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直並列グラフの最小ペア支配集合を求める線形時間アルゴリズム

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概要： グラフ G の支配集合 D は，D が誘導する部分グラフが完全マッチングを持つとき，ペア支配集合であるという．ペア支配集合は Haynes, Slater によって導入され，さらに一般のグラフに対して最小のペア支配集合を求める問題は NP-Hard であることが証明された．本論文では，直並列グラフの最小ペア支配集合を求める線形時間アルゴリズムを提案する．

A linear time algorithm for finding the minimum paired-dominating set in series-parallel graphs

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Abstract: For a graph G, a subset D of vertices is a paired-dominating set of G if every vertex not in D is adjacent to a vertex in D and the subgraph induced by D has a perfect matching. The paired-domination number of a graph is the minimum cardinality of paired-dominating sets. Haynes and Slater proved that the problem of determining the paired-domination number is NP-hard. In this paper, we give a linear time algorithm for computing the paired-domination number of a series-parallel graph. Our algorithm is based on the work by Kikuno, Yoshida, and Kakuda which is an algorithm for computing an ordinary dominating set of a series-parallel graph.

1 Introduction

Let $G = (V, E)$ be a graph without isolated vertices with vertex set V and edge set E. The *open neighborhood* of a vertex x is defined by $N_G(x) = \{y \mid xy \in E\}$ and the *closed neighborhood* of x is $N_G[x] = N_G(x) \cup \{x\}$. A set $D \subseteq V$ is a *dominating set* if every vertex not in D is adjacent to a vertex in D. The *domination number* of G is the minimum cardinality of a dominating set, and denoted by $\gamma(G)$.

Paired-domination was introduced by Haynes and Slater [4]. A set $D \subseteq V$ is a *paired-dominating set* if D is a dominating set and the induced subgraph $\langle D \rangle$ has a perfect matching. If $xy \in M$, where M is a perfect matching in $\langle D \rangle$, we say that x and y are *paired* in D. The *paired-domination number* $\gamma_{\rm pr}(G)$ is the minimum cardinality of a paired-dominating set in G. By the definition, clearly $\gamma(G) \leq \gamma_{\rm pr}(G)$ for any graph G.

Paired-domination was proposed with the following application in mind. If we think of each $x \in D$ as the location of a guard that can protect vertex in $N_G[x]$, then domination requires every vertex to be protected. In paired-domination, each guard is assigned another adjacent one, and they are designed as backup for each other.

Every graph without isolated vertices has a paired-dominating set because the end vertices of any maximal matching form a paired-dominating set. After the paper [4] was published, some graph-theoretical results for the paired-domination problem have been studied, e.g. [1, 2, 6, 10]. From algorithmic point of view, the following was also shown in [4].

Theorem 1.1 (Haynes and Slater [4]). *Deciding, for a given graph G and positive integer k, "Is $\gamma_{\mathrm{pr}}(G) \leq k$?" is NP-complete.*

Since the problem of determining the paired-domination number of an arbitrary graph is NP-hard, it is important to consider algorithms of paired-domination number in special graphs. However, as far as we know, only few efficient algorithms are proposed for the paired-domination problem. Qiao, Kang, Cardei and Du [9] presented a linear time algorithm computing the paired-domination number for trees. Kang, Soh, and Cheng [6] presented a linear time algorithm for inflated graphs.

In this paper, we consider the paired-domination problem for series-parallel graphs. It is known that many NP-hard problems can be efficiently solved for series-parallel graphs [11]. Kikuno, Yoshida, and Kakuda [7] gave a linear time algorithm for finding a minimum dominating set in a series-parallel graph. Linear time algorithms for solving the problems of independent domination and total domination [8], weighted perfect domination [12] for series-parallel graphs were presented. Efficient edge domination problem for series-parallel graphs were studied by Grinstead, Slater, Sherwani and Holmes [3].

2 Series-Parallel Graphs

2.1 Definition of series-parallel graphs

The class of (two terminal) series-parallel graphs can be defined in several ways. The following is a recursive definition.

Definition 2.1. *Series-parallel graphs are obtained only by the following recursive rules:*

1. *An edge xy is a series-parallel graph with terminals x and y.*

2. *Suppose that G_1 with terminals x and y, and G_2 with terminals z and w are series-parallel graphs.*

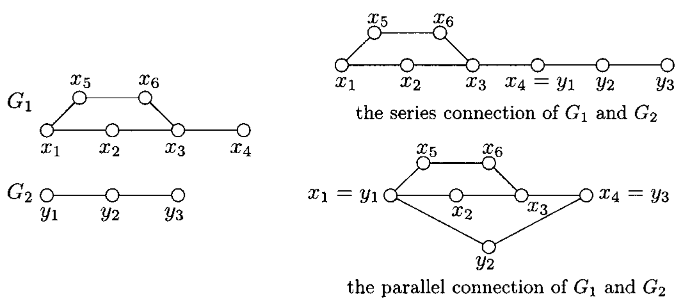


Figure 1: Two series-parallel graphs G_1 and G_2, and the series and parallel connection of them.

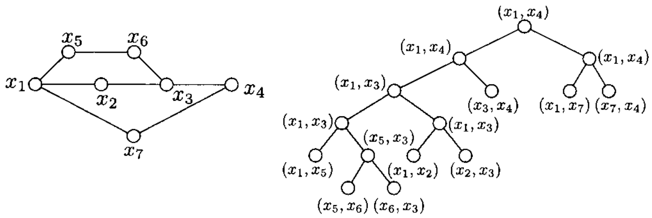


Figure 2: A binary decomposition tree of a series-parallel graph.

(a) *The* series connection *of G_1 and G_2 by identifying y with z is a series-parallel graph with terminals x and w.*

(b) *The* parallel connection *of G_1 and G_2 by identifying x with z, and y with w is a series-parallel graph with terminals x and y.*

Fig. 1 shows two series-parallel graphs G_1 and G_2, and their series and parallel connections.

In [5, 7], a linear time algorithm to recognize whether a given graph is a series-parallel graph was given. A *binary decomposition tree* T of a series-parallel graph $G = (V, E)$ is a binary tree in which there are $|E|$ leaf nodes and every internal node has exactly two children. Each edge $xy \in E$ corresponds to a leaf node of T which is labeled by (x, y). Each internal node of T corresponds to the series-parallel graph obtained from a series or parallel connection of its two children, and labeled by the terminals of the graph. The root of the tree T labeled (x, y) represents the series-parallel graph G. Fig. 2 shows a binary decomposition tree of a series-parallel graph.

2.2 Paired-dominating set in series-parallel graphs

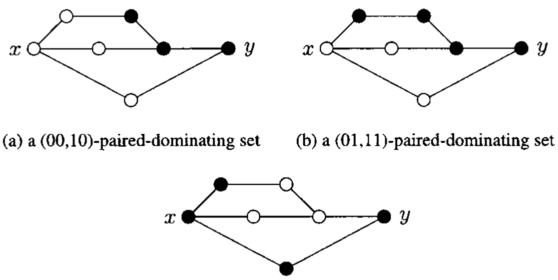
For technical reasons, we introduce a definition that is slightly modified by relaxing the restrictions of paired-domination with respect to terminals.

Let $G = (V, E)$ be a series-parallel graph with terminals x and y. For $\alpha, \beta \in \{00, 01, 10, 11\}$, a subset $D \subseteq V$ is an *(α, β)-paired-dominating set* if it is a dominating set of $G - A$ for $A \subseteq \{x, y\}$, where

- $x \in A$ and $N_G[x] \cap D = \emptyset$ iff $\alpha = 00$,
- $x \notin D$ and $x \notin A$ iff $\alpha = 01$,
- $x \in D$ if and only iff $\alpha = 10$ or 11,
- $y \in A$ and $N_G[y] \cap D = \emptyset$ iff $\beta = 00$,
- $y \notin D$ and $y \notin A$ iff $\beta = 01$,
- $y \in D$ iff $\beta = 10$ or 11,

and the induced subgraph $\langle D \setminus B \rangle$ for $B \subseteq \{x, y\} \setminus A$ has a perfect matching, where

- $x \in B$ iff $\alpha = 10$,
- $y \in B$ iff $\beta = 10$.



(a) a (00,10)-paired-dominating set (b) a (01,11)-paired-dominating set

(c) a (11,11)-paired-dominating set

Figure 3: Examples of (α, β)-paired-dominating sets D (vertices in D are colored in black).

The (α, β)*-paired-domination number* $\gamma_{\text{pr}}(G, \alpha, \beta)$ of G is the minimum cardinality of an (α, β)-paired-dominating set. If there is no (α, β)-paired-dominating set for given α and β, we define $\gamma_{\text{pr}}(G, \alpha, \beta) = \infty$.

That is, a terminal does not have to be dominated by a vertex in D if $\alpha = 00$ or $\beta = 00$. Also, if a terminal is in D, it does not have to have a paired vertex if $\alpha = 10$ or $\beta = 10$. For example, (α, β)-paired-dominating sets are shown in Fig. 3.

Example. Let G be a series-parallel graph shown in Fig. 3.

- Consider the set of vertices colored in black in Fig. 3(a). In this figure, $N_G[x] \cap D = \emptyset$, and the induced subgraph $\langle D \rangle - \{y\}$ has a perfect matching. Hence it is a $(00, 10)$-dominating set. Since it is a minimum $(00, 10)$-dominating set, we have $\gamma_{\text{pr}}(G, 00, 10) = 3$.
- The dominating set in Fig. 3(b) is a minimum $(01, 11)$-dominating set, and thus $\gamma_{\text{pr}}(G, 01, 11) = 4$.
- The dominating set in Fig. 3(c) is a minimum $(11, 11)$-dominating set, and thus $\gamma_{\text{pr}}(G, 11, 11) = 4$.
- G has no $(00, 00)$-dominating set, and hence $\gamma_{\text{pr}}(G, 00, 00) = \infty$.

It should be noted that an (α, β)-paired-dominating set is a paired-dominating set if and only if $\alpha, \beta \in \{01, 11\}$.

3 Algorithm

In this section, we describe an algorithm for determining the paired-domination number of a given series-parallel graph. Our algorithm calculates the (α, β)-paired-domination number of the series-parallel graphs corresponding to the nodes in a binary decomposition tree from leaf node to the root.

Informally, the values of α and β imply the following assertions:

- $\alpha = 00$ means that x is not dominated by any vertex.
- $\alpha = 01$ means that $x \notin D$ and is already dominated.
- $\alpha = 10$ means that $x \in D$, but it needs to be dominated by a vertex outside to make a perfect matching.
- $\alpha = 11$ means that $x \in D$ and it already has a paired vertex.

Lemma 3.1. *Let G be a series-parallel graph consisting of two vertices x and y and an edge between them. Then*

$$\begin{aligned}
\gamma_{\text{pr}}(G,00,00) &= 0, & (1)\\
\gamma_{\text{pr}}(G,01,10) &= 1, & (2)\\
\gamma_{\text{pr}}(G,10,01) &= 1, & (3)\\
\gamma_{\text{pr}}(G,10,10) &= 2, & (4)\\
\gamma_{\text{pr}}(G,11,11) &= 2, & (5)\\
\gamma_{\text{pr}}(G,\alpha,\beta) &= \infty, \textit{ for other } \alpha \textit{ and } \beta
\end{aligned}$$

The dominating set corresponds to (1) is the empty set, (2) is $\{y\}$, (3) is $\{x\}$, (4) and (5) are $\{x, y\}$.

Lemma 3.2. *Suppose that G_1 is the series-parallel graph with terminals x and y, and G_2 with terminals z and w. Let G be the series connection of G_1 and G_2 by identifying y and z. Then*

$$\gamma_{\text{pr}}(G,\alpha,\beta) = \min\left\{\begin{array}{l}
\gamma_{\text{pr}}(G_1,\alpha,00)+\gamma_{\text{pr}}(G_2,01,\beta),\\
\gamma_{\text{pr}}(G_1,\alpha,01)+\gamma_{\text{pr}}(G_2,00,\beta),\\
\gamma_{\text{pr}}(G_1,\alpha,01)+\gamma_{\text{pr}}(G_2,01,\beta),\\
\gamma_{\text{pr}}(G_1,\alpha,10)+\gamma_{\text{pr}}(G_2,11,\beta)-1,\\
\gamma_{\text{pr}}(G_1,\alpha,11)+\gamma_{\text{pr}}(G_2,10,\beta)-1,
\end{array}\right\}. \qquad (6)$$

Proof. Let D be a minimum (α,β)-paired-dominating set in G and $D_1 = D \cap V(G_1)$ and $D_2 = D \cap V(G_2)$. Then either $y = z \in D$ or $y = z \notin D$.

Assume that $y = z \in D$. So, $y \in D_1$ and $z \in D_2$. If D_1 is an $(\alpha, 10)$-paired-dominating set in G_1, then y has no paired vertex in D_1. Since $\langle D \rangle$ has a perfect matching, the terminal z of G_2 must have paired vertex in D_2. Hence D_2 is a $(11,\beta)$-paired-dominating set. Clearly $|D| = |D_1| + |D_2| - 1$, and D_1 and D_2 are a minimum $(\alpha, 10)$- and $(11, \beta)$-paired-dominating sets, respectively. Hence we obtain $\gamma_{\text{pr}}(G,\alpha,\beta) = \gamma_{\text{pr}}(G_1,\alpha,10) + \gamma_{\text{pr}}(G_2,11,\beta) - 1$. By considering the case that D_1 is an $(\alpha, 11)$-paired-dominating set, we show the equation $\gamma_{\text{pr}}(G,\alpha,\beta) = \gamma_{\text{pr}}(G_1,\alpha,11) + \gamma_{\text{pr}}(G_2,10,\beta) - 1$ holds by a similar argument.

Next assume that $y = z \notin D$. Then D_1 is either an $(\alpha, 00)$- or an $(\alpha, 01)$-paired-dominating set. If D_1 is an $(\alpha, 00)$-paired-dominating set, then D_2 must be a $(01,\beta)$-paired-dominating set. Hence we have $\gamma_{\text{pr}}(G,\alpha,\beta) = \gamma_{\text{pr}}(G_1,\alpha,00) + \gamma_{\text{pr}}(G_2,01,\beta)$. If D_1 is an $(\alpha, 01)$-paired-dominating set, then D_2 is either $(00,\beta)$- or a $(01,\beta)$-paired-dominating set. It is similarly shown that the remaining equations hold. □

Lemma 3.3. *Suppose that G_1 is a series-parallel graph with terminals x and y, and G_2 with terminals* *z and w. Let G be the parallel connection of G_1 and G_2 by identifying x with z, and y with w. Then*

$$\gamma_{\rm pr}(G,00,00) = \gamma_{\rm pr}(G_1,00,00)+\gamma_{\rm pr}(G_2,00,00), \tag{7}$$
$$\gamma_{\rm pr}(G,00,01) = \min_{k+l\geq 1}\{\gamma_{\rm pr}(G_1,00,0k)+\gamma_{\rm pr}(G_2,00,0l)\}, \tag{8}$$
$$\gamma_{\rm pr}(G,00,10) = \gamma_{\rm pr}(G_1,00,10)+\gamma_{\rm pr}(G_2,00,10)-1, \tag{9}$$
$$\gamma_{\rm pr}(G,00,11) = \min_{k+l=1}\{\gamma_{\rm pr}(G_1,00,1k)+\gamma_{\rm pr}(G_2,00,1l)-1\}, \tag{10}$$
$$\gamma_{\rm pr}(G,01,00) = \min_{i+j\geq 1}\{\gamma_{\rm pr}(G_1,0i,00)+\gamma_{\rm pr}(G_2,0j,00)\}, \tag{11}$$
$$\gamma_{\rm pr}(G,01,01) = \min_{i+j\geq 1,k+l\geq 1}\{\gamma_{\rm pr}(G_1,0i,0k)+\gamma_{\rm pr}(G_2,0j,0l)\}, \tag{12}$$
$$\gamma_{\rm pr}(G,01,10) = \min_{i+j\geq 1}\{\gamma_{\rm pr}(G_1,0i,10)+\gamma_{\rm pr}(G_2,0j,10)\}, \tag{13}$$
$$\gamma_{\rm pr}(G,01,11) = \min_{i+j\geq 1,k+l=1}\{\gamma_{\rm pr}(G_1,0i,1k)+\gamma_{\rm pr}(G_2,0j,1l)-1\}, \tag{14}$$
$$\gamma_{\rm pr}(G,10,00) = \gamma_{\rm pr}(G_1,10,00)+\gamma_{\rm pr}(G_2,10,00)-1, \tag{15}$$
$$\gamma_{\rm pr}(G,10,01) = \min_{k+l\geq 1}\{\gamma_{\rm pr}(G_1,10,0k)+\gamma_{\rm pr}(G_2,10,0l)-1\}, \tag{16}$$
$$\gamma_{\rm pr}(G,10,10) = \gamma_{\rm pr}(G_1,10,10)+\gamma_{\rm pr}(G_2,10,10)-2, \tag{17}$$
$$\gamma_{\rm pr}(G,10,11) = \min_{k+l=1}\{\gamma_{\rm pr}(G_1,10,1k)+\gamma_{\rm pr}(G_2,10,1l)-2\}, \tag{18}$$

$$\gamma_{\rm pr}(G,11,00) = \min_{i+j=1}\{\gamma_{\rm pr}(G_1,1i,00)+\gamma_{\rm pr}(G_2,1j,00)-1\}, \tag{19}$$
$$\gamma_{\rm pr}(G,11,01) = \min_{i+j=1,k+l\geq 1}\{\gamma_{\rm pr}(G_1,1i,0k)+\gamma_{\rm pr}(G_2,1j,0l)-1\}, \tag{20}$$
$$\gamma_{\rm pr}(G,11,10) = \min_{i+j=1}\{\gamma_{\rm pr}(G_1,1i,10)+\gamma_{\rm pr}(G_2,1j,10)-2\}, \tag{21}$$
$$\gamma_{\rm pr}(G,11,11) = \min_{i+j=1,k+l=1}\{\gamma_{\rm pr}(G_1,1i,1k)+\gamma_{\rm pr}(G_2,1j,1l)-2\}. \tag{22}$$

Proof. In this proof, only the value of α is considered. Let D be a minimum (α,β)-paired-dominating set in G and $D_1 = D\cap V(G_1)$ and $D_2 = D\cap V(G_2)$. Then either $x=z\in D$ or $x=z\notin D$.

(Case 1) $\alpha = 00$. In this case, $x\notin D_1$ and $z\notin D_2$ and the two vertices has no adjacent vertex in D. Hence D_1 and D_2 are $(00,\beta_1)$- and $(00,\beta_2)$-paired-dominating set of G_1 and G_2, respectively, for appropriate β_1 and β_2.

(Case 2) $\alpha = 01$. In this case, $x\notin D_1$ and $z\notin D_2$ and these are dominated by D. Suppose that D_1 is $(00,\beta_1)$-paired-dominating set of G_1 for some β_1. Then z must be dominated by a vertex in D_2. Hence D_2 is a $(01,\beta_2)$-paired-dominating set for some β_2. If D_1 is $(01,\beta_1)$-paired-dominating set of G_1, then D_2 is either a $(00,\beta_2)$- or $(01,\beta_2)$-paired-dominating set.

(Case 3) $\alpha = 10$. In this case, $x\in D_1$ and $z\in D_2$ and the two vertices has no paired vertex in D. Hence D_1 and D_2 are $(10,\beta_1)$- and $(10,\beta_2)$-paired-dominating set of G_1 and G_2, respectively, for appropriate β_1 and β_2.

(Case 4) $\alpha = 11$. In this case, $x\in D_1$ and $z\in D_2$. Suppose that D_1 is $(10,\beta_1)$-paired-dominating set of G_1 for some β_1. Then z must have paired vertex in D_2. Hence D_2 is a $(11,\beta_2)$-paired-dominating set for some β_2. Similarly, if D_1 is a $(11,\beta_1)$-paired-dominating set of G_1, then D_2 is a $(10,\beta_2)$-paired-dominating set.

From the discussions of the above four cases, we can show the sixteen equations (7)-(22) hold. □

If (α,β)-paired-domination numbers for each α and β is calculated, we can obtain the paired-dominating number. The following lemma is clearly holds.

Lemma 3.4. *For a series parallel graph G,*

$$\gamma_{\rm pr}(G) = \min\{\gamma_{\rm pr}(G, 01, 01), \gamma_{\rm pr}(G, 01, 11), \gamma_{\rm pr}(G, 11, 01), \gamma_{\rm pr}(G, 11, 11)\}.$$

Based on Lemma 3.1, 3.2, 3.3 and 3.4, we now describe an algorithm for computing paired-domination number of a series-parallel graph in Algorithm 1 and 2. These algorithms compute a binary decomposition tree of a given series-parallel graph, and then calculate the (α, β)-paired-domination numbers for each α and β recursively by using equations in Lemma 3.1, 3.2, and 3.3. Finally, after the (α, β)-paired-domination numbers of the entire series-parallel graph are obtained, the algorithm computes the paired-domination number by Lemma 3.4, and outputs it.

Algorithm 1 MIN-PD-SP(G)

```
Input: A series-parallel graph G
Output: The paired-domination number γpr(G)
  Construct a binary decomposition tree T of G
  T* ← T {T* is the root of T}
  gp ← CALC-PARTIAL-PD(T*) {described in Algorithm 2}
  return min{gp[01,01], gp[01,11], gp[11,01], gp[11,11]}
```

Algorithm 2 CALC-PARTIAL-PD(T^*)

```
Input: a node of a binary decomposition tree T;
Output: two dimensional array gp[α, β] stores the (α, β)-paired-domination numbers
  if T* is a leaf node then
    gp[00,00] ← 0
    gp[01,10] ← 1
    gp[10,01] ← 1
    gp[10,10] ← 2
    gp[11,11] ← 2
    return gp
  else
    gp1 ← CALC-PARTIAL-PD(left-child of T*)
    gp2 ← CALC-PARTIAL-PD(right-child of T*)
    if T* is the series connection of its children then
      foreach α, β ∈ {00, 01, 10, 11} do
        calculate gp[α, β] from gp1 and gp2 by applying the equation (6) in Lemma 3.2
      end for
    else
      foreach α, β ∈ {00, 01, 10, 11} do
        calculate gp[α, β] from gp1 and gp2 by applying the equations (7)-(22) in Lemma 3.3
      end for
    end if
    return gp
  end if
```

Theorem 3.5. *Algorithm MIN-PD-SP computes the paired-domination number of the given series-parallel graph with n vertices in time $O(n)$.*

Proof. The construction of the binary decomposition tree T takes $O(n)$ time. For leaf nodes and internal nodes of T, the (α, β)-domination number for the node is calculated in $O(1)$ time. Since the number of nodes in T is $2|E|-1$ and any series-parallel graph has at most $3n-6$ edges, the time complexity of our algorithm needs $O(n)$ time. □

4 Conclusion

Our algorithm described in Section 3 computes the paired-domination number of a series-parallel graph in linear time. This algorithm is easily modified so that it outputs a minimum paired-dominating set instead of the paired-domination number. Studying the paired-domination in k-trees, or the class of perfect graphs are also interesting.

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