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中立地帯を持ったボロノイ図に関する考察

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概要　中立地帯のあるボロノイ図の概念の提案を与える。従来のボロノイ図が空間の分割だったのに対し、中立地帯のあるボロノイ図では、空間内の母点に対応する領域は互いに隣接せず、様々なパターンの生成を可能にする。数学的な定義を与え、その存在性と幾何学的性質、及び構築アルゴリズムに関して議論する。

On Voronoi Diagrams with Neutral Zones

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Abstract. We introduce a new notion of "Voronoi diagram with neutral zone". In contrast to the ordinary Voronoi diagram to give a spatial tessellation, our new Voronoi diagram creates regions around given points such that no pair of regions are adjacent to each other and this enables to simulate more flexible patterns. We give a mathematical definition, and discuss existence and its geometric/algorithmic properties.

1 Introduction

Voronoi diagram is one of the most popular structures in computational geometry. It is frequently used as a mathematical model to represent a natural/artificial pattern created by a competitive growth process where many bodies grow simultaneously to form a geometric structure together; e.g., cell structure of biological tissue, crystal-lattice structure, geographic/geological pattern, economic/political regional equilibrium, and gravity/electronic/magnetic field. There are several variations of Voronoi diagrams, and their geometric properties and computational complexities are widely studied [2, 3, 4]. A common feature of those variations of Voronoi diagram is that they give partitions of space into regions (called *Voronoi cells*) each of which is the dominating region of an input point/object.

However, we sometimes observe in the nature a geometric structure in which the union of cells has a nonempty complement region (called *neutral zone* in this paper) in the plane. We can regard such a structure as a result of growth process in which the growth terminates before the cell boundaries meet each other, and the termination is due to some non-contact action of other regions. In this paper, we propose *Voronoi diagram with neutral zone* (*N-Voronoi diagram* in short) to model such a structure.

Fig. 1 shows an N-Voronoi diagram of seven points, where six points among them form a regular hexagon and the remaining point is located at the center of the hexagon. Note that none of the points has infinite Voronoi region.

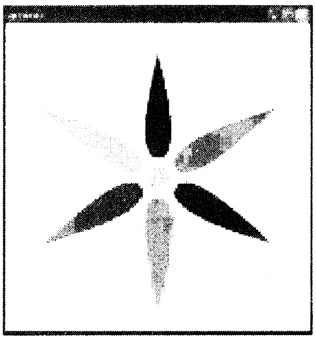

Figure 1: A "flower" obtained as an N-Voronoi Diagram.

We can explain the idea of defining an N-Voronoi diagram by using the following story on equilibrium in the "age of wars". Suppose that there are n kingdoms that are hostile to each other. Each kingdom has a territory around its castle (the position of the castle is given), and if its territory is attacked by another kingdom, a troop departs the castle to intercept. We assume that the interception succeeds if and only if the troop arrives the attacking point on the border of the territory earlier than the enemy. However, an attacker can confidentially move his troop inside his territory, and the defense is only able to start when the attacker departs from a point on the boundary of the attacker's territory.

Naturally, if there is no neutral zone, there always exist two kingdoms sharing border, and peace never comes. Our N-Voronoi diagram is a configuration to give an equilibrium so that every kingdom can guard the territory and no kingdom can grow without risk of invasion by other kingdoms. We prove existence and uniqueness of such an equilibrium for two points, and discuss the properties of curves defining the N-Voronoi diagram. We also discuss existence and give an algorithm for comptuting an N-Voronoi diagram of n points in a plane approximately.

2 Definition of Voronoi diagram with neutral zone

Given a set $X \subseteq \mathbb{R}^d$ and a point $p \in \mathbb{R}^d$, we define the *dominance region* $dom_\alpha(p, X)$ of p with respect to X as the set of all points that are (non-strictly) closer to p than X. That is,
$dom(p, X) = \{x \in \mathbb{R}^d : d(x, p) \leq d(x, X)\}$,
where $d(\cdot, \cdot)$ denotes the Euclidean distance.

We are given a set $S = \{p_1, p_2, \ldots, p_n\}$ of n points in the space. A set $\{R_1, R_2, \ldots, R_n\}$ of regions is called a *Voronoi diagram with neutral zone* (N-Voronoi diagram) of S if $R_i = dom(p_i, \cup_{j \neq i} R_j)$ for each $i = 1, 2, \ldots, n$. R_i is the *N-Voronoi region* of p_i, and denoted by $Vor(p_i)$.

Since an N-Voronoi region R_i is represented as an intersection of halfspaces $\cap_{q \in \cup_{j \neq i} R_j} dom(p_i, q)$, it is a convex set. They are clearly mutually disjoint. However, although we define an N-Voronoi diagram as above, neither its existence nor uniqueness is obvious.

From now on, we focus on the case where $d = 2$; that is, N-Voronoi diagram in the Euclidean plane.

3 N-Voronoi diagram of two points

Let us consider the case where S has only two points. Even this case is nontrivial, and has an interesting motivation [1]; namely, the problem to draw equally-spaced curves between two points.

Here, equally-spaced means that a curve is the bisector of its adjacent two curves (the input points are also considered as curves). This problem occurs in VLSI layout (personal communication with Dr. Hiroshi Murata, Kitakyusyu University), and if we draw only one curve, it is the perpendicular bisector line to give the ordinary Voronoi diagram. We can also draw three equally spaced curves, where the center curve is the perpendicular bisector and each of the other two curves is the bisector parabola of a point and the center curve. However, if we want to draw two equally spaced curves, we have difficulty. Indeed, the problem of drawing of two such curves is equivalent to the construction of the N-Voronoi diagram on two points.

We use capital letters for representing planar points in this section, since it is common in elementary Euclidean geometry. Suppose that two points are located at $P = (0, 1)$ and $Q = (0, -1)$. A curve defined by a continuous function $y = f(x)$ is denoted by $C(f)$. $C^+(f)$ and $C^-(f)$ are the regions defined by $y > f(x)$ and $y < f(x)$, respectively. For a pair of curves $y = f(x)$ and $y = g(x)$, we write $f \geq g$ if $f(x) \geq g(x)$ for any x.

Theorem 3.1 *There exists a curve $C(f) : y = f(x)$ satisfying that $C(f)$ is the bisector of P and $C(-f)$, and simultaneously $C(-f)$ is the bisector of Q and $C(f)$.*

We prove the existence of such an f by a constructive method. We define $y = f_1(x)$ to be the x-axis and $y = g_1(x)$ to be the bisector of x-axis and $(0, 1)$. We consider functions $y = f_i(x)$ and $y = g_i(x)$ for $i = 2, 3 \ldots$ such that $f_i(x)$ is defined as the bisector of $y = -g_{i-1}(x)$ and the point $P = (0, 1)$, while $g_i(x)$ is the bisector of $y = -f_{i-1}(x)$ and P. See Figure 2.

Lemma 3.2 (Monotonicity) $g_i \geq f_i$, $f_i \geq f_{i-1}$, *and* $g_i \leq g_{i-1}$.

Proof Routine. □

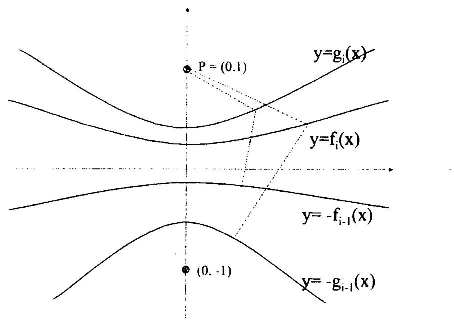

Figure 2: Curves in the iterative construction.

Lemma 3.3 (Tangent lines) *$y = f_i(x)$ and $y = g_i(x)$ are convex functions. Moreover, a tangent line to $C(f_i)$ is given as the perpendicular bisector of P and the nearest point A on $C(-g_{i-1})$ from the point to the tangent point T (see Figure 3). Analogous statement holds for a tangent line to $C(g_i)$.*

Proof Since $C(f_i)$ is the bisecting curve of $C(-g_{i-1})$ and P, $d(A, T) = d(T, P)$. Let us consider the perpendicular bisector ℓ of P and A. Consider any point B on ℓ. Then, $d(B, P) = d(B, A) \geq d(B, C(g_{i-1}))$. Thus, B is below or on the bisector curve $C(f_i)$ of $C(-g_{i-1})$ and P; that is, $B \in C^-(f_i) \cup C(f_i)$. This means that ℓ is the tangent line. This also implies convexity. □

From the convexity, it is clear that $f_i(x)$ and $g_i(x)$ are nondecreasing in the range $x > 0$.

Lemma 3.4 *The functions $f_i(x)$ and $g_i(x)$ are differentiable.*

Proof We prove by induction on i. The functions $f_1(x)$ and $g_1(x)$ are clearly differentiable. We assume that g_{i-1} is differentiable, and show that $f_i(x)$ is differentiable. It suffices to consider the curves in the range $x \geq 0$, since the curves are symmetric about y-axis.

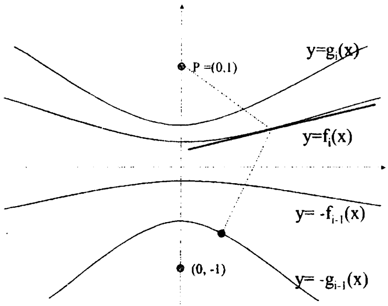

Figure 3: Tangent lines to the curves are bisectors.

Since $y = -g_{i-1}(x)$ is continuous, differentiable, and concave, the angle $\theta(x)$ of the normal vector of $C(-g_{i-1}(x))$ is continuous and decreasing. Let $\ell(x)$ be the corresponding halfline normal to the tangent line at $(x, g_{i-1}(x))$ emanated from the tangent point towards $C^+(-g_{i-1}(x))$. Then, no pair of halflines cross each other. Thus, if we define $B(x)$ to be the nearest point on $C(-g_{i-1}(x))$ from the point $(x, f_i(x))$, the x-value of $B(x)$ is increasing and continuous. Thus, because of differentiability of $g_{i-1}(x)$, the angle of the perpendicular bisector of $B(x)$ and P is continuous. This means that $y = f_i(x)$ is differentiable. □

We denote the differential of $f(x)$ by $f'(x)$.

Lemma 3.5 *If* $x > 0$, $g_i'(x) > f_i'(x)$.

Proof We prove by induction on i. See Figure 4 to get intuition. We can assume that $g_{i-1}'(x) > f_{i-1}'(x)$. Now, suppose on the contrary that there exists points $A = (x_0, f_i(x_0))$ and $B = (x_0, g_i(x_0))$ such that $f_i'(x_0) \geq g_i'(x_0)$. Let C be the nearest point on $C(-g_{i-1})$ from A and let D be the nearest point on $C(-f_{i-1})$ from B. We claim that the point D is located in the right halfplane of the line PC. This means that the line PD has a larger slope than PC, and hence perpendicular bisector of PD has a larger slope than that of PC. Thus, the tangent at B to $C(g_i)$ has a larger slope than that to $C(f_i)$ at A.

Suppose that the claim is false, and D is on the left of the line PC. The slope of PB is larger than that of PA, and because of convexity the slope of DB is larger than that of CA. This means that the perpendicular bisector of PD has a larger slope than that of PC. Thus, the tangent at B is steeper than that at A, and contradicts to our assumption that $f_i'(x_0) \geq g_i'(x_0)$. □

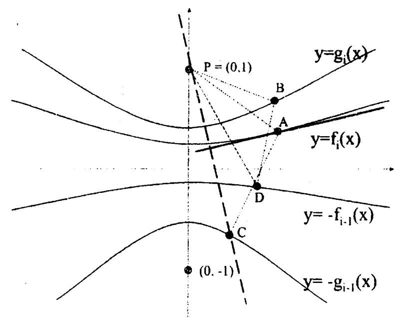

Figure 4: Comparison of tangent slopes.

Lemma 3.6 *If* $x > 0$, $g_i'(x) \leq g_{i-1}'(x)$ *and* $f_i'(x) \geq f_{i-1}'(x)$.

Proof The argument is same as the one in the previous lemma. □

Lemma 3.7 *Consider an interval* $[-c, c]$ *of* x *for any fixed constant* $c > 0$. *Then, the series of functions* $(g_i)_{i=1,2,\ldots}$ *and* $(f_i)_{i=1,2,\ldots}$ *uniformly converges to continuous functions* $g(x)$ *and* $f(x)$ *in this interval.*

Proof We show the uniform convergence of $(g_i)_{i=1,2,\ldots}$, since that for $(f_i)_{i=1,2,\ldots}$ is analogously proven. Since $g_1(x) = (x^2+1)/2$, it follows from Lemma 3.2 that $g_i(x) \leq g_1(c) = (c^2+1)/2$, thus it is uniformly bounded. Moreover, from Lemma 3.6 and the convexity, $|g_i'(x)| \leq g_1'(c) = c$. Thus, we have Lipschitz condition with the uniform constant c. Although the rest of the proof is routine (Ascoli-Arzelà's theorem [5] Th.9.8), we will give it for reader's convenience. For any given constant $\epsilon > 0$, we should show that there exists a natural number N such that $|g_i(x) - g_j(x)| < \epsilon$ for any $i \geq j \geq N$ for any $x \in [-c, c]$. We take $4c\epsilon^{-1}+1$ values $-c = p_1 < p_2 < \ldots < p_m = c$ in $[-c, c]$ such that they subdivide the interval evenly into subintervals of length $\epsilon c^{-1}/4$. In each point x in a subinterval $[p_k, p_{k+1}]$, $|g_i(x) - g_i(p_k)| < \epsilon/4$ because of the Liphshitz condition. For each point p_k, because of monotonicity and boundedness, there is $N_k > 0$ such that $|g_i(p_k) - g_j(p_k)| < \epsilon/2$ for any $i \geq j \geq N_k$. We set $N = \max_{1 \leq k \leq m} N_k$ to have our assertion. □

Now, in order to prove the theorem, it suffices to show that $g(x) = f(x)$. Clearly, the curve $C(g)$ is the bisector of P and $C(-f)$, whereas $C(f)$ is the bisector of Q and $C(-g)$.

We can assume that c is sufficiently large such that $g(c) > 1$. Let us consider a satisfying $g(a) = 1$, and consider the interval $[0, a]$. We first show that $f(x) = g(x)$ in $[0, a]$. Then, by using the bisection property, symmetry and convexity, we will show $g(x) = f(x)$ in the wider interval $[-c, c]$.

Lemma 3.8 $g(x) = f(x)$ *in the subinterval* $[0, a]$.

Proof

Figure 5 illustrates the idea of the proof. It is clear that $g(x) \geq f(x)$. Since $g_i'(x) \geq f_i'(x) > 0$ in $(0, a]$, $g_i(x) - f_i(x)$ is nondecreasing, and hence, $g(x) - f(x)$ is nondecreasing in the interval $[0, a]$.

Now, assume that $A = (a, g(a))$ and $A' = (a, f(a))$ are different. Let the nearest point from A on $y = -f(x)$ be $B = (b, -f(b))$. We can see that $b < a$, since the line AB must be normal to the decreasing curve $y = -f(x)$. Since $C(g)$ is the bisector of P and $C(-f)$, $PA = AB$. Since $f(a) < g(a) = 1$, $PA' > PA$. Now, draw a parallel (and same length) segment ℓ to AB from A'. Because of nondecreasing property of $g(x) - f(x)$, ℓ intersects the curve $y = -g(x)$. This means that the distance δ between A' and the curve $y = -g(x)$ is at most AB, while $\delta = PA' > PA = AB$. This is contradiction. □

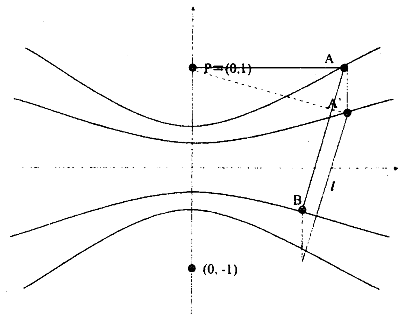

Figure 5: Distance from A' becomes shorter than the length of ℓ

Proposition 3.9 $f(x) = g(x)$.

Proof $f(x)$ (resp. $g(x)$) is the bisector of $-g(x)$ (resp. $-f(x)$) and P. We have already shown that $f(x) = g(x)$ on $[0, a]$. Let a_1 be the supremum of values b such that the x-coordinate of the point on $C(-g)$ nearest to $(b, f(b))$ lies in $[0, a]$. Then, clearly $f(x) = g(x)$ on $[0, a_1]$. The line through $(a, -f(a))$ and $(a_1, f(a_1))$ must be normal to $y = -f(x)$, and let $\rho > 0$ be the slope of the line.

Now, assume that $f(x) = g(x)$ on $[0, a_i]$ for an $a_i \geq a_1$. Then we claim that $f(x) = g(x)$ on $[0, a_i + 2\rho^{-1}]$. If we define $a_{i+1} = a_i + 2\rho^{-1}$, the sequence a_i $(i = 1, 2, \ldots,)$ is increasing and tends to infinity. Thus, we have $f(x) = g(x)$ if the above claim holds.

For proving the claim, it suffices that the x-value of the nearest point $T = (x_T, y_T)$ on $C(-g(x))$ from $S = (x_S, y_S) = (a_i + \rho, f(a_i + \rho))$ is in $[0, a_i]$. Clearly, $x_T > a$, thus the slope of the line ST is smaller than ρ, because of concavity of $y = -g(x)$. Now, $y_S - y_T > 1 - (-1) = 2$ since $a_i > a$. Thus, $x_S - x_T > 2\rho^{-1}$. □

Now we have proven existence of regions satisfying the condition of Voronoi diagram with neutral zone; indeed, they are $C^+(f)$ and $C^-(-f)$. Moreover, we have the uniqueness as follows:

Corollary 3.10 *For any given pair of points in the plane, the Voronoi diagram with neutral zone uniquely exists.*

Proof Since the statement is invariant under rotation, translation and scaling, we can assume that the points are $P = (0, 1)$ and $Q = (0, -1)$. Let $Vor(P)$ be the neutral-Voronoi region for P. We can prove by induction that $C^+(g_i) \subseteq Vor(P) \subseteq C^+(f_i)$ for each $i = 1, 2, \ldots$. Thus, $C^+(g) \subseteq Vor(P) \subseteq C^+(f)$, and hence $Vor(P) = C^+(f)$. □

3.1 Behavior at infinity

For every point $(x, f(x))$ on $C(f)$, let $(t(x), -f(t(x))$ be the unique nearest point on $C(-f)$. By definition, the distance between $(x, f(x))$ and $(t(x), -f(t(x))$ equals to the distance between $(x, f(x))$ and $P = (0, 1)$. Thus,

$$(t(x) - x)^2 + (f(t(x)) + f(x))^2 = x^2 + (f(x) - 1)^2. \qquad (1)$$

and since $(t(x), -f(t(x))$ is the nearest point,

$$t(x) - x + (f(x) + f(t(x)))f'(t(x)) = 0. \qquad (2)$$

where f' is derivative of f. It is easy to see that $f'(t(x))$ is nondecreasing. Furthermore, we have the following:

Lemma 3.11 $f'(t(x)) \leq 1$.

Proof Suppose on the contrary that $f'(t(a)) > 1$ for a value $a > 0$. Since $C(f)$ is convex, $f'(a) > 1$. On the other hand, the slope of the line segment s connecting $A = (a, f(a))$ and $T = (t(a), f(t(a))$ is less than 1, since it is normal of $C(-f)$ at T. Thus, s lies above $C(f)$ in the neighbor of A, and intersect with $C(f)$ at a point $B = (b, f(b))$ for an $b < a$. It is easy to observe that $PB > TB$ contradicting the definition of f. □

Since both of functions t and f' are nondecreasing, $f'(t(x))$ converges, and also we have $\lim_{x\to\infty} f'(x) \geq 1$. Thus, two possible cases can be considered for the behavior of f at $x \to \infty$. Case 1: $\lim_{x\to\infty} f'(x) = 1$ and $f(x) = x - o(x)$, or Case 2: $\lim_{x\to\infty} f'(x) > 1$, $t(x)$ converges to a constant c, and $C(f)$ approaches from above to the perpendicular bisector of P and $(c, -f(c))$. We do not know which is the correct one currently.

4 General case

We can characterize the N-Voronoi diagram of a given set S as a fixed point of an operator. Let $\mathcal{R}$ be the set of all n-tupples $\mathbf{R} = (R_1, R_2, \ldots, R_n)$ of sets with $p_i \in R_i$. We define a partial order $\preceq$ on $\mathcal{R}$ by component wise inclusion. We define an operator $\mathbf{D} : \mathcal{R} \to \mathcal{R}$; the i-th component of $\mathbf{D}(\mathbf{R})$ is the dominance region of p_i with respect to the set $\cup_{j\neq i} R_j$. We let $Fix(\mathbf{D}) = \{\mathbf{R} \in \mathcal{R}, \mathbf{D}(\mathbf{R}) = \mathbf{R}\}$. denote the set of all fixed points of $\mathbf{D}$. By definition, $\mathbf{R}$ is an N-Voronoi diagram of S if and only if it is an element in $Fix(\mathbf{D})$.

Theorem 4.1 *For any point set S, $Fix(\mathbf{D}) \neq \emptyset$.*

The proof of the above theorem uses the following well-known theorem in functional analysis (See [6] Th.3.3), which is an infinite-dimensional version of the famous Brauer's fixed point theorem.

Theorem 4.2 (Schauder's fixed point theorem) *Let Z be a Banach space and let $K \subset Z$ be a nonempty, compact, and convex set. Then, any* *continuous operator $F : K \to K$ has at least one fixed point.*

We represents $\mathcal{R}$ as a set of radial functions, introduce a measure in the set $\mathcal{R}$, and define a convex set K to apply the above theorem. We omit it in this version.

We note that the uniqueness of the fixed point is not obtained from Shauder's fixed point theorem, although we believe that N-Voronoi diagram for a given point set is unique.

5 Algorithms

Even for the two-point case, the bisector $y = f(x)$ seems to be a non-rational function. Thus, it seems to be hard to compute the N-Voronoi diagram precisely in an algebraic model. Thus, we compute an approximate N-Voronoi diagram, where we fix a small constant $\epsilon > 0$ and compute a family of convex regions R_i $(i = 1, 2, \dots, n)$ such that $(1-\epsilon)d(x, \cup_{j\neq i}R_j) \le d(x, p_i) \le d(x, \cup_{j\neq i}R_j)$ for every point x on the boundary of R_i.

5.1 Iterative Construction Scheme

We use the operator $\mathbf{D}$ defined in the previous section. Starting from $\mathbf{P}^{(0)} = S = \{p_1, p_2, \cdots, p_n\}$, we define $\mathbf{P}^{(k+1)} = \mathbf{D}(\mathbf{P}^k)$. Note that $\mathbf{P}^{(1)}$ is the ordinary Voronoi diagram. We have $\mathbf{P}^0 \preceq \mathbf{P}^2 \preceq \dots \preceq \mathbf{R} \preceq \dots \preceq \mathbf{P}^3 \preceq \mathbf{P}^1$ if N-Voronoi diagram $\mathbf{R}$ exists. Thus, if we find k such that $(1-\epsilon)d(x, \cup_{j\neq i}R_j^{(2k)}) \le d(x, p_i) \le d(x, \cup_{j\neq i}R_j^{(2k)})$ for each $i = 1, 2, \dots n$, we can output $\mathbf{P}^{(2k)} = \{R_1^{(2k)}, R_2^{(2k)}, \dots R_n^{(2k)}\}$ as an approximate N-Voronoi diagram of S.

If the distances $d(p_i, p_j)$ between input points satisfies that $\epsilon d(p_i, p_j)/6$ is sufficiently larger than the precision for computing nearest distance of curves considered in our algorithm, we can check the above termination condition if $\mathbf{P}^{(2k)}$ converges to the real N-Voronoi diagram.

Figure 6 shows $\mathbf{P}^0$ (input points), $\mathbf{P}^1$ (ordinary Voronoi diagram), and $\mathbf{P}^2$ together with the output N-Voronoi diagram (drawn by using bold lines).

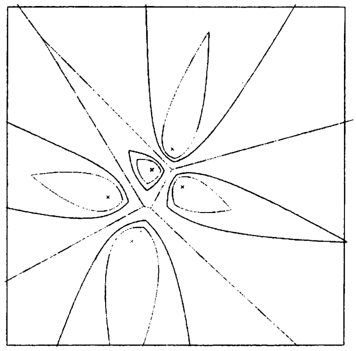

Figure 6: An approximate N-Voronoi diagram and some curves generated during runtime of our algorithm.

5.2 Difficulty on designing efficient algorithms

For an ordinary Voronoi diagram on n sites, if we draw bisectors between all pairs of input sites independently, the skeleton (i.e., union of boundary polygonal curves of regions) of the Voronoi diagram is a subset of the union of bisectors. This feature holds for almost all variants of Voronoi diagrams, such as furthest neighbor Voronoi diagram, Voronoi diagrams of lines and convex objects, and the power diagram. This enables to design $O(n^2)$ time algorithm for constructing those Voronoi diagrams provided that bisectors between two cites can be computed in $O(1)$ time.

Since we have investigated the two-point case precisely, if we use the bisector computation of two-points as a black box, it seems that we can also design an $O(n^2)$ time algorithm for the N-Voronoi diagram on n points. Unfortunately, this argument does not apply to our case. Fig. 7 shows an N-Voronoi diagram of three points to give us a counterexample.

Suppose on the contrary the argument above can be correctly applied to this case. Let A be the left site and B and C be two sites in the right (B is above C). Let C_{AB} be are bounding curves of N-Voronoi region of A in the two-points set $\{A, B\}$ without considering the third point C. We analogously define C_{BA}, C_{BC} etc. The Voronoi region of $V(A)$ should be equal to $\tilde{V}(A) = H(A,B) \cap H(A,C)$, where $H(A,B)$ and $H(A,C)$ are regions bounded by the curve C_{AB} and C_{AC}, respectively. In other words, $H(A,B)$ is the N-Voronoi region of A in the two-points set $\{A, B\}$.

Let P be the rightmost point on the boundary of $\tilde{V}(A)$. It must have images one on C_{BA} and the other on C_{CA}. Since it is on C_{BA}, the nearest point Q on C_{BA} satisfies that $d(A,P) = d(P,Q)$. Now, if the above argument holds, Q must be on the boundary of the N-Voronoi region $V(B)$ or $V(C)$ in the three-points set. However, in the configuration of the figure, Q is in the region $(H(B,A) \cup H(C,A)) \setminus (V(B) \cup V(C))$ to give contradiction. Indeed, the boundary curve of $V(A)$ is piecewise linear in a neighborhood of its rightmost point, and $V(A)$ is strictly larger than $\tilde{V}(A)$.

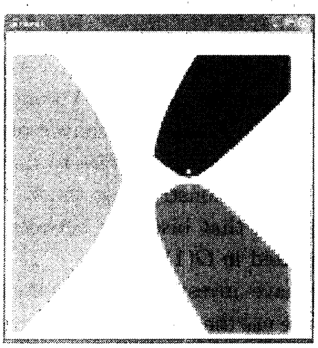

Figure 7: N-Voronoi Diagram with non-pairwise effect.

6 Concluding Remarks

There are several open problems: Uniqueness and a constructive proof for the existence of N-Voronoi diagrams, analysis of convergence of algorithms, time complexity analysis in terms of dependence on n, and extension to N-Voronoi diagrams of general objects. Also, the n-equally spaced curve problem has been only solved for the case $n = 1, 2$, and 3.

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