}(v) \\
&\geq 2t\delta_w \sum_{v\in V(G)\setminus V(K)} w_v.
\end{aligned}$$

Thus we can prove this theorem in the same manner as Theorem 5.2. □

5.3 Algorithm

In this section we propose two algorithms: WGSA, whose approximation ratio is a function of the weighted average degree, and WGSI, whose approximation ratio is a function of the weighted inductiveness.

WGSA is the following algorithm. We get an independent set by applying WG. Independently, we apply the algorithm given by Theorem 5.1 to the induced subgraph by the vertices whose degrees are smaller than $2t\overline{d}_w$ to obtain another independent set. We output the one with larger weight.

Theorem 5.4 WGSA *can achieve approximation ratio* $O(\overline{d}_w \log\log \overline{d}_w / \log \overline{d}_w)$ *for the weighted independent set problem.*

Proof: When $t \geq W(G)/\alpha_w(G)$, we can approximate within $O(t^2 \overline{d}_w^{1-1/(8t)})$ from Theorem 5.2. On the other hand, when $t \leq W(G)/\alpha_w(G)$, the approximation ratio of WG is $O(\overline{d}_w/t)$ from Theorem 3.2. These two functions cross when $t = \frac{1}{24}\log \overline{d}_w / \log \log \overline{d}_w$. Thus the approximation ratio is $O(\overline{d}_w \log\log \overline{d}_w / \log \overline{d}_w)$. □

WGSI is the following algorithm. We get an independent set by applying WG. Independently, we apply the algorithm given by Theorem 5.1 to the induced subgraph by the vertices whose right degrees in a smallest-first ordering are smaller than $2t\delta_w$ to obtain another independent set. We output the one with larger weight.

Theorem 5.5 WGSI *can achieve approximation ratio* $O(\delta_w \log\log \delta_w / \log \delta_w)$ *for the weighted independent set problem.*

Proof: From Theorems 3.3 and 5.3, we can prove this theorem in the same way as Theorem 5.4. □

6 Conclusion

In this paper, we defined the weighted average degree $\overline{d}_w$ and the weighted inductiveness δ_w, and proved the lower bound of the weight of the independent set obtained by the weighted greedy algorithm. Combining with LP, we obtained the approximation ratio $\min((\overline{d}_w+1)/2, (\delta_w+1)/2)$. Also combining with SDP, we proved that approximation ratio can attain $O(\overline{d}_w \log\log \overline{d}_w / \log \overline{d}_w)$ and $O(\delta_w \log\log \delta_w / \log \delta_w)$.

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