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正方行列上に一様に整数を配置する方法の提案とディジタルハーフトーニングへの応用

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要旨 本論文では，$n \times n$ の正方行列上に 0 から n^2-1 までの整数をなるべく一様に配置する問題を考える．一様性を評価するためにディスクレパンシに基づいた基準を導入する．すなわち，$k \times k$ の領域上での要素の和に注目し，その最大値と最小値の差をディスクレパンシとして定義する．n と k が共に偶数ならばディスクレパンシを 0 にする方法が知られている．本文では，n が奇数，$k=2$ という基本的な場合について，ディスクレパンシを $2n$ にまで下げられることを示す．これは，従来の限界 $4n$ を改良したものになっている．数学のラテン方陣や魔方陣の理論で展開されている基本的な考え方をヒントとしている．この問題はディジタル・ハーフトーニングと密接な関係がある．低ディスクレパンシの行列を構成することができれば，オーダードディザ法の性能を向上できることが期待できる．

Distributing Distinct Integers Uniformly over a Square Matrix with Application to Digital Halftoning

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Abstract This paper considers how to distribute n^2 integers between 0 and n^2-1 as uniformly as possible over an $n \times n$ square matrix. We introduce a discrepancy-based measure to evaluate the uniformity. More precisely, we take a sum of matrix elements over every $k \times k$ contiguous submatrix and define the discrepancy of the matrix as the largest difference among those sums. It is known that if n and k are both even integers then we can construct zero-discrepancy matrices. In this paper we present a scheme for achieving a new discrepancy bound $2n$ when n is odd and k is 2. This is an improvement from the previous bound $4n$. We borrow basic ideas behind orthogonal Latin squares and semi-magic squares. An n-ary number system also plays an important part.

This problem is closely related to digital halftoning. Low discrepancy matrices would improve the quality of commonly used Ordered Dither Algorithm.

1 Introduction

Digital Halftoning is an important technique for the rendition of continuous-tone pictures on displays that can only produce two levels. There have been a great number of methods for digital halftoning. One of the most popular methods is *Ordered Dithering* which determines an output level at each pixel by comparison with a threshold in a predetermined table called *Dither Matrix*. The performance of the algorithm heavily depends on the Dither matrix.

A Dither matrix is an $n \times n$ square matrix containing integers $0, \ldots, n^2-1$. It is good when those integers are *uniformly* distributed. To evaluate the uniformity we introduce a discrepancy-based measure. More precisely, we take a sum of matrix elements over every $k \times k$ contiguous submatrix (region) and define the discrepancy of the matrix as the largest difference among those sums. This measure reflects human eye perception usually modeled using weighted sum of intensity levels with Gaussian coefficients over square regions around each pixel [2]. It is known by experience that a matrix with low discrepancy frequently produces good-looking pictures. This is the reason why we are interested in finding a good matrix with low discrepancy.

The analogous geometric problem of distributing n points uniformly in a unit square has been studied extensively in the literature [6, 9]. Usually, a family of regions is introduced to evaluate the uniformity of a point distribution. If the points of an n-point set P are uniformly distributed, for any region R in the family the number of points in R should be close to $\frac{1}{n}$ area(R), where $\frac{1}{n}$ is the point density of P in the entire square. Thus, the *discrepancy of P in a region R* is defined as the difference between this value and the actual number of points of P in R. The *discrepancy of the point distribution P* with respect to the family of regions is defined by the maximum such difference, over all regions. The problem of establishing discrepancy bounds for various classes of regions has been studied extensively [7]. One of the simplest families is that of axis-parallel rectangles for which $\Theta(\log n)$ bound is known [6, 9].

For the problem of establishing discrepancy bounds for families of regions (contiguous submatrices), some preliminary observations are obtained in [1]. One basic observation is that we can construct an $n \times n$ matrix of zero-discrepancy for a family of 2×2 regions if n is even. A space-efficient algorithm is also presented in [1] for constructing a $k^m \times k^m$ matrix of zero-discrepancy for a family of $k \times k$ regions. More precisely, given any matrix index (i, j) we can compute the corresponding matrix element of the index in constant time using only $O(k^2)$ working space instead of $O(k^{2m})$ required to store an output matrix. It is also shown in [1] that zero-discrepancy cannot be achieved if n is odd and k is even, and only trivial bound has been obtained for the discrepancy in that case. In this paper we present a new scheme for achieving a new discrepancy bound $2n$ when $n \geq 5$ is odd and k is 2. This is an improvement from the previous bound $4n$ [1]. Basic tools and theories are orthogonal Latin squares, semi-magic squares, and the n-ary number system.

2 Preliminary Definitions

For integers $n > 1$, let $\mathbb{Z}_n(n)$ be the class of all $n \times n$ integer matrices such that all the integers ranging from 0 to $n-1$ are included exactly n times and let $\mathbb{Z}_1(n)$ be that of all $n \times n$ matrices which contain every value $0, \ldots, n^2 - 1$ exactly once. In this paper we only deal with square matrices consisting of an odd number of rows (and columns) unless otherwise specified.

A contiguous $k \times k$ submatrix (or *region*, hereafter) $R_{i,j} = R_{i,j}^{(k)}$ with its upper left corner at (i, j) is defined by $R_{i,j}^{(k)} = \{(i', j') \mid i' = i, \ldots, i + k - 1 \text{ and } j' = j, \ldots, j + k - 1\}$, where indices are calculated modulo n. [1] Given a matrix P and a $k \times k$ region $R_{i,j}$, $P(R_{i,j})$ denotes the sum of the elements of P in locations given by $R_{i,j}$. The $k \times k$*-discrepancy* $\mathcal{D}_{k,n}(P)$ of an $n \times n$ matrix P for the family $\mathcal{F}_{k,n}$ of all $k \times k$ regions is defined as

$$\mathcal{D}_{k,n}(P) = \max_{R \in \mathcal{F}_{k,n}} P(R) - \min_{R' \in \mathcal{F}_{k,n}} P(R').$$

Let $\mathbb{N}(k, n)$ be the set of all such zero-$k \times k$-discrepancy matrices of order (k, n).

Theorem 1 *[1] The set $\mathbb{N}(k, n)$ of zero-$k \times k$-discrepancy matrices of order (k, n) has the following properties:*

(a) $\mathbb{N}(k, n)$ is non-empty if k and n are both even.

[1]Throughout this paper, index arithmetic is performed modulo matrix size n unless otherwise noted.

(b) $\mathbb{N}(k,n)$ *is empty if* k *and* n *are relatively prime.*

(c) $\mathbb{N}(k,n)$ *is empty if* k *is odd and* n *is even.*

(d) $\mathbb{N}(k,k^m)$ *is non-empty for any integers* k *and* m, $k \geq 2, m \geq 2$.

It follows from the theorem that zero-discrepancy cannot be achieved in a basic case of n odd and k even. In this paper we consider how much we can reduce the discrepancy of such a matrix in this basic case. One simple question is whether we can achieve a $\Theta(\log n)$ bound as in geometric discrepancy problems.

3 Basic Construction Schemes

A goal here is to design a low discrepancy matrix. We begin with some basic schemes for constructing matrices having some nice properties.

3.1 n-Ary Number System

¿From now on, we assume that n is an odd number not less than 3, and our target $n \times n$ matrix $C \in \mathbb{Z}_1(n)$. Since all such integers ranging from 0 and n^2-1 can be represented by two digits in the n-ary number system, we can associate to each $C \in \mathbb{Z}_1(n)$ two square matrices A and B representing upper and lower digits. That is, each element $c_{i,j}$ of the matrix C is given by

$$c_{i,j} = n \times a_{i,j} + b_{i,j},\ \ 0 \leq a_{i,j}, b_{i,j} < n,\ \ i,j = 0,1,\ldots,n-1.$$

Observe that $C \in \mathbb{Z}_1(n)$ if and only if the two matrices A and B are in the class $\mathbb{Z}_n(n)$ and are mutually orthogonal, that is, no ordered pair $(a_{i,j}, b_{i,j})$ occurs more than once. So, we need schemes for generating two mutually orthogonal matrices in the class $\mathbb{Z}_n(n)$. One is called *Alternating Diagonal Sequencing* and the other *Diagonal Repeating.*

3.2 Alternating Diagonal Sequencing

Alternating Diagonal Sequencing is a scheme for generating two mutually orthogonal matrices $A_n^{(+)} = (a_{i,j}^{(+)})$ and $A_n^{(-)} = (a_{i,j}^{(-)})$ in the class $\mathbb{Z}_n(n)$ as follows.

$$a_{i,j}^{(+)} = \begin{cases} i, & i+j \text{ is odd,} \\ n-1-i, & \text{otherwise,} \end{cases} \qquad a_{i,j}^{(-)} = \begin{cases} j, & i+j \text{ is odd,} \\ n-1-j, & \text{otherwise.} \end{cases}$$

Lemma 2 *The two matrices generated by Alternating Diagonal Sequencing are mutually orthogonal.*

Proof: We omit the proof for space. □

The lemma guarantees that combining these two matrices $A_n^{(+)}$ and $A_n^{(-)}$ by $a_{i,j} = n \times a_{i,j}^{(+)} + a_{i,j}^{(-)}$ or $a_{i,j} = n \times a_{i,j}^{(-)} + a_{i,j}^{(+)}$ results in an $n \times n$ matrix A_n containing all the numbers between 0 and n^2-1. Figure 1 shows how a matrix is obtained by combining two matrices produced by this scheme. Here note that $a_{i,j}^{(+)} = a_{j,i}^{(-)}$ holds due to the symmetry of their definitions.

Let $R_{i,j}^{(2)}$ be a 2×2 contiguous region on an $n \times n$ matrix, i.e., $R_{i,j}^{(2)} = \{(i,j), (i+1,j), (i,j+1), (i+1,j+1)\}$ where index additions are done modulo n. By $A_n^{(+)}(R_{i,j}^{(2)})$ we denote the sum of elements of $A_n^{(+)}$ in the region $R_{i,j}^{(2)}$, that is,

$$A_n^{(+)}(R_{i,j}^{(2)}) = a_{i,j}^{(+)} + a_{i+1,j}^{(+)} + a_{i,j+1}^{(+)} + a_{i+1,j+1}^{(+)}.$$

$$A_5^{(+)} \times 5 + A_5^{(-)} = \begin{bmatrix} 4&0&4&0&4 \\ 1&3&1&3&1 \\ 2&2&2&2&2 \\ 3&1&3&1&3 \\ 0&4&0&4&0 \end{bmatrix} \times 5 + \begin{bmatrix} 4&1&2&3&0 \\ 0&3&2&1&4 \\ 4&1&2&3&0 \\ 0&3&2&1&4 \\ 4&1&2&3&0 \end{bmatrix} = \begin{bmatrix} 24&1&22&3&20 \\ 5&18&7&16&9 \\ 14&11&12&13&10 \\ 15&8&17&6&19 \\ 4&21&2&23&0 \end{bmatrix}.$$

Figure 1: A matrix given by combination of two 5×5 matrices $A_5^{(+)}$ and $A_5^{(-)}$ generated by Alternating Diagonal Sequencing.

Lemma 3 *The* 2×2 *discrepancy of* $A_n = nA_n^{(+)} + A_n^{(-)}$ *and* $A'_n = nA_n^{(-)} + A_n^{(+)}$ *are* $4n$.

Proof: Recall that n is an odd number not less than 3. Then, we have

$$A_n^{(+)}(R_{i,j}^{(2)}) = \begin{cases} 2n & \text{if } i < n-1 \text{ is even and } j = n-1, \\ 2n-4 & \text{if } i \text{ is odd and } j = n-1, \\ 2n-2 & \text{otherwise,} \end{cases}$$

$$A_n^{(-)}(R_{i,j}^{(2)}) = \begin{cases} 2n & \text{if } i = n-1 \text{ and } j < n-1 \text{ is even,} \\ 2n-4 & \text{if } i = n-1 \text{ and } j \text{ is odd,} \\ 2n-2 & \text{otherwise.} \end{cases}$$

Observe that there is no pair (i, j) such that $A_n^{(+)}(R_{i,j}^{(2)})$ and $A_n^{(-)}(R_{i,j}^{(2)})$ are both $2n$ or both $2n-4$. Thus, $A_n(R_{i,j}^{(2)})$s and $A'_n(R_{i,j}^{(2)})$s are at most $2n \times n + 2n - 2 = 2n^2 + 2n - 2$ and are at least $(2n-4) \times n + 2n - 2 = 2n^2 - 2n - 2$. Indeed, $A_n(R_{0,n-1}^{(2)}) = A'_n(R_{n-1,0}^{(2)}) = 2n^2 + 2n - 2$ and $A_n(R_{1,n-1}^{(2)}) = A'_n(R_{n-1,1}^{(2)}) = 2n^2 - 2n - 2$. Therefore, the 2×2 discrepancy of A_n and A'_n are $4n$. □

Observe that, when n is even, the 2×2 discrepancy of A_n and A'_n become 0, because we have $A_n^{(+)}(R_{i,j}^{(2)}) = A_n^{(-)}(R_{i,j}^{(2)}) = 2n - 2$ for every (i, j) when n is even. Notice that the non-emptyness of $\mathbb{N}(2, n)$ for even n is already shown in (a) of Theorem 1.

3.3 Diagonal Repeating

We define another scheme called *Diagonal Repeating* for generating low discrepancy matrices. We partition n^2 elements of an $n \times n$ matrix into n disjoint sets $L_0^{(+)}, \ldots, L_{n-1}^{(+)}$ along 45 degree lines as follows:

$$L_s^{(+)} = \{(i, j) \mid (i + j) \bmod n = s\}, \; s = 0, \ldots, n-1.$$

In a similar way we also define sets $L_0^{(-)}, \ldots, L_{n-1}^{(-)}$ along -45 degree lines:

$$L_s^{(-)} = \{(i, j) \mid (i - j) \bmod n = k\}, \; s = 0, \ldots, n-1.$$

Now, we define two $n \times n$ matrices $D_n^{(+)} = (d_{i,j}^{(+)})$ and $D_n^{(-)} = (d_{i,j}^{(-)})$ as follows:

$$d_{i,j}^{(+)} = \begin{cases} k & \text{if } (i, j) \in L_s^{(+)} \text{ with } s \text{ is even,} \\ n-1-k & \text{otherwise,} \end{cases}$$

$$d_{i,j}^{(-)} = \begin{cases} k & \text{if } (i, j) \in L_s^{(-)} \text{ with } s \text{ is even,} \\ n-1-k & \text{otherwise.} \end{cases}$$

Figure 2 shows the two 5×5 matrices $D_5^{(+)}$ and $D_5^{(-)}$ defined above. As is seen in the figure, the same number is repeated along 45 degree lines in the upper-digit matrix A and along -45 degree lines in the lower-digit matrix B. So, it is called *Diagonal Repeating*.

$$D_5^{(+)} = \begin{bmatrix} 0 & 3 & 2 & 1 & 4 \\ 3 & 2 & 1 & 4 & 0 \\ 2 & 1 & 4 & 0 & 3 \\ 1 & 4 & 0 & 3 & 2 \\ 4 & 0 & 3 & 2 & 1 \end{bmatrix}, \quad D_5^{(-)} = \begin{bmatrix} 0 & 4 & 1 & 2 & 3 \\ 3 & 0 & 4 & 1 & 2 \\ 2 & 3 & 0 & 4 & 1 \\ 1 & 2 & 3 & 0 & 4 \\ 4 & 1 & 2 & 3 & 0 \end{bmatrix}.$$

Figure 2: Two mutually orthogonal Latin squares $D_5^{(+)}$ (left) and $D_5^{(-)}$ (right) generated by Diagonal Repeating.

Lemma 4 *Diagonal Repeating produces two mutually orthogonal Latin squares.*

Proof: Recall that n is odd. Then, for every two pairs (i,j) and (i',j'), $d_{i,j}^{(+)} = d_{i',j'}^{(+)}$ if and only if $i+j \equiv i'+j' \mod n$, and $d_{i,j}^{(-)} = d_{i',j'}^{(-)}$ if and only if $i-j \equiv i'-j' \mod n$. Hence, $D_n^{(+)}$ and $D_n^{(-)}$ are Latin squares. Moreover, the two sets $L_s^{(+)}$ and $L_t^{(-)}$ intersect at a single place for every $s,t = 0,\ldots,n-1$, and thus, $D_n^{(+)}$ and $D_n^{(-)}$ are mutually orthogonal. □

The following lemma shows that Diagonal Repeating can produce lower discrepancy matrices than Alternating Diagonal Sequencing.

Lemma 5 *The 2×2 discrepancy of $D_n = nD_n^{(+)} + D_n^{(-)}$ and $D'_n = nD_n^{(-)} + D_n^{(+)}$ are $2n+2$.*

Proof: By definition, we have

$$D_n^{(+)}(R_{i,j}^{(2)}) = \begin{cases} 2n-1 & \text{if } (i,j) \in L_{n-2}^{(+)}, \\ 2n-3 & \text{if } (i,j) \in L_{n-1}^{(+)}, \\ 2n-2 & \text{otherwise,} \end{cases} \qquad D_n^{(-)}(R_{i,j}^{(2)}) = \begin{cases} 2n-1 & \text{if } (i,j) \in L_{n-1}^{(-)}, \\ 2n-3 & \text{if } (i,j) \in L_0^{(-)}, \\ 2n-2 & \text{otherwise.} \end{cases} \quad (1)$$

Thus, the maximum and minimum values of $D_n(R_{i,j}^{(2)})$s are respectively $(2n-1) \times n + 2n - 1 = 2n^2 + n - 1$ and $(2n-3) \times n + 2n - 3 = 2n^2 - n - 3$, and the same holds for $D'_n(R_{i,j}^{(2)})$s. □

The discrepancy bound $2n+2$ achieved by Diagonal Repeating is much better than the bound $4n$ done by Alternating Diagonal Sequencing. Unfortunately, this bound is not optimal. In fact, we reduce the discrepancy bound further to $2n$ in the next section. Nevertheless, this bound looks near optimal because of the following lemma establishing a lower bound 2 on the 2×2 discrepancy for matrices in the class $\mathbb{Z}_n(n)$. In fact, the lemma shown a stronger result.

Lemma 6 *Let k be any integer such that $2 \leq k < n$. If n and k are relatively prime, then the $k \times k$ discrepancy of each $A \in \mathbb{Z}_n(n)$ is at least 2.*

Proof: Let $A \in \mathbb{Z}_n(n)$. Observe that each matrix element is included in exactly k^2 different $k \times k$ regions, and each integer between 0 and $n-1$ appears exactly n times in A. Thus,

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} A(R_{i,j}^{(2)}) = k^2(0+1+\cdots+n-1) \times n = k^2 n^2 \frac{(n-1)}{2}.$$

The lemma is proven by contradiction.

Suppose the $k \times k$ discrepancy of A is 0 for some $A \in \mathbb{Z}_n(n)$. It follows that that every $k \times k$ region has the same sum $k^2(n-1)/2$. Define a row sum $r_{i,j}$ by $r_{i,j} = a_{i,j} + \cdots + a_{i,j+k-1}$. In terms of $r_{i,j}$s, we have $r_{i,j} + r_{i+1,j} + \cdots + r_{i+k-1,j} = k^2(n-1)/2$ for $i,j = 0,1,\ldots,n-1$, which implies $r_{i,j} = r_{i+k,j}$. Since n and k are relatively prime, we have $r_{i,0} = r_{i,1} = \cdots = r_{i,n-1}$ for each $i = 0,1,\ldots,n$, and thus, $r_{i,j} = k(n-1)/2$ for each $i,j = 0,1,\ldots,n-1$. By applying the same arguments to elements of A, it follows that all the elements of the matrix A are $(n-1)/2$, which contradicts to the assumption $A \in \mathbb{Z}_n(n)$.

$$9D_9^{(+)} + M_9^{(+)} = \begin{bmatrix} 0&7&2&5&4&3&6&1&8\\ 7&2&5&4&3&6&1&8&0\\ 2&5&4&3&6&1&8&0&7\\ 5&4&3&6&1&8&0&7&2\\ 4&3&6&1&8&0&7&2&5\\ 3&6&1&8&0&7&2&5&4\\ 6&1&8&0&7&2&5&4&3\\ 1&8&0&7&2&5&4&3&6\\ 8&0&7&2&5&4&3&6&1 \end{bmatrix} \times 9 + \begin{bmatrix} 0&8&0&8&8&0&8&0&8\\ 7&1&7&7&1&7&1&7&1\\ 2&6&6&2&6&2&6&2&6\\ 5&5&3&5&3&5&3&5&3\\ 4&4&4&4&4&4&4&4&4\\ 5&3&5&3&5&3&5&3&3\\ 2&6&2&6&2&6&2&2&6\\ 7&1&7&1&7&1&1&7&1\\ 0&8&0&8&0&0&8&0&8 \end{bmatrix} = \begin{bmatrix} 0&71&18&53&44&27&62&9&80\\ 70&19&52&43&28&61&10&79&1\\ 20&51&42&29&60&11&78&2&69\\ 50&41&30&59&12&77&3&68&21\\ 40&31&58&13&76&4&67&22&49\\ 32&57&14&75&5&66&23&48&39\\ 56&15&74&6&65&24&47&38&33\\ 16&73&7&64&25&46&37&34&55\\ 72&8&63&26&45&36&35&54&17 \end{bmatrix}$$

Figure 3: Low discrepancy 9×9 matrix (discrepancy $= 18$).

Next, suppose the $k \times k$ discrepancy of A is 1 for some $A \in \mathbb{Z}_n(n)$. Then, there are only two different $k \times k$ sums S_0 and S_1 with $S_1 = S_0 + 1$. For $\ell = 0, 1$, we denote by R_ℓ the number of $k \times k$ regions whose sums are S_ℓ. Then, we have $R_0 + R_1 = n^2$ and $0 < R_0, R_1 < n^2$. Thus, we have $R_0 S_0 + R_1 S_1 = k^2 n^2 (n-1)/2$. From $R_0 + R_1 = n^2$ and $S_1 = S_0 + 1$, we have

$$R_1 = k^2 n^2 \frac{(n-1)}{2} - n^2 S_0 = n^2 \left(k^2 \frac{(n-1)}{2} - S_0 \right).$$

Since n is odd and S_0 is an integer, R_1 is a multiple of n^2, which contradicts $0 < R_1 < n^2$. □

Observe from (1) that $D_n^{(+)}$ and $D_n^{(-)}$ have 2×2 discrepancy 2, and according the theorem, $D_n^{(+)}$ and $D_n^{(-)}$ has the lowest 2×2 discrepancy among all matrices in $\mathbb{Z}_n(n)$.

4 Combined Strategy for Improving Discrepancy

We have presented two schema for generating low discrepancy matrices, i.e., Alternating Diagonal Sequencing and Diagonal Repeating. The discrepancy bound achieved is $2n + 2$ for those matrices in the class $\mathbb{Z}_1(n)$ with n odd. In this section we propose yet another scheme, called *Modified Alternating Diagonal Sequencing*, based on these two strategies to achieve a better bound, $2n$.

Here we take $D_n^{(+)}$ or $D_n^{(-)}$ as matrices for upper digits, each of which has the lowest 2×2 discrepancy among all matrices in $\mathbb{Z}_n(n)$. Recall that all the elements on of $D_n^{(+)}$ ($D_n^{(-)}$, resp.) in the location given by $L_s^{(+)}$ ($L_s^{(-)}$, resp.) have the same value, s.

For lower digits, we take two matrices $M_n^{(+)} = (m_{i,j}^{(+)})$ and $M_n^{(-)} = (m_{i,j}^{(-)})$ defined as follows:

$$m_{i,j}^{(+)} = \begin{cases} i & \text{if } (i,j) \in L_s^{(+)} \text{ such that } s \text{ is odd and } s < \lfloor \frac{n}{2} \rfloor, \\ i & \text{if } (i,j) \in L_s^{(+)} \text{ such that } s \text{ is even and } s > \lfloor \frac{n}{2} \rfloor, \\ n-1-i & \text{otherwise,} \end{cases}$$

$$m_{i,j}^{(-)} = \begin{cases} i & \text{if } (i,j) \in L_s^{(-)} \text{ such that } s \text{ is odd and } s < \lfloor \frac{n}{2} \rfloor, \\ i & \text{if } (i,j) \in L_s^{(-)} \text{ such that } s \text{ is even and } s > \lfloor \frac{n}{2} \rfloor, \\ n-1-i & \text{otherwise.} \end{cases}$$

By the definition of the matrix, all the numbers $0, \ldots, n-1$ appear on the set $L_s^{(+)}$ on the matrix $M_n^{(+)}$ for each $s = 0, \ldots, n-1$. In the matrix $M_n^{(-)}$, each set $L_s^{(-)}$ contains all the integers $0, \ldots, n-1$. This proves the following lemma.

Lemma 7 *The matrix $D_n^{(+)}$ ($D_n^{(-)}$, resp.) is orthogonal to $M_n^{(+)}$ ($M_n^{(-)}$, resp.).*

The lemma implies that the two $n \times n$ matrices $nD_n^{(+)} + M_n^{(+)}$ and $nD_n^{(-)} + M_n^{(-)}$ belong to $\mathbb{Z}_1(n)$. Figure 3 shows a 9×9 matrix constructed in this way. In the following we show that the 2×2 discrepancy of these two matrices are $2n$.

Recall from (1) that the largest and the smallest 2×2 sums are $2n-1$ and $2n-3$. For $M_n^{(+)}$ and $M_n^{(-)}$, we have the following lemma.

Lemma 8 *The largest and smallest 2×2 sums are $3n-3$ and $n-1$, respectively, in the matrices $M_n^{(+)}$ and $M_n^{(-)}$.*

Proof: Let $s = \lfloor n/2 \rfloor$. By the definition, the set $L_s^{(+)}$ corresponds to the decreasing sequence $(n-1, \ldots, 0)$. The sequence for $L_{s-1}^{(+)}$ is decreasing if s is odd and increasing otherwise. It follows from the figure that the 2×2 sum is $3n-3$ at $(n-1, s-1)$ if s is even and at $(n-1, s)$ otherwise, and that it is $n-1$ at $(n-1, s)$ if s is even and at $(n-1, s+1)$ otherwise. Careful case study leads to the observation that they are in fact the largest and smallest 2×2 sums.

The proof for $M_n^{(-)}$ proceeds similarly. □

In order to show the 2×2 discrepancy of $nD_n^{(+)} + M_n^{(+)}$ and $nD_n^{(-)} + M_n^{(-)}$, let us introduce the concept of bad elements. Let T be any matrix in the class $\mathbb{Z}_n(n)$. An (i,j) element of T is called *good* if the 2×2 sum $T(R_{i,j}^{(2)})$ is equal to $2n-2$. All other elements are called *bad*. If all the elements are *good* then the 2×2 discrepancy of the matrix is 0. We denote a set of all bad elements of T by $Bad(T)$.

Observation 9 *All the bad elements of the matrices $D_n^{(+)}$ and $D_n^{(-)}$ have been located before, that is, $Bad(D_n^{(+)}) \subseteq L_{n-2}^{(+)} \cup L_{n-1}^{(+)}$ and $Bad(D_n^{(-)}) \subseteq L_{n-1}^{(-)} \cup L_0^{(-)}$.*

Lemma 10 *The distributions of bad elements are disjoint between $D_n^{(+)}$ and $M_n^{(+)}$ and also between $D_n^{(-)}$ and $M_n^{(-)}$. That is, $Bad(D_n^{(+)}) \cap Bad(M_n^{(+)}) = \emptyset$ and $Bad(D_n^{(-)}) \cap Bad(M_n^{(-)}) = \emptyset$.*

Proof: Recall the definition of the matrix $M_n^{(+)}$. Matrix elements are arranged increasingly or decreasingly in each line $L_s^{(+)}$ and increasing sequence and decreasing one appear alternately. Only illegularity happens between the two lines $L_{\lfloor n/2 \rfloor -1}^{(+)}$ and $L_{\lfloor n/2 \rfloor -1}^{(+)}$. More precisely, for any (i,j) such that $R_{i,j}^{(2)} \cap (L_{\lfloor n/2 \rfloor}^{(+)} \cup L_{\lfloor n/2 \rfloor -1}^{(+)}) = \emptyset$, the four elements in $R_{i,j}^{(2)}$ consist of $i, n-1-i, n-1-(i+1)$, and $i+1$, which implies that the element (i,j) is *good* since $M_n^{(+)}(R_{i,j}^{(2)}) = 2n-2$. This also implies that all the bad elements of $M_n^{(+)}$ are included in the union $L_{\lfloor n/2 \rfloor}^{(+)} \cup L_{\lfloor n/2 \rfloor -1}^{(+)} \cup L_{\lfloor n/2 \rfloor -2}^{(+)}$. Thus, from Observation 9, we have $Bad(D_n^{(+)}) \cap Bad(M_n^{(+)}) = \emptyset$ if $n \geq 5$. The remaining proof is symmetric. □

Theorem 11 *The 2×2 discrepancy of the two matrices $nD_n^{(+)} + M_n^{(+)}$ and $nD_n^{(-)} + M_n^{(-)}$ are $2n$ for any odd integer $n \geq 5$.*

Proof: Let us show the 2×2 discrepancy of $nD_n^{(+)} + M_n^{(+)}$, and that of $nD_n^{(-)} + M_n^{(-)}$ can be shown in the same way. According to Lemma 10, for each (i,j), we have the following three cases.

- Suppose $(i,j) \in Bad(D_n^{(+)})$ and $(i,j) \notin Bad(M_n^{(+)})$. Then, from (1), the 2×2 sum in $nD_n^{(+)} + M_n^{(+)}$ at (i,j) is at least $(2n-3) \times n + (2n-2) = 2n^2 - n - 2$ and at most $(2n-1) \times n + (2n-2) = 2n^2 + n - 2$.

- Suppose $(i,j) \in Bad(M_n^{(+)})$ and $(i,j) \notin Bad(D_n^{(+)})$. Then, from Lemma 8, the 2×2 sum in $nD_n^{(+)} + M_n^{(+)}$ at (i,j) is at least $(2n-2) \times n + (n-1) = 2n - n - 1$ and at most $(2n-2) \times n + (3n-3) = 2n + n - 3$.

- Suppose $(i,j) \notin Bad(D_n^{(+)}) \cup Bad(M_n^{(+)})$. Then, the 2×2 sum in $nD_n^{(+)} + M_n^{(+)}$ at (i,j) is exactly $(2n-2) \times n + (2n-2) = 2n^2 - 2$.

Hence, the 2×2 discrepancy of $nD_n^{(+)} + M_n^{(+)}$ is at most $(2n^2 + n - 2) - (2n^2 - n - 2) = 2n$. Indeed, each $(i, j) \in L_{n-2}^{(+)}$ achieves a 2×2 sum in $nD_n^{(+)} + M_n^{(+)}$ with value $2n^2 + n - 2$, and each $(i, j) \in L_{n-1}^{(+)}$ achieves a 2×2 sum in $nD_n^{(+)} + M_n^{(+)}$ with value $2n^2 - n - 2$. Therefore, the 2×2 discrepancy of $nD_n^{(+)} + M_n^{(+)}$ is $2n$. □

5 Conclusions

In this paper we have devoted to minimizing the discrepancy against a family of 2×2 regions. The region size may be too small for practical application to digital halftoning. We could define a combined discrepancy measure, that is, sum of discrepancy bounds for several families of regions such as $\mathcal{D}_{2,n}(P) + \mathcal{D}_{3,n}(P) + \cdots + \mathcal{D}_{k,n}(P)$. What is know so far is that there is no matrix P for which the combined discrepancy bound $\mathcal{D}_{2,n}(P) + \mathcal{D}_{3,n}(P)$ is zero.

One technical open problem is to prove optimality of the discrepancy bound $2n$ we have established in this paper. It is not so easy even for a small value of n. For example, when n is five, we have 24! different matrices to check.

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