2006－AL－109 (5)
2006／11／21

辺連結度制約と次数制約をもつネットワーク設計問題

福永拓郎, 永持仁
京都大学情報学研究科数理工学専攻

概要

本報告では，辺連結度と各節点の次数に制約が与えられているネットワーク設計問題について考える．具体的には，節点集合 V, メトリック辺コスト c, 正整数 k, 指定次数 b が入力として与えられているとする．このとき，節点 $v \in V$ の次数が $b(v)$ となるような V 上のコスト最小 k-辺連結グラフを求める問題を考える．この問題は，巡回セールスマン問題の一般化となっている．我々は，各節点 $v \in V$ に対し $b(v) \geq 2$ が成り立つという仮定のもと，近似アルゴリズムを与える．このアルゴリズムの近似率は，k が偶数のときは 2.5，k が奇数のときには $2.5 + 1.5/k$ である．

Network design with edge-connectivity and degree constraints

Takuro Fukunaga, Hiroshi Nagamochi
Department of Applied Mathematics and Physics,
Graduate School of Informatics, Kyoto University

abstract

We consider the following network design problem; Given a vertex set V with a metric cost c on V, an integer $k \geq 1$, and a degree specification b, find a minimum cost k-edge-connected multigraph on V under the constraint that the degree of each vertex $v \in V$ is equal to $b(v)$. This problem generalizes metric TSP. In this paper, we propose that the problem admits a ρ-approximation algorithm if $b(v) \geq 2$, $v \in V$, where $\rho = 2.5$ if k is even, and $\rho = 2.5 + 1.5/k$ if k is odd.

1 Introduction

It is a main concern in the field of network design to construct a graph of the least cost which satisfies some connectivity requirement. Actually many results on this topic have been obtained so far. In this paper, we consider a network design problem that asks to find a minimum cost k-edge-connected multigraph on a metric edge cost under degree specification. This provides a natural and flexible framework for treating many network design problems. For example, it generalizes the vehicle routing problem with m vehicles (m-VRP) [3, 7], which will be introduced below, and hence contains a well-known metric traveling salesperson problem (TSP), which has already been applied to numerous practical problems [8].

Let $\mathbb{Z}_+$ and $\mathbb{Q}_+$ denote the sets of non-negative integers and non-negative rational numbers, respectively. Let $G = (V, E)$ be a multigraph with a vertex set V and an edge set E, where a multigraph may have some parallel edges but is not allowed to have any loops. For two vertices u and v, an edge joining u and

v is denoted by uv. Since we consider multigraphs in this paper, we distinguish two parallel edges $e_1 = uv$ and $e_2 = uv$, which may be simply denoted by uv and uv. For a non-empty vertex set $X \subset V$, $d(X;G)$ (or $d(X)$) denotes the number of edges whose one end vertex is in X and the other is in $V - X$. In particular $d(v;G)$ (or $d(v)$) denotes the degree of vertex v in G. The edge-connectivity $\lambda(u,v;G)$ (or $\lambda(u,v)$) between u and v is the maximum number of edge-disjoint paths between them in G. The edge-connectivity $\lambda(G)$ of G is defined as $\min_{u,v \in V} \lambda(u,v;G)$. If $\lambda(G) \geq k$ for some $k \in \mathbb{Z}_+$, then G is called *k-edge-connected*. For a function $r : \binom{V}{2} \to \mathbb{Z}_+$, G is called *r-edge-connected* if $\lambda(u,v;G) \geq r(u,v)$ for every $u,v \in V$. Edge cost $c : \binom{V}{2} \to \mathbb{Q}_+$ is called *metric* if it obeys the triangle inequality, i.e., $c(uv) + c(vw) \geq c(uw)$ for every $u,v,w \in V$.

For a degree specification $b : V \to \mathbb{Z}_+$, a multigraph G with $d(v;G) = b(v)$ for all $v \in V$ is called a *perfect b-matching*. In this paper, we focus on the following network design problem.

k-edge-connected multigraph with degree specification (k-ECMDS):
A vertex set V, a metric edge cost $c : \binom{V}{2} \to \mathbb{Q}_+$, a degree specification $b : V \to \mathbb{Z}_+$, and a positive integer k are given. We are asked to find a minimum cost perfect b-matching $G = (V,E)$ of edge-connectivity k. □

In this paper, we suppose that $b(v) \geq 2$ for all $v \in V$ unless stated otherwise, and propose approximation algorithms to k-ECMDS in this case.

Problem k-ECMDS is a generalization of m-VRP, which asks to find a minimum cost set of m cycles, each containing a designated initial city s, such that each of the other cities is covered by exactly one cycle. Observe that this problem is 2-ECMDS where $b(s) = 2m$ for the initial city $s \in V$ and $b(v) = 2$ for every $v \in V - s$. If $m = 1$, then m-VRP is exactly TSP. Since TSP is known to be NP-hard [11] even if a given cost is metric (metric TSP), k-ECMDS is also NP-hard. If a given cost is not metric, TSP cannot be approximated unless P = NP [11]. For m-VRP, there is a 2-approximation algorithm based on the primal-dual method [7].

It is well studied to find a minimum cost multigraph either with k-edge-connectivity or with degree specification. It is known that finding a minimum cost k-edge-connected graph is NP-hard since it is equivalent to metric TSP when $k = 2$ and a given edge cost is metric. On the other hand, it is known that a minimum cost perfect b-matching can be constructed in polynomial time (for example, see [10]). As a prior result on problems equipped with both edge-connectivity requirements and degree constraints, Frank [1] showed that it is polynomially solvable to find a minimum cost r-edge-connected multigraph G with $\ell(v) \leq d(v;G) \leq u(v)$, $v \in V$ for degree lower and upper bounds $\ell, u : V \to \mathbb{Z}_+$ and a metric edge cost c such that $c(uv)$ is defined by $w(u) + w(v)$ for some weight $w : V \to \mathbb{Q}_+$ (in particular, $c(uv) = 1$ for every $uv \in \binom{V}{2}$). Recently Fukunaga and Nagamochi [4] presented approximation algorithms for a network design problem with a general metric edge cost and some degree bounds; For example, they presented a $(2 + 1/\lfloor \min_{u,v \in V} r(u,v)/2 \rfloor)$-approximation algorithm for constructing a minimum cost r-edge-connected multigraph that meets a local-edge-connectivity requirement r with $r(u,v) \geq 2$, $u,v \in V$ under a uniform degree upper bound. Afterwards Fukunaga and Nagamochi [5] gave a 3-approximation algorithm for the case where $r(u,v) \in \{1,2\}$ for every $u,v \in V$ and $\ell(v) = u(v)$ for each $v \in V$. In this paper, we extend the 3-approximation result [5] to k-ECMDS. Concretely, we prove that k-ECMDS is ρ-approximable if $b(v) \geq 2$, $v \in V$, where $\rho = 2.5$ if k is even and $\rho = 2.5 + 1.5/k$ if k is odd. To design our algorithms for k-ECMDS, we take a similar approach with famous 2- and 1.5-approximation algorithms for metric TSP.

2 Algorithm for k-ECMDS

For some degree specification b, there is no perfect b-matching. The following theorem shows provides a necessary and sufficient condition for a degree specification to admit a perfect b-matching. Note that $b(v)$ can be 1 in this theorem.

Theorem 1 *Let V be a vertex set with $|V| \geq 2$ and $b: V \to \mathbb{Z}_+$ be a degree specification. Then there exists a perfect b-matching if and only if $\sum_{v \in V} b(v)$ is even and $b(v) \leq \sum_{u \in V-v} b(u)$ for each $v \in V$.*

Proof: The necessity is trivial. We show the sufficiency by constructing a perfect b-matching. We let $V = \{v_1, \ldots, v_n\}$ and $B = \sum_{\ell=1}^{n} b(v_\ell)/2$. For $j = 1, \ldots, B$, we define i_j as the minimum integer such that $\sum_{\ell=1}^{i_j} b(v_\ell) \geq j$, and i'_j as the minimum integer such that $\sum_{\ell=1}^{i'_j} b(v_\ell) \geq B + j$. Notice that $\sum_{\ell=1}^{i_j-1} b(v_\ell) < j$ holds by the definition if $i_j \geq 2$. Then we can see that $i_j \neq i'_j$ since otherwise we would have $b(v_{i_j}) = \sum_{\ell=1}^{i_j} b(v_\ell) - \sum_{\ell=1}^{i_j-1} b(v_\ell) > (B + j) - j = B$ if $i_j \geq 2$ and $b(v_{i_j}) \geq B + j > B$ otherwise, which contradicts to the assumption.

Let $M = \{e_j = v_{i_j} v_{i'_j} \mid j = 1, \ldots, B\}$. Then M contains no loop by $i_j \neq i'_j$. Moreover G_M is a perfect b-matching since $|\{j \mid i_j = \ell \text{ or } i'_j = \ell\}| = b(v_i)$, as required. □

Theorem 1 does not mention the edge-connectivity. For existence of connected perfect b-matchings, we additionally need the condition that $\sum_{v \in V} b(v) \geq 2(|V| - 1)$ [5]. This is always satisfied if $b(v) \geq 2$, $v \in V$, which we assume for 1-ECMDS. For $k \geq 2$, the conditions in Theorem 1 and $b(v) \geq k$, $v \in V$ are sufficient for the existence of k-edge-connected perfect b-matchings as our algorithm will construct such b-matchings under the conditions.

Now we describe our algorithm to k-ECMDS. Let (V, b, c, k) be an instance of k-ECMDS. The conditions appeared in Theorem 1 and $b(v) \geq k$ for all $v \in V$ can be verified in polynomial time, where they are apparently necessary for an instance to have k-edge-connected perfect b-matchings. Hence our algorithm checks them, and if some of them are violated, it outputs message "INFEASIBLE". In the following, we suppose the existence of perfect b-matchings with $b(v) \geq k$ for all $v \in V$. If $2 \leq |V| \leq 3$, then every perfect b-matching is k-edge-connected because any non-empty vertex set $X \subset V$ is $\{v\}$ or $V - \{v\}$ for some $v \in V$, and then $d(X) = d(v) \geq k$. Hence we can assume without loss of generality that $|V| \geq 4$.

For an edge set F on V, we denote graph (V, F) by G_F. Let M be a minimum cost edge set such that G_M is a perfect b-matching. In addition, let H be an edge set of a Hamiltonian cycle spanning V constructed by the 1.5-approximation algorithm for TSP due to Christofides [11].

Initialization: After testing the feasibility of a given instance, our algorithm first prepares M and $k' = \lceil k/2 \rceil$ copies $H_1, \ldots, H_{k'}$ of H. Let E denote the union $M \cup H_1 \cup \cdots \cup H_{k'}$ of them. Notice that G_E is $2k'$-edge-connected by the existence of edge-disjoint k' Hamiltonian cycles. We call a vertex v in a handling graph G an *excess vertex* if $d(v; G) > b(v)$ (otherwise a *non-excess vertex*). In G_E, all vertices are excess vertices since $d(v; G_E) = b(v) + 2k'$. In the following steps, the algorithm reduces the degree of excess vertices until no excess vertex exists while generating no loops and keeping k-edge-connectivity (Notice that $k < 2k'$ if k is odd). This is achieved by two phases, Phase 1 and Phase 2, as follows.

Phase 1: In this phase, we modify only edges in M while keeping edges in $H_1, \ldots, H_{k'}$ unchanged. We define the following two operations on an excess vertex $v \in V$.

Operation 1: If v has two incident edges xv and yv in M with $x \neq y$, replace xv and yv by new edge xy.

Operation 2: If v has two parallel edges uv in M with $d(u) > b(u)$, remove those edges.

Phase 1 repeats Operations 1 and 2 until none of them is executable. For avoiding ambiguity, we let M' denote M after executing Phase 1, and M denote the original set in what follows. Moreover, let $E' = M' \cup H_1 \cup \cdots \cup H_{k'}$. Note that $d(v) - b(v)$ is always a non-negative even integer throughout (and after) these operations because $d(v;G_E) - b(v) = 2k'$ and each operation decreases the degree of a vertex by 2. If no excess vertex remains in $G_{E'}$, then we are done. We consider the case in which there remain some excess vertices, and show some properties on M' before describing Phase 2.

Claim 1 *Every excess vertex in $G_{E'}$ has at least one incident edge in M' and its neighbors in $G_{M'}$ are unique.*

Proof: Omitted due to the space limitation. □

For an excess vertex v in $G_{E'}$, let $n(v)$ denote the unique neighbor of v in $G_{M'}$. If $n(v)$ is also an excess vertex in $G_{E'}$, we call the pair $\{v, n(v)\}$ by a *strict pair.*

Claim 2 *Let $\{v, n(v)\}$ be a strict pair. Then $d(v; G_{M'}) = d(n(v); G_{M'}) = 1$, k is odd, and $b(v) = b(n(v)) = k$.*

Proof: By Claim 1, $d(v; G_{M'}) = d(n(v); G_{M'})$. If $d(v; G_{M'}) = d(n(v); G_{M'}) > 1$, Operation 2 can be applied to v and $n(v)$, a contradiction. Hence $d(v; G_{M'}) = d(n(v); G_{M'}) = 1$ holds. Let $u \in \{v, n(v)\}$. Then it holds that $d(u; G_{E'}) = d(u; G_{H_1 \cup \cdots \cup H_{k'}}) + d(u; G_{M'}) = 2k' + 1 = 2\lceil k/2 \rceil + 1$. Since $d(u; G_{E'}) - b(u)$ is even, $b(u)$ must be odd. This fact and $d(u, G_{E'}) > b(u) \geq k$ indicates that $b(u) = k$ and k is odd. □

By definition, the existence of excess vertices which are in no strict pairs indicate that of some non-excess vertices. Upon completion of Phase 1, let N denote the set of non-excess vertices in $G_{E'}$, and S denote the set of strict pairs in $G_{E'}$. If $N = \emptyset$, all excess vertices are in some strict pairs. By Claim 2, k is an odd integer in this case, and furthermore $k \geq 3$ by the assumption that $b(v) \geq 2$, $v \in V$ if $k = 1$. From this fact and $|V| \geq 4$, $N = \emptyset$ implies that at least two strict pairs exist (i.e., $|S| \geq 2$).

Phase 2: Now we describe Phase 2. First, we deal with a special case in which V consists of only two strict pairs.

Claim 3 *If V consists of two strict pairs after Phase 1, we can transform $G_{E'}$ into a k-edge-connected perfect b-matching without increasing the cost.*

Proof: Let $V = \{u, v, w, z\}$ and $H = \{uv, vw, wz, zu\}$. Now $E' = M' \cup H_1 \cup \cdots \cup H_{k'}$ ($k \geq 2$). Then either $M' = \{uv, wz\}$ (or $\{vw, zu\}$) or $M' = \{uw, vz\}$ holds. In both cases, we replace $M' \cup H_1 \cup H_2$ by $E'' = \{uv, vw, wz, zu, uw, vz\}$ (see Fig. ??). Then, we can see that $d(v; G_{E''}) = 3$ for all $v \in V$ and $G_{E''}$ is 3-edge-connected. Since $d(v; G_{H_i}) = 2$ for $v \in V, i = 3, \ldots, k'$ and G_{H_i} is 2-edge-connected for $i = 3, \ldots, k'$, it holds that $d(v; G_{E'' \cup H_3 \cup \cdots \cup H_{k'}}) = 3 + 2(k' - 2) = k = b(v)$ for $v \in V$ and the edge-connectivity of $G_{E'' \cup H_3 \cup \cdots \cup H_{k'}}$ is $3 + 2(k' - 2) = k$ (The existence of strict pair implies that k is odd by Claim 2.).

Hence it suffices to show that $c(E'') \leq c(M') + c(H_1) + c(H_2)$. If $M' = \{uw, vz\}$ (or $\{vw, zu\}$), then it is obvious since $E'' = M' \cup H_1 \subseteq M' \cup H_1 \cup H_2$. Let us consider the other case, i.e., $M' = \{uv, wz\}$. From $M' \cup H_1 \cup H_2$, remove $\{uv, uv\}$, replace $\{wz, zu\}$ by $\{wu\}$, and replace $\{vw, wz\}$ by $\{vz\}$. Then the edge set becomes E'' without increasing edge cost, as required. □

In the following, we assume that $|S| \geq 3$ when $N = \emptyset$. In this case, Phase 2 modifies only edges in H_i, $i = 1, \ldots, k'$ while keeping the edges in M' unchanged. Let $V(H_i)$ denote the set of vertices spanned by H_i. We define *detaching v from cycle H_i* to be an operation that replaces the pair $\{uv, vw\} \subseteq H_i$ of edges incident to v by a new edge uw. Note that this decreases $d(v)$ by 2, but H_i remains a cycle on $V(H_i) := V(H_i) - \{v\}$. For each excess vertex v in $G_{E'}$, Phase 2 reduces $d(v)$ to $b(v)$ by detaching v from $(d(v; G_{E'}) - b(v))/2$ cycles in $H_1, \ldots, H_{k'}$. We notice that $(d(v; G_{E'}) - b(v))/2 \leq k'$ by $d(v; G_{E'}) - b(v) \leq d(v; G_E) - b(v) = 2k'$. One important point is to keep $|V(H_i)| \geq 2$ for each $i = 1, \ldots, k'$ during Phase 2. In other words, we always select H_i with $|V(H_i)| \geq 3$ to detach an excess vertex. This is necessary because, if we detach a vertex from H_i with $V(H_i) = 2$, then H_i becomes a loop. In addition, we detach the two excess vertices u and v in a strict pair from different cycles in $H_1, \ldots, H_{k'}$, respectively. This is in order to maintain the k-edge-connectivity of $G_{E'}$ as will be explained below.

Claim 4 *It is possible to decrease the degree of each excess vertex v in $G_{E'}$ to $b(v)$ by detaching from some cycles in $H_1, \ldots, H_{k'}$ so that $|V(H_i)|$ remains at least 2 for $i = 1, \ldots, k'$ and the two excess vertices in each strict pair are detached from H_i and H_j with $i \neq j$, respectively.*

Proof: First, let us consider the case of $S \neq \emptyset$. Recall $k \geq 3$ and $k' = \lceil k/2 \rceil \geq 2$ in this case. For each strict pair $\{u, v\} \in S$, we detach u and v from different cycles in $H_1, \ldots, H_{k'}$. On the other hand, we detach excess vertex z from arbitrary $(d(z; G_{E'}) - b(z))/2$ cycles. After this, each of $H_1, \ldots, H_{k'}$ is incident to at least one vertex of any strict pair in S in addition to all non-excess vertices in N. By the relation between $|S|$ and $|N|$ we explained in the above, it holds that $|V(H_i)| \geq |S| + |N| \geq 2$ for each $i = 1, \ldots, k'$, as required.

Next, let us consider the case of $S = \emptyset$. As explained in the above, $|N| \geq 1$ holds for this case. If $|N| \geq 2$, the claim is obvious since each of $H_1, \ldots, H_{k'}$ is always incident to all vertices in N. Hence suppose that $|N| = 1$, and let x be the unique non-excess vertex in N. Then all edges in M' are incident to x, since otherwise $S = \emptyset$ implies that Operation 1 or 2 would be applicable to some vertex in $V - x$. In other words, $b(x) = d(x; G_{E'}) = |M'| + 2k'$ holds before Phase 2. Moreover $\sum_{v \in V - x} b(v) \geq b(x)$ also holds by the assumption that perfect b-matchings exist. Now assume that we have converted some excess vertices in $G_{E'}$ into non-excess vertices by detaching them from some of $H_1, \ldots, H_{k'}$ while keeping $|V(H_i)| \geq 2$, $i = 1, \ldots, k'$, and yet an excess vertex $y \in V - x$ remains. Hence $\sum_{v \in V} d(v) > \sum_{v \in V} b(v)$. Then there remains a cycle H_i with $|V(H_i)| > 2$ because

$$\begin{aligned} 2 \sum_{1 \leq i \leq k'} |V(H_i)| &= \sum_{v \in V} d(v; G_{H_1 \cup \cdots \cup H_{k'}}) \\ &= \sum_{v \in V} d(v) - 2|M'| \\ &> \sum_{v \in V - x} b(v) + b(x) - 2|M'| \\ &\geq 2(b(x) - |M'|) \\ &\geq 4k'. \end{aligned}$$

Therefore we can detach an excess vertex y from such H_i as long as such a vertex exists. This implies that the claim holds also for $|N| = 1$. □

In the following, we let H'_i denote H_i after Phase 2, and H_i denote the original Hamiltonian cycle for $i = 1, \ldots, k'$. Moreover let $E'' = M' \cup H'_1 \cup \cdots \cup H'_{k'}$. The algorithm outputs $G_{E''}$. The entire algorithm is described as follows.

Algorithm UNDIRECT(k)

Input: A vertex set V, a degree specification $b : V \to \mathbb{Z}_+$, a metric edge cost $c : V \to \mathbb{Q}_+$, and a positive integer k

Output: A k-edge-connected perfect b-matching or "INFEASIBLE"

```
if ∑_{v∈V} b(v) is odd, ∃v : b(v) > ∑_{u∈V−v} b(u) or k > b(v) then
    Output "INFEASIBLE" and halt
end if;
Compute a minimum cost perfect b-matching G_M;
if |V| ≤ 3 then
    Output G_M and halt
end if;
Compute a Hamiltonian cycle G_H on V by Christofides' algorithm;
k' := ⌈k/2⌉; Let H_1, ..., H_k' be k' copies of H;
# Phase 1
M' := M;
while Operation 1 or 2 is applicable to a vertex v ∈ V
with d(v; G_{M'∪H_1∪···∪H_k'}) > b(v) do
```

if $\exists\{xv, vy\} \subseteq M'$ such that $x \neq y$ **then**
  $M' := (M' - \{xv, vy\}) \cup \{xy\}$ # Operation 1
else
  if $\exists\{xv, vx\} \subseteq M'$ such that $d(x; G_{M' \cup H_1 \cup \cdots \cup H_{k'}}) > b(x)$ **then**
    $M' := M' - \{xv, vx\}$ # Operation 2
  end if
end if
end while;
# Phase 2
if V consists of two strict pairs **then**
  Rename vertices so that $H = \{uv, vw, wz, zu\}$;
  $H'_2 := \emptyset$; $M' := \{uw, vz\}$;
  Output $G_{M' \cup H'_1 \cup \cdots \cup H'_{k'}}$ and halt
end if;
$H'_i := H_i$ for each $i = 1, \ldots, k'$;
while $\exists v \in V$ with $d(v; G_{M' \cup H'_1 \cup \cdots \cup H'_{k'}}) > b(v)$ **do**
  if v and $n(v)$ forms a strict pair **then**
    Detach v from H'_i and $n(v)$ from H'_j, where $i \neq j$
  else
    Detach v from H'_i with $V(H'_i) > 2$
  end if
end while;
$E'' := M' \cup H'_1 \cup \cdots \cup H'_{k'}$;
Output $G_{E''}$

Claim 5 *$G_{E''}$ is a k-edge-connected perfect b-matching.*

Proof: We have already seen the case in which V consists of two strict pairs. Hence we suppose the other case in the following. Moreover we have already observed that $d(v; G_{E''}) = b(v)$ holds for each $v \in V$. Furthermore $G_{E''}$ is loopless since G_E is loopless and no operations in the algorithm generate loops. Hence we prove the k-edge-connectivity of $G_{E''}$ below.

Let $u, v \in V$. (i) First suppose that u and v are in some (possibly different) strict pairs in $G_{E'}$. Moreover, let $u \notin V(H'_i)$ and $v \notin V(H'_j)$ (hence $u \in V(H'_{i'})$ for $i' \neq i$ and $v \in V(H'_{j'})$ for $j' \neq j$). For each $\ell \in \{1, \ldots, k'\} - \{i, j\}$, $\lambda(u, v; G_{H'_\ell}) = 2$ holds because $u, v \in V(H'_\ell)$. If $i = j$, $\lambda(u, v; G_{H'_i \cup M'}) = 1$ holds because $d(u; G_{M'}) = d(v; G_{M'}) = 1$ and $n(u), n(v) \in V(H'_i)$. Then it holds that $\lambda(u, v; G_{E''}) = 2(k' - 1) + 1 = k$ in this case (Recall that the existence of strict pairs implies that k is odd by Claim 2). Hence we let $i \neq j$, and show that $\lambda(u, v; G_{H'_i \cup H'_j \cup M'}) \geq 3$ from now on, from which $\lambda(u, v; G_{E''}) \geq 2(k' - 2) + 3 = k$ can be derived.

Let N and S denote the sets of non-excess vertices and strict pairs in $G_{E'}$ after Phase 1, respectively. Suppose that $V(H'_i) \cap V(H'_j) = \emptyset$. In this case, it can be seen that $N = \emptyset$, and hence $|S| \geq 3$ by the assumption about the relation between N and S. Since at least one vertex of each strict pair is spanned by each cycle in $H'_1, \ldots, H'_{k'}$, we can see that M' contains at least three vertex-disjoint edges that join vertices in $V(H'_i)$ and in $V(H'_j)$, two of which are u and v. This indicates that $\lambda(u, v; G_{H'_i \cup H'_j \cup M'}) \geq 3$ holds (see the graph of Figure 1 (b)).

Let us consider the case of $V(H'_i) \cap V(H'_j) \neq \emptyset$ in the next. By the existence of u and v, $|S| \geq 1$ holds. If u and v forms a strict pair (i.e., $uv \in M'$), $\lambda(u, v; G_{M'}) = 1$ holds. Since $V(H'_i) \cap V(H'_j) \neq \emptyset$ implies $\lambda(G_{H'_i \cup H'_j}) \geq 2$, we see that $\lambda(u, v; G_{H'_i \cup H'_j \cup M'}) \geq 3$ in this case. Thus let u and v belong to different strict pairs (i.e., $|S| \geq 2$). Then there exists two vertex-disjoint edges in M' joins vertices in $V(H'_i)$ and in $V(H'_j)$ (see Figure 1 (a)). If we split each vertex $w \in V(H'_i) \cap V(H'_j)$ into two vertices w' and w'' so that H'_i and H'_j are vertex-disjoint cycles, and add new edges $w'w''$ joining those two split vertices to M', then we can reduce this case to the case of $V(H'_i) \cap V(H'_j) = \emptyset$, in which $\lambda(u, v; G_{H'_i \cup H'_j \cup M'}) \geq 3$ has already been observed in the above (see Figure 1). Accordingly, we have $\lambda(u, v; G_{H'_i \cup H'_j \cup M'}) \geq 3$ if u and v are in some strict pairs, as required.

(ii) In the next, let u and v be not in any strict pairs. For $z \in \{u, v\}$, let $n'(z)$ denote z itself if $z \in N$, and $n(z)$ otherwise. Notice that $n'(z) \in N$ for any $z \in \{u, v\}$, i.e., it is

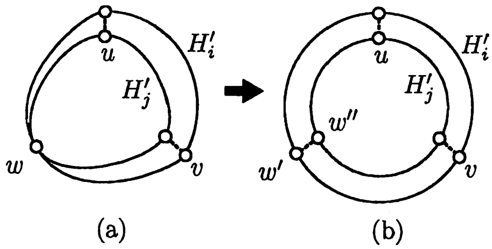

Figure 1: Reduction to the case of $V(H_i') \cap V(H_j') = \emptyset$

spanned by $H_1', \ldots, H_{k'}'$. If $z \in \{u, v\}$ is not spanned by $p > 0$ cycles in $H_1', \ldots, H_{k'}'$ (and hence z is an excess vertex in $G_{E'}$), then z has at least $k - 2(k' - p)$ incident edges in M' because $d(z; G_{M'}) = b(z) - d(z; G_{H_1' \cup \cdots \cup H_{k'}'}) \geq k - 2(k' - p)$. Hence $\lambda(z, n'(z); G_{E''}) \geq 2(k' - p) + k - 2(k' - p) = k$ holds for each $z \in \{u, v\}$, where we define $\lambda(z, z; G_{E''}) = +\infty$. Moreover it is obvious that $\lambda(n'(u), n'(v); G_{E''}) \geq 2k'$. Therefore, it holds that

$$\begin{aligned} \lambda(u, v; G_{E''}) \geq \min\{&\lambda(u, n'(u); G_{E''}), \\ &\lambda(n'(u), n'(v); G_{E''}), \lambda(n'(v), v; G_{E''})\} \\ &\geq k. \end{aligned}$$

(iii) Finally, let us consider the remaining case, i.e., u is in a strict pair and v is a vertex which is not in any strict pair. Let us define $n'(v)$ as in the above. Then $\lambda(v, n'(v); G_{E''}) \geq k$ holds. Without loss of generality, let u be detached from H_1', and spanned by $H_2', \ldots, H_{k'}'$. Since $un(u) \in M'$ and $n(u), n'(v) \in V(H_1')$, it holds that $\lambda(u, n(u); G_{M' \cup H_1'}) = 1$, and $\lambda(n(u), n'(v); G_{M' \cup H_1'}) \geq 2$. Then,

$$\begin{aligned} &\lambda(u, n'(v); G_{E''}) \geq \\ &\quad \min\{\lambda(u, n(u); G_{M' \cup H_1'}), \\ &\qquad \lambda(n(u), n'(v); G_{M' \cup H_1'})\} \\ &\qquad + \lambda(u, n'(v); G_{H_2' \cup \cdots \cup H_{k'}'}) \\ &\qquad \geq 1 + 2(k' - 1) = 2k' - 1 = k. \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda(u, v; G_{E''}) \geq \min\{&\lambda(u, n'(v); G_{E''}), \\ &\lambda(v, n'(v); G_{E''})\} \geq k, \end{aligned}$$

holds, as required. □

Let us consider the cost of the graph $G_{E''}$. The following theorem on the Christofides' algorithm gives us an upper bound on $c(H)$. Here, we let $\delta(U)$ denote the set of edges whose one end vertex is in U and the other is in $V - U$ for nonempty $U \subset V$.

Theorem 2 ([6, 12]) *Let*

$$\begin{aligned} &OPT_{TSP} = \\ &\min \quad \textstyle\sum_{e \in E} c(e)x(e) \\ &s.\ t. \quad \textstyle\sum_{e \in \delta(U)} x(e) \geq 2 \quad \textit{for each } \emptyset \neq U \subset V, \\ &\qquad\quad x(e) \geq 0 \qquad\qquad \textit{for each } e \in E. \end{aligned}$$

Christofides' algorithm for TSP always outputs a solution of cost at most $1.5 OPT_{TSP}$. □

Claim 6 $c(E'')$ *is at most* $1 + 3\lceil k/2 \rceil / k$ *times the optimal cost of k-ECMDS.*

Proof: No operation in Phases 1 and 2 increases the cost of the graph since the edge cost is metric. Hence it suffices to show that $c(M \cup H_1 \cup \cdots \cup H_{k'})$ is at most $(1 + 3\lceil k/2 \rceil / k) \cdot c(G)$, where G denotes an optimal solution of k-ECMDS. Since G is a perfect b-matching, $c(M) \leq c(G)$ obviously holds. Thus it suffices to show that $c(H_i) \leq 3c(G)/k$ for $1 \leq i \leq k'$, from which the claim follows.

Let $x_G : \binom{V}{2} \to \mathbb{Z}_+$ be the function such that $x_G(uv)$ denotes the number of edges joining u and v in G. Since G is k-edge-connected, $\sum_{e \in \delta(U)} x_G(e) \geq k$ holds for every nonempty $U \subset V$. Hence $2x_G/k$ is feasible for the linear programming in Theorem 2, which means that $\mathrm{OPT}_{TSP} \leq 2c(G)/k$. By Theorem 2, $c(H_i) \leq 1.5\mathrm{OPT}_{TSP}$. Therefore we have $c(H_i) \leq 3c(G)/k$, as required. □

Claims 5 and 6 establish the next.

Theorem 3 *Algorithm* UNDIRECT(k) *is a* ρ*-approximation algorithm for k-ECMDS, where* $\rho = 2.5$ *if k is even and* $\rho = 2.5 + 1.5/k$ *if k is odd.* □

Algorithm UNDIRECT(k) always outputs a solution for $k \geq 2$ as long as there exists a perfect b-matching and $b(v) \geq k$ for all $v \in V$. This fact and Theorem 1 imply the following corollary.

Corollary 1 *For $k \geq 2$, there exists a k-edge-connected perfect b-matching if and only if $\sum_{v \in V} b(v)$ is even and $k \leq b(v) \leq \sum_{u \in V-v} b(u)$ for all $v \in V$.* □

We close this section with a few remarks. The operations in Phases 1 and 2 are equivalent to a graph transformation called *splitting*, followed by removing generated loops if any. There are many results on the conditions for splitting to maintain the edge-connectivity [2, 9]. However, the splittings in these results may generate loops. Hence algorithm UNDIRECT(k) needs to specify a sequence of splitting so that removing loops does not make the degrees lower than the degree specification.

One may consider that a perfect $(b-2k')$-matching is more appropriate than a perfect b-matching as a building block of our algorithm, since there is no excess vertex for the union of a perfect $(b-2k')$-matching and k' Hamiltonian cycles. However, there is a degree specification b that admits a perfect b-matching, and no perfect $(b-2k')$-matching. Furthermore, even if there exits a perfect $(b-2k')$-matching, the minimum cost of the perfect $(b-2k')$-matching may not be a lower bound on the optimal cost of k-ECMDS. Therefore we do not use a perfect $(b-2k')$-matching in general case. When a degree specification b is uniform, we can show that a perfect $(b-2k')$-matching exists and its cost can be estimated. By using this, we can improve the approximation factor of our algorithm in this case.

References

[1] A. Frank, Augmenting graphs to meet edge-connectivity requirements, SIAM Journal on Discrete Mathematics 5 (1992) 25–53.

[2] A. Frank, On a theorem of Mader, Discrete Mathematics 191 (1992) 49–57.

[3] G. N. Frederickson, M. S. Hecht, C. E. Kim, Approximation algorithms for some routing problems, SIAM Journal of Computing 7 (1978) 178–193.

[4] T. Fukunaga, H. Nagamochi, Approximating minimum cost multigraphs of specified edge-connectivity under degree bounds, Proceedings of the 9th Japan-Korea Joint Workshop on Algorithm and Computation (2006) 25-32.

[5] T. Fukunaga, H. Nagamochi, Approximating a generalization of metric TSP, IEICE Transactions on Information and Systems, to appear.

[6] M. X. Goemans, D. J. Bertsimas, Survivable networks, linear programming relaxations and the parsimonious property, Mathematical Programming 60 (1993) 145–166.

[7] M. X. Goemans, D. P. Williamson, The primal-dual method for approximation algorithms and its application to network design problems, PWS, 1997, Ch. 4, pp. 144–191.

[8] E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy Kan, D. B. Shmoys (Eds.), The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization, John Wiley & Sons, 1985.

[9] W. Mader, A reduction method for edge-connectivity in graphs, Annals of Discrete Mathematics 3 (1978) 145–164.

[10] A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency, Springer, 2003.

[11] V. Vazirani, Approximation Algorithm, Springer, 2001.

[12] L. A. Wolsey, Heuristic analysis, linear programming and branch and bound, Mathematical Programming Study 13 (1980) 121–134.