On Decision Time of classical Logic

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Abstract

It is interesting to research, how long time spend to decide whether a given sequent is provable or not on classical logic. In this paper, we show that it is possible by exponential time.

1 Preliminary

We cannot calculate all functions even if we restrict its domain the natural numbers, because that the set of all functions is uncountable, and the other hand the set of all computable functions is countable. However, a computable function that we say, is only computable theoretically, but also may spend a million years until to get the answer. The Ackermann function is one of the typical example. The Ackermann function is computable, but its computing time grows extremely bigger as n grows bigger. According to our experiment using a computer, we get immediately the answer until $n = 3$, but the computer does not move for $n = 4$ (of course, it moves underground). It seems that the computing time may spend a million years for $n = 10$ or $n = 100$.

The reading head must scan input data to answer, if the size of the input data is n, the machine needs at least n steps. Of course, it seems that the machine spends many steps according that n grows. Therefore, it is clear that $t(n) > n$ if t(n) means the computing time as the function of n. So, we must argue the $t(n)$ less than polynomial, or less than exponential, and so on. Now, for the small n, the computing time is less than some constant, and so we neglect small n, and argue sufficiently bigger n.

if $t(n)$ is the polynomial with degree i, we can write $t(n) = a_0n^i + a_1n^{i-1} + \cdots$, but we get $t(n) < (a_0+1)n^i$ for sufficiently bigger n. So, we write $t(n) < O(n^i)$ for this fact. We write $t(n) < O(2^n)$ similarly, in the case of exponential. Classical logic has two method, so called Hilbert style and Gentzen style, and so we use Gentzen style. Our logical symbols are $\neg$(not), $\wedge$(and), $\vee$(or), $\rightarrow$(if$\cdots$then$\cdots$), $\forall$(all), $\exists$(exist). If Γ and Δ are finite sequences respectively, we call $\Gamma \vdash \Delta$ sequent. It means intuitively, that if we assume all of Γ we deduce at least one of Δ. If A is a formula, the sequent $A \vdash A$ is only one axiom. The inference rule are following:

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \textit{Thinning}\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ \vdash\textit{Thinning}$$

$$\frac{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}\ \textit{Interchange}\vdash \qquad \frac{\Gamma \vdash \Delta_1, B, A, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2}\ \vdash\textit{Interchange}$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \textit{Contraction}\vdash \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ \vdash\textit{Contraction}$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ \neg\vdash \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \vdash\neg$$

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \wedge\vdash \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \wedge\vdash \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ \vdash\wedge$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ \vee\vdash \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ \vdash\vee \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ \vdash\vee$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{A \rightarrow B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \rightarrow\vdash \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}\ \vdash\rightarrow$$

$$\frac{A(t), \Gamma \vdash \Delta}{\forall x A(x), \Gamma \vdash \Delta}\ \forall\vdash \qquad \frac{\Gamma \vdash \Delta, A(a)}{\Gamma \vdash \Delta, \forall x A(x)}\ \vdash\forall$$

$$\frac{A(a), \Gamma \vdash \Delta}{\exists x A(x), \Gamma \vdash \Delta} \ \exists\vdash \qquad\qquad \frac{\Gamma \vdash \Delta, A(t)}{\Gamma \vdash \Delta, \exists x A(x)} \ \vdash\exists$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \ cut$$

where t in the inference rule of $\forall$ and $\exists$ is any term, and a is any free variable not occurring lower sequent.

2 Main Theorem

Provability is equivalent to provability without cut, and so it is sufficient to argue the decision of the provability without cut. In the proof figure without cut, we attend that the logical symbols including upper sequent are fewer than them including lower sequent. We negrect $\forall$ and $\exists$ during short time, because that these symbols are compliceted. Let be n is the length of a given sequent. We will show that we decide whether the sequent is provable or not within $O(2^n)$ steps, by using induction on n. (This is the time to spend decision of provability of the sequent, but not the time to prove it.)

Case 1: No logical symbol occuring in $\Gamma \vdash \Delta$.
In the case that Γ and Δ have a same formula, it is provable, and in the other case, it is unprovable. We can decide it within $O(n)$ steps.

Case 2: $\Gamma_1, \neg A, \Gamma_2 \vdash \Delta$.
Provability of this is equivalent one of $\Gamma_1, \Gamma_2 \vdash \Delta, A$. The length of $\Gamma_1, \Gamma_2 \vdash \Delta, A$ is less than n. So, the provability of this can decide within $O(2^{n-1})$ steps. And, we can decide whether it is the form $\Gamma_1, \neg A, \Gamma_2 \vdash \Delta$ or not within $O(n)$ steps. So, provability of this sequent can be desided within $O(n \cdot 2^{n-1}) \leq O(2^n)$ steps.

Case 3: $\Gamma \vdash \Delta_1, \neg A, \Delta_2$.
similarly.

Case 4: $\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta$.
Provability of this is equivalent one of $\Gamma_1, A, B, \Gamma_2 \vdash \Delta$. Because that if $\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta$ is provable,

(i) $\Gamma_1, A, \Gamma_2 \vdash \Delta$ is provable

or

(ii) $\Gamma_1, B, \Gamma_2 \vdash \Delta$ is provable.

The case of (i), the proof figure of this sequent is of the form

$$\begin{array}{ccc} \underline{A \vdash A} & \underline{B \vdash B} & \underline{C \vdash C} \\ \ddots & \vdots & \therefore \\ \hline & \Gamma_1, A, \Gamma_2 \vdash \Delta & \end{array} \quad .$$

Then,

$$\begin{array}{ccc} \dfrac{A \vdash A}{B, A \vdash A} & \dfrac{B \vdash B}{B, B \vdash B} & \dfrac{C \vdash C}{B, C \vdash C} \\ \ddots & \vdots & \therefore \\ \hline & \Gamma_1, A, B, \Gamma_2 \vdash \Delta & \end{array}$$

is a proof figure of $\Gamma_1, A, B, \Gamma_2 \vdash \Delta$.

The case of (ii) is similarly.

Conversely, if there is a proof figure of $\Gamma_1, A, B, \Gamma_2 \vdash \Delta$

$$\dfrac{\dfrac{\dfrac{\ddots \quad \vdots \quad \therefore}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}}{\Gamma_1, A \wedge B, A \wedge B, \Gamma_2 \vdash \Delta}}{\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta}$$

is a proof figure of $\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta$. where, if we consider that $A \wedge B$ is the abbreviation of $(A \wedge B)$, the length of $\Gamma_1, A, B, \Gamma_2 \vdash \Delta$ is less than n. Remainder is similar to Case 2.

Case 5: $\Gamma \vdash \Delta_1, A \wedge B, \Delta_2$.
Provability of this is equivalent to provability of both of $\Gamma \vdash \Delta_1, A, \Delta_2$ and $\Gamma \vdash \Delta_1, B, \Delta_2$. The decision of the former is of $O(2^{n-1})$, and one of the latter $O(2^{n-1})$, and therefore the decision of $\Gamma \vdash \Delta_1, A \wedge B, \Delta_2$ is of $O(n \cdot 2^{n-1} + n \cdot 2^{n-1}) \leq O(2^n)$ as a whole.

Case 6: $\Gamma_1, A \vee B, \Gamma_2 \vdash \Delta$.
Similarly to Case 5.

Case 7: $\Gamma \vdash \Delta_1, A \vee B, \Delta_2$.
Similarly to Case 4.

Case 8: $\Gamma_1, A \to B, \Gamma_2 \vdash \Delta$.
Because that if $\Gamma_1, A \to B, \Gamma_2 \vdash \Delta$ is provable, there is $\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22}, \Delta_1, \Delta_2$ such that $\Gamma_{11} \subseteq \Gamma_1, \Gamma_{12} = \Gamma_1 - \Gamma_{11}, \Gamma_{21} \subseteq \Gamma_2, \Gamma_{22} = \Gamma_2 - \Gamma_{21}, \Delta_1 \subseteq \Delta, \Delta_2 = \Delta - \Delta_1$, and further $\Gamma_{11}, \Gamma_{21} \vdash \Delta_1, A$ and $\Gamma_{12}, B, \Gamma_{22} \vdash \Delta_2$ are provable. If

$$\begin{array}{c}\underline{A\vdash A}\quad \underline{B\vdash B}\quad \underline{C\vdash C}\\ \ddots \qquad \vdots \qquad \cdot^{\cdot^{\cdot}} \\ \hline \Gamma_{11},\Gamma_{21}\vdash\Delta_1,A\end{array}$$

is a proof figure of $\Gamma_{11}, \Gamma_{21} \vdash \Delta_1, A$, then

$$\begin{array}{c} \dfrac{A\vdash A}{\Gamma_{12},A,\Gamma_{22}\vdash\Delta_2,A} \quad \dfrac{B\vdash B}{\Gamma_{12},B,\Gamma_{22}\vdash\Delta_2,B} \quad \dfrac{C\vdash C}{\Gamma_{12},C,\Gamma_{22}\vdash\Delta_2,C}\\ \ddots \qquad \vdots \qquad \cdot^{\cdot^{\cdot}} \\ \hline \Gamma_1,\Gamma_2\vdash\Delta,A \end{array}$$

is a proof figure of $\Gamma_1, \Gamma_2 \vdash \Delta, A$.

Similarly, if $\Gamma_{11}, B, \Gamma_{21} \vdash \Delta_1$ is provable, then $\Gamma_1, B, \Gamma_2 \vdash \Delta$ is also provable.

Conversely, if $\Gamma_1, \Gamma_2 \vdash \Delta, A$ and $\Gamma_1, B, \Gamma_2 \vdash \Delta$is provable, then

$$\dfrac{\dfrac{\dfrac{\ddots\ \vdots\ \cdot^{\cdot^{\cdot}}}{\Gamma_1,\Gamma_2\vdash\Delta,A}\quad\dfrac{\ddots\ \vdots\ \cdot^{\cdot^{\cdot}}}{\Gamma_1,B,\Gamma_2\vdash\Delta}}{\Gamma_1,\Gamma_1,A\to B,\Gamma_2,\Gamma_2\vdash\Delta,\Delta}}{\Gamma_1,A\to B,\Gamma_2\vdash\Delta}$$

is a proof figure of $\Gamma_1, A \to B, \Gamma_2 \vdash \Delta$.

$\Gamma_1, \Gamma_2 \vdash \Delta, A$ and $\Gamma_1, B, \Gamma_2 \vdash \Delta$ can be decided $O(2^{n-1})$ steps, respectively, and hence $\Gamma_1, A \to B, \Gamma_2 \vdash \Delta$ is decided within $O(n \cdot 2^{n-1} + n \cdot 2^{n-1}) \leq O(2^n)$ steps, wholly.

Case 9: $\Gamma \vdash \Delta_1, A \to B, \Delta_2$.
Provability of this is equivalent to provability of $A, \Gamma \vdash \Delta_1, B, \Delta_2$, and the latter can be decided within $O(2^{n-1})$ steps, and therefore the former can be decided within $O(n \cdot 2^{n-1}) \leq O(2^n)$ steps.

Unless $\forall$ or $\exists$, we get the same result in spite of an order of elimination of logical symbols. But we will fail to prove a provable sequent, if we make a mistake the order of elimination. This comes from very strong condition that a in $\vdash \forall$ or $\exists \vdash$ must be an free variable not occurring in the lower sequent.

We select the most alternating formula of $\forall$ and $\exists$ for a given sequent. On the occasion, we count alternated quantifiers for $\forall$ or $\exists$ in $\neg$, or in the leftside of $\to$. Last, we alternate all quantifiers of the leftside of $\vdash$, again. In the case that most alternating formulas are plural, we select the formula begining $\forall$. In the case that such formulas yet are plural, we may select any formula. The time which select such a formula, can be within $O(n^2)$steps.

Case 10: $\Gamma_1, \forall x A(x), \Gamma_2 \vdash \Delta$. The provability of this is equivalent to provability of $\Gamma_1, A(t), \Gamma_2 \vdash \Delta$ for some term t. But, the latter may not be shorter than the former contrary the cases until now. So, in this case we devise as following. We notice no inference rule to change contents of the term t in general predicate logic. (This argument does not good in a special predicate logic, for example, natural number theory.) Namely, when a term t occur anywhere in the proof figure, corresponding part remain t in all sequent over that place. Therfore, the one replaced the term t occuring in the upper sequent of $\forall \vdash$ or $\exists \vdash$ in anywhere in the proof figure, to a free variable a not occuring in the lower sequent throughout whole the proof figure, is yet a proof figure. For example, leftside figure is before replacing and rightside figure is after replacing as following:

$$\dfrac{P(f(c)) \vdash P(f(c))}{\forall x P(x) \vdash P(f(c))} \qquad \dfrac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)}$$

$$\dfrac{P(f(c)) \vdash P(f(c))}{\forall x P(f(x)) \vdash P(f(c))} \qquad \dfrac{P(f(a)) \vdash P(f(a))}{\forall P(f(x)) \vdash P(f(a))}$$

The provability of $\Gamma_1(t), \forall x A(x,t), \Gamma_2(t) \vdash \Delta(t)$ is equivalent to one of $\Gamma_1(t), A(t,t), \Gamma_2(t) \vdash \Delta(t)$ and moreover this is equvalent to one of $\Gamma_1(a), A(a,a), \Gamma_2(a) \vdash \Delta(a)$. And the last sequent is shorter than the first sequent. Therefore, the last sequent can be desided within $O(2^{n-1})$ steps. And, the selections of t are at most $O(n)$, and so the first sequent can be decided within $O(n^4 \cdot 2^{n-1}) \leq O(2^n)$ steps.

Case 11: $\Gamma \vdash \Delta_1, \forall x A(x), \Delta_2$. The provability of this is equivalent to one of $\Gamma \vdash \Delta_1, A(a), \Delta_2$ for a free variable a not occurring in the sequent. And since the length of the latter is shorter than one of the former, it can be decided within $O(n^3 \cdot 2^{n-1}) \leq O(2^n)$ steps.

Case 12: $\Gamma_1, \exists x A(x), \Gamma_2 \vdash \Delta$. Similarly to Case 11.

Case 13: $\Gamma \vdash \Delta_1, \exists x A(x), \Delta_2$. Similarly to Case 10.

We can get the following theorem, from the above results.

Main Theorem 1 *In the classical logic, we can deside that* $\Gamma \vdash \Delta$ *is provable or not, within exponential time.*

3 conclusion

In this paper, we argue by Gentzen style, but it is well known that the provability by Gentzen style and one by Hilbert style are equivalent. On the other hand, it is well known that the provability of a sequent and the validity of it are equivalent. Therefore, the decision time of provability and one of validity are same. In the propositional logic, it is trivial that validity of a sequent can be decided within exponential time, since it is valid if it is true for all combination of the truth values of the atomic formulas contained in the sequent. But, in the predicate logic, it is not trivial, that the truth value of $\forall x A(x)$ or $\exists x A(x)$ can be decided within exponential time. It is due that we must examine the truth value of $A(t)$ for all term t. In this paper, we showed that it is sufficient within exponential time, but not necessary within it. It may be decided within polynomial time. We are remainned to show immposibility of it within polynomial time. We began about most easy classical logic at the start, and we also will examine about non-classical logic.