正多角形領域に対するオンライン追跡問題

藤原 洋志 関西学院大学 h-fujiwara@ksc.kwansei.ac.jp
米澤 弘毅 北海道大学 yonezawa@meme.hokudai.ac.jp
岩間 一雄 京都大学 iwama@kuis.kyoto-u.ac.jp

概要 我々は動かすべきサーバが 1 個しかないサーバ配置問題を考察する。もしサーバが要求点そのものに移動しなければならないのであれば選択の余地がないので問題として成立しない。しかし、サーバの移動を要求点そのものでなく、その「近く」でも良いとするなら話は違ってくる。つまり要求に対して一定の領域を対応させ、サーバはその領域内のいずれかに移動させれば良いとする（オンライン追跡問題)。本稿では領域が正 n 角形の場合に注目する。まず貪欲アルゴリズムの解析を行い、競合比が $O(n)$ であることを示す。次に幾つかの入力例に対する仕事関数アルゴリズムの性能評価を行い、適切な設定の下で定数競合比を達成すると予想する。

Online Chasing Problems for Regular Polygons

Hiroshi Fujiwara, Kwansei Gakuin University
Kouki Yonezawa, Hokkaido University
Kazuo Iwama, Kyoto University

Abstract. We consider a server location problem with only one server to move. If each request must be served on the exact position, there is no choice for the online player and the problem is trivial. In *the online chasing problem* a request is given as a region such that the service can be done anywhere inside. Namely, for each request an online algorithm chooses an arbitrary point in the region and moves the server there. In this paper we focus on a regular n-gon. We prove that the greedy algorithm is $O(n)$-competitive. We also give a preliminary observation of the work function algorithm and conjecture that it achieves a constant competitive ratio with an appropriate setting of the parameter.

1 Introduction

In the k-server problem the player manages k servers by changing their location so that at least one of them serves the request at each time step [9]. In this paper we study a 1-server problem on the xy-plane. One may think that if the player owns only one server, there is no choice in server location and therefore the competitive ratio is always one. This is obviously true if the player has to move the server to the exact position of the request. However, if it is enough to move it somewhere near the request, then the player does have a choice. Namely, in *the online chasing problem*, each request has a certain region such that the request can be served if the server moves or stays inside the region. The single server can choose an arbitrary point in the region in order to reduce the total travel distance. Natural applications include allocation of a relay broadcasting car to follow up consecutive incidents and that of a taxi in an old city with severe traffic restrictions.

Friedman and Linial first studied this problem and proved that there exists a competitive online algorithm for the online chasing problem for any convex region [6]. They began with a line chasing and extended the analysis to a half plane and general convex bodies. However, they were not interested in any specific shape of the region or specific values of the competitive ratio (only such a result is an upper bound of 28.53 for the competitive ratio of the line chasing). The problem has not appeared in the literature since then.

Our Contribution. In this paper we focus our attention on a regular n-gon (without rotation) as the shape of the region and investigate how we can take advantage of the information of the specific shape. It is shown that: (i) The greedy algorithm (GRD, which moves the server to the nearest position in the request region from the previous position) works quite well for a small n. For instance, it achieves competitive ratios of 2, $\sqrt{2}$, and 3.24 for a regular triangle, a square, and a regular pentagon, respectively. Especially for a square region, the ratio equals the lower bound for an arbitrary convex region [6], that is, GRD is optimal. Our result for a general n is $1/\sin\frac{\pi}{2n}$ for odd n and $1/\sin\frac{\pi}{n}$ for even n. Our analysis is tight, namely, there are request sequences for which the competitive ratio of GRD coincides the above values. (ii) We give a preliminary observation of the work function algorithm WFA_α, which minimizes the weighted sum of the distance to the next position and the length of the optimal tour that terminates at the next position. Please note that the competitive ratio of GRD is $O(n)$ for a regular n-gon. We consider some input sequences which seem to be cruel against WFA_α and conjecture that WFA_α achieves a constant competitive ratio by setting the parameter α appropriately.

Related Work. In a broader sense, the CNN problem [8, 11] can be regarded as a special case of the online chasing problem; One can just set the region $\{(x,y) \mid x = r_x \text{ or } y = r_y\}$ for a request (scene) on $\mathrm{R}(r_x, r_y)$. Although the upper bound remains still large, [7] showed that with a nontrivial restriction the competitive ratio decreases to 9. Chrobak and Sgall provided a 5-competitive work function algorithm for the weighted 2-server problem which corresponds to a special case of the CNN problem [5]. [1] illustrated the application of the work function algorithm to some problems belonging to metrical service systems (MMS). It is also famous that the work function algorithm is $(2N-1)$-competitive and optimal for general metrical task systems (MTS) where the size of the space is N [4]. See [2] for the recent progress in the k-server problem.

2 Greedy Algorithm

In the online chasing problem a request $\mathrm{R}_i = (x_i, y_i)$ is given somewhere on the xy-plane at each time step $i = 1, 2, \ldots, m$. Then an online algorithm ALG sets the sole server on a point A_i in region D_i that is associated with R_i. For an input sequence $\sigma = (\mathrm{R}_1, \mathrm{R}_2, \ldots, \mathrm{R}_m)$, the cost of ALG is defined as

$$ALG(\sigma) = \overline{\mathrm{SA}_1} + \sum_{i=2}^{m} \overline{\mathrm{A}_{i-1}\mathrm{A}_i}, \tag{1}$$

where S is the initial location of the server and $\overline{\mathrm{A}_{i-1}\mathrm{A}_i}$ denotes the Euclidean distance between A_{i-1} and A_i. The offline problem, i.e., minimization of $\overline{\mathrm{SA}_1} + \sum_{i=2}^{m} \overline{\mathrm{A}_{i-1}\mathrm{A}_i}$ subject to $\mathrm{A}_i \in D_i$ given in advance for all i, is solved in polynomial time if every region D_i is convex [10]. In this paper we let region D_i be the union of a regular polygon with centroid R_i and its interior. The polygon does not rotate and therefore we can assume without loss of generality that its bottom side is always parallel to the x-axis. We set the length of sides in the polygon as one. It should be noticed that the size of the polygon does not matter to the online competitiveness. We use the definition of *the competitive ratio* as in [12, 3]: The competitive ratio of an online algorithm ALG is c if there exists a constant b such that, for all input sequences σ, $ALG(\sigma) - c \cdot OPT(\sigma) \leq b$, where OPT is an optimal offline algorithm. We call the value of $ALG(\sigma)/OPT(\sigma)$ for some specific σ *the cost ratio.*

The greedy algorithm for the online chasing problem is defined as below. For a regular polygon one can easily see that there are two types on the server's behavior: The server arrives on one of the sides after vertical movement against that side, or on one of the vertices.

Algorithm GRD: For each request i, (i) if the server's previous position A_{i-1} is not in D_i, then move the server to $\mathrm{X} \in D_i$ such that minimizes $\overline{\mathrm{A}_{i-1}\mathrm{X}}$. (ii) Otherwise, do not move the server.

Lemma 1. *For a regular n-gon ($n \geq 3$) the competitive ratio of* GRD *is at most $1/\sin\frac{\pi}{2n}$ for odd n and $1/\sin\frac{\pi}{n}$ for even n.*

Proof. We begin with an odd n. Let us investigate the behavior of GRD and an arbitrary offline algorithm OFF for an input sequence σ. We write the positions of GRD's and OFF's server as $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_m$ and $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_m$, respectively. Since it may occur for some of the requests that GRD moves the server while OFF does not, a simple piecewise comparison does not work. Therefore, we must adopt some potential function for an amortized analysis. One can observe that the server of GRD always approaches the server of OFF when OFF does not change the position. So some kind of distance between the two servers appears to help, but it turns out that simple ones are insufficient. Our choice of the potential function is

$$\Phi(x,y) = \sum_{k=0}^{n-1} \left| x \sin\frac{2k\pi}{n} + y \cos\frac{2k\pi}{n} \right|, \tag{2}$$

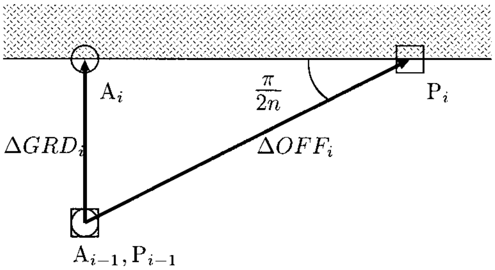

Figure 1: Case where E is maximized (n odd.)

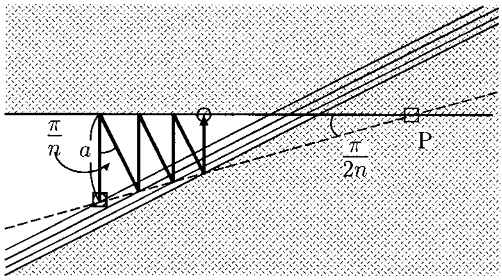

Figure 2: GRD for σ_1 (n odd.)

where (x, y) is the displacement from the position of OFF's server to that of GRD's. $\Phi(x, y)$ represents the total length of all projections of vector (x, y) to the normal of each side of the n-gon. We use the notation of ΔGRD_i and ΔOFF_i as the cost of GRD to serve the i-th request and that of OFF, respectively, and Φ_i as the value of $\Phi(x, y)$ immediately after processing the i-th request. We can obtain a competitive ratio as a positive constant of c such that

$$E := \Delta GRD_i + \Phi_i - \Phi_{i-1} - c \cdot \Delta OFF_i \leq 0 \tag{3}$$

for $2 \leq i \leq m$. If OFF does not move the server for the i-th request (that is $P_{i-1} = P_i$), GRD's server goes closer to P_i by traveling no more than $(\Phi_{i-1} - \Phi_i)$. Thus (3) holds if $\Delta OFF_i = 0$. As for $\Delta OFF_i > 0$, without loss of generality, we assume that OFF stops the server immediately after reaching region D_i. One can see that the value of E is maximized to

$$\Delta GRD_i + \frac{2\cos^2 \frac{\pi}{2n}}{\sin \frac{\pi}{n}} \sqrt{\Delta OFF_i^2 - \Delta GRD_i^2} - c \cdot \Delta OFF_i, \tag{4}$$

when GRD moves the server from P_{i-1} to A_i that is on the same side as P_i by a vertical movement with the side. Note that (4) is derived from $\Phi(\lambda x, \lambda y) = |\lambda| \cdot \Phi(x, y)$ and $\Phi(1, 0) = \sum_{k=0}^{n-1} \left|\sin \frac{2k\pi}{n}\right| = 2\sum_{k=1}^{\frac{n-1}{2}} \sin \frac{2k\pi}{n} = \frac{2\cos^2 \frac{\pi}{2n}}{\sin \frac{\pi}{n}}$. Besides, for such a GRD's action, (4) is maximized to

$$\left(1 - \sqrt{c^2 + 1 - \frac{1}{\sin^2 \frac{\pi}{2n}}}\right) \cdot \Delta GRD_i \tag{5}$$

when $\Delta GRD_i : \Delta OFF_i = \sqrt{c^2 - \left(2\cos^2 \frac{\pi}{2n} / \sin \frac{\pi}{n}\right)^2} : c$. (5) is non-positive if $c \geq 1/\sin \frac{\pi}{2n}$. (If $c = 1/\sin \frac{\pi}{2n}$, E is maximized to zero when $\angle A_{i-1}P_iA_i = \frac{\pi}{2n}$ as shown in Figure 1.)

For an even n we make a similar analysis applying $\Phi'(x, y) = \sum_{k=0}^{n/2-1} \left|x \sin \frac{2k\pi}{n} + y \cos \frac{2k\pi}{n}\right|$ as a potential function instead of Φ. Φ' represents the total length of projections of vector (x, y) to the normal of each side as well. However, it is sufficient to sum up half of projections since there are $\frac{n}{2}$ pairs of parallel sides in an even n-gon. As a result, it is obtained that the competitive ratio is at most $1/\sin \frac{\pi}{n}$ for even n. □

In particular for $n = 4$, the lemma below implies that GRD is an optimal online algorithm.

Lemma 2 ([6]). *There exists no online algorithm whose competitive ratio is smaller than $\sqrt{2}$ for any convex region on $\mathbf{R}^2$.*

Although GRD may not be optimal for other polygons, we show that the analysis in Lemma 1 is tight.

Lemma 3. *For a regular n-gon ($n = 3$ or $n \geq 5$) the competitive ratio of* GRD *is at least $1/\sin \frac{\pi}{2n}$ for odd n and $1/\sin \frac{\pi}{n}$ for even n.*

Table 1: Competitive ratio of GRD for regular n-gons.

n	3	4	5	6	7	8	9	10
Competitive ratio	2.00	1.41	3.24	2.00	4.49	2.61	5.76	3.24

Sketch of proof. For odd n, consider input sequence σ_1 consisting of $\mathrm{R}_i(\frac{1}{2}, \frac{1}{2\tan\frac{\pi}{n}})$ for odd i's and $\mathrm{R}_i(\frac{3}{4} + \frac{a}{\tan\frac{\pi}{2n}}(1-\cos^{i/2}\frac{\pi}{n}), -\frac{1}{2\sin\frac{\pi}{n}} + \frac{1}{4}\tan\frac{\pi}{n})$ for even i's and the server's initial location $\mathrm{S}_1(\frac{1}{2} - \frac{a}{\tan\frac{\pi}{n}}, -a)$ for $0 < a < \frac{1}{2}\sin\frac{\pi}{n}$. It is seen that the server of GRD moves drawing a zigzag between two sides and approaches $\mathrm{P}(\frac{1}{2} + \frac{a}{\sin\frac{\pi}{n}}, 0)$ (see Figure 2.) On the other hand, the optimal offline algorithm moves the server directly to P. The reason why the lower polygon slightly slides is to maximize the cost ratio. Consequently, we have $GRD(\sigma_1) = 2a\left(1 + \cos\frac{\pi}{n} + \cos^2\frac{\pi}{n} + \cdots\right) \to 2a/(1-\cos\frac{\pi}{n})$ and $OPT(\sigma_1) = a/\sin\frac{\pi}{2n}$. For even n, consider the input sequence σ_1' of $\mathrm{R}_i(\frac{1}{2}, \frac{1}{2\tan\frac{\pi}{n}})$ for odd i's and $\mathrm{R}_i(1 + \frac{a}{\tan\frac{\pi}{2n}}(1-\cos^{i/2}\frac{\pi}{n}), -\frac{1}{2\sin\frac{\pi}{n}})$ for even i's and the server's initial location $\mathrm{S}_1'(\frac{1}{2} - \frac{a'}{\tan\frac{2\pi}{n}}, -a')$ for $0 < a' < \frac{1}{2}\sin\frac{2\pi}{n}$. □

Theorem 1. *For the online chasing problem the tight competitive ratio of the greedy algorithm is $1/\sin\frac{\pi}{2n}$ for odd n and $1/\sin\frac{\pi}{n}$ for even n, if the request region is a regular n-gon ($n \geq 3$). Especially for the case of a square, the greedy algorithm is an optimal online algorithm.*

3 Work Function Algorithms

We showed that the competitive ratio of GRD is $O(n)$ for a regular n-gon. In this section we attempt to improve the performance by applying *the work function algorithm*, which plays a significant role for server location problems [5, 1] as well as for metrical task systems (MTS) [4]. For the online chasing problem the work function $w_i(\mathrm{X})$ is defined as the length of the optimal tour that terminates at $\mathrm{X} \in D_i$ for request i, that is, the minimum cost of serving all requests $1, 2, \ldots, i-1$, and i finally reaching the position X. Formally, given $\mathrm{X} \in D_i$,

$$w_i(\mathrm{X}) = \min_{\mathrm{B}_j \in D_j, 1 \leq j \leq i-1} \left(\overline{\mathrm{SB}_1} + \sum_{j=2}^{i-1} \overline{\mathrm{B}_{j-1}\mathrm{B}_j} + \overline{\mathrm{B}_{i-1}\mathrm{X}} \right). \tag{6}$$

For a given $\alpha \geq 0$, the work function algorithm is defined as follows.
Algorithm WFA$_\alpha$: For each request i, (i) if the server's previous position A_{i-1} is not in D_i, then move the server to $\mathrm{X} \in D_i$ such that minimizes $\overline{\mathrm{A}_{i-1}\mathrm{X}} + \alpha \cdot w_i(\mathrm{X})$. (ii) Otherwise, do not move the server.

Note that the case of $\alpha = 0$ is equivalent to GRD. Also we define the retrospective-greedy algorithm, which is so called in [3]. One can see that the behavior of WFA$_\alpha$ approaches that of this algorithm as $\alpha \to \infty$.
Algorithm RTR: For each request i, (i) if the server's previous position A_{i-1} is not in D_i, then move the server to $\mathrm{X} \in D_i$ such that minimizes $w_i(\mathrm{X})$. (ii) Otherwise, do not move the server.

We consider the following input sequences σ_2 and σ_3 (and also σ_3' for even n) for WFA$_\alpha$ with different settings of the parameter α. Intuitively speaking, WFA$_\alpha$ with a large α is weak against σ_2 since the end point of the optimal tour jumps far for each request. For the sequence σ_3, WFA$_\alpha$ with a small α performs badly. Note that σ_3 is almost the same as σ_1 in the proof of Lemma 3, except that the lower polygon does not slide.

- Input sequence σ_2: $\mathrm{R}_i(\frac{1}{2} + \frac{i-1}{2} \cdot (1 - \cos\frac{2\pi}{n}), \frac{1}{2\tan\frac{\pi}{n}} + \frac{i-1}{2}\sin\frac{\pi}{n})$ for odd i's and $\mathrm{R}_i(\frac{1}{2} - \cos\frac{2\pi}{n} + \frac{i-1}{2} \cdot (1 - \cos\frac{2\pi}{n}), \frac{1}{2\tan\frac{\pi}{n}} + \frac{i}{2}\sin\frac{\pi}{n})$ for even i's. The server's initial location $\mathrm{S}_2(-\frac{\sin\frac{2\pi}{n}}{2\tan\frac{\pi}{n}}, -\frac{\cos\frac{2\pi}{n}}{2\tan\frac{\pi}{n}})$.
- Input sequence σ_3 for odd n: $\mathrm{R}_i(\frac{1}{2}, \frac{1}{2\tan\frac{\pi}{n}})$ for odd i's and $\mathrm{R}_i(\frac{1}{2}, -\frac{1}{2\sin\frac{\pi}{n}})$ for even i's. The server's initial location $\mathrm{S}_3(\frac{1}{2} - \frac{a}{\tan\frac{\pi}{n}}, -a)$ for $0 < a < \sin\frac{\pi}{n}$.

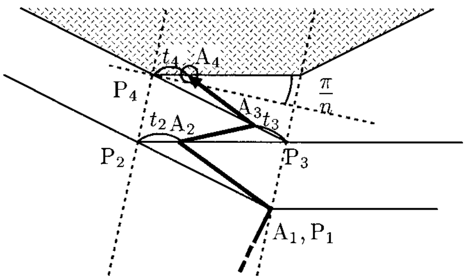

Figure 3: WFA_α for σ_2.

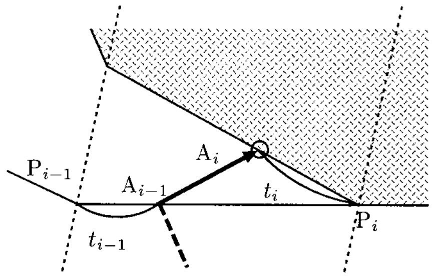

Figure 4: WFA_α for a request in σ_2.

- Input sequence σ_3' for even n: $\mathrm{R}_i(\frac{1}{2}, \frac{1}{2\tan\frac{\pi}{n}})$ for odd i's and $\mathrm{R}_i(1, -\frac{1}{2\sin\frac{\pi}{n}})$ for even i's. The server's initial location $\mathrm{S}_3'(\frac{1}{2} - \frac{a'}{\tan\frac{2\pi}{n}}, -a')$ for $0 < a' < \sin\frac{2\pi}{n}$.

Proposition 1. *Let $n \geq 5$ and the server start from $\mathrm{S} = \mathrm{S}_2$. Then, the cost ratio $\frac{WFA_\alpha(\sigma_2)}{OPT(\sigma_2)}$ approaches*

$$\begin{cases} \dfrac{\sqrt{1-\alpha^2\sin^2\frac{\pi}{n}}}{\cos\frac{\pi}{n} - \alpha\sin^2\frac{\pi}{n}\left(\alpha\cos\frac{\pi}{n} + \sqrt{1-\alpha^2\sin^2\frac{\pi}{n}}\right)}, & 0 \leq \alpha < \frac{\cos\frac{2\pi}{n}}{\sin\frac{\pi}{n}}; \\ \dfrac{1}{\sin\frac{\pi}{n}}, & \frac{\cos\frac{2\pi}{n}}{\sin\frac{\pi}{n}} \leq \alpha. \end{cases} \tag{7}$$

Proof. Without loss of generality we denote the server's position immediately after serving the i-th request by $\mathrm{A}_i(\frac{i-1}{2}\cdot(1-\cos\frac{2\pi}{n}) - t_i\cos\frac{2\pi}{n}, \frac{i-1}{2}\sin\frac{\pi}{n} - t_i\sin\frac{2\pi}{n})$ for odd i's and $\mathrm{A}_i(\frac{i-1}{2}\cdot(1-\cos\frac{2\pi}{n}) - \cos\frac{2\pi}{n} + t_i, \frac{i}{2}\sin\frac{\pi}{n})$ for even i's (see Figure 3). Note that $0 \leq t_i \leq 1$ holds for all i since the length of a side is one. For any $\varepsilon > 0$, there exist i_0 and $\{W_i\}$ such that for all $i \geq i_0$,

$$W_{i-1} + x\sin\frac{\pi}{n} < w_i(\mathrm{X}) < W_{i-1} + x\sin\frac{\pi}{n} + \frac{\varepsilon}{\alpha}, \tag{8}$$

where X is the position given by substituting x for t_i in A_i. Suppose that WFA_α just completes processing the $(i-1)$-th request. By definition, WFA_α chooses the next position X that minimizes $\overline{\mathrm{A}_{i-1}\mathrm{X}} + \alpha\cdot w_i(\mathrm{X})$. We have $f(x, t_{i-1}) < \overline{\mathrm{A}_{i-1}\mathrm{X}} + \alpha\cdot w_i(\mathrm{X}) < f(x, t_{i-1}) + \varepsilon$, where

$$f(x,t) := \sqrt{x^2 + (1-t)^2 - 2x(1-t)\cdot\cos\frac{2\pi}{n}} + \alpha\cdot\left(W_{i-1} + x\cdot\sin\frac{\pi}{n}\right). \tag{9}$$

(See Figure 4.) In what follows we also use

$$g(x,t) := \frac{\partial}{\partial x}f(x,t) = \frac{x - (1-t)\cdot\cos\frac{2\pi}{n}}{\sqrt{x^2 + (1-t)^2 - 2x(1-t)\cdot\cos\frac{2\pi}{n}}} + \alpha\sin\frac{\pi}{n}. \tag{10}$$

Suppose that $0 \leq \alpha < \cos\frac{2\pi}{n}/\sin\frac{\pi}{n}$. Consider the sequence $\{s_i\}$ $(i \geq i_0)$ defined by $s_{i_0} = t_{i_0}$ and s_i is the value of $x \in [0,1]$ that minimizes $f(x, s_{i-1})$ for $i \geq i_0 + 1$. We write it as $s_i = h(s_{i-1})$. Let γ denote the unique root of the equation $g(\gamma,\gamma) = 0$ in $(0, 1/(\cos^2\frac{\pi}{n})]$. Firstly, we show that s_i converges to γ. Since $g(x,t)$ is a monotonically increasing function with respect to x, the unique x that minimizes $f(x, s_{i-1})$ is the root of $g(x, s_{i-1}) = 0$ in $(0, 1/(\cos^2\frac{\pi}{n})]$. The condition $0 \leq \alpha < \cos\frac{2\pi}{n}/\sin\frac{\pi}{n}$ guarantees the existence. Suppose that $s_{i-1} < \gamma$. The mean value theorem implies that there exists $\beta \in (s_{i-1}, \gamma)$ such that $s_i - \gamma = h(s_{i-1}) - h(\gamma) = h'(\beta)\,(s_{i-1} - \gamma)$. Since $0 = \frac{d}{ds}g(h(s), s) = \frac{\partial g}{\partial x}h'(s) + \frac{\partial g}{\partial t}$, we have $h'(s) = -\frac{\partial g}{\partial t}/\frac{\partial g}{\partial x} = -\frac{h(s)}{1-s}$. It turns out that $h(s)/(1-s) < \cos\frac{2\pi}{n}$ since $h(s)$ is chosen so as to minimize $f(x,s)$. (Otherwise the both terms in $f(x,s)$ become larger.) We have $|s_i - \gamma| < \cos\frac{2\pi}{n}\cdot|s_{i-1} - \gamma|$. Also for the case of $s_{i-1} > \gamma$ the same statement holds. Therefore, $s_i \to \gamma$ as $i \to \infty$.

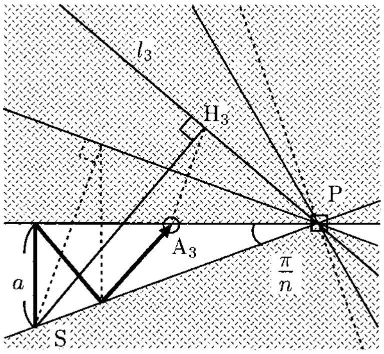

Figure 5: RTR achieves a constant competitive ratio for σ_3.

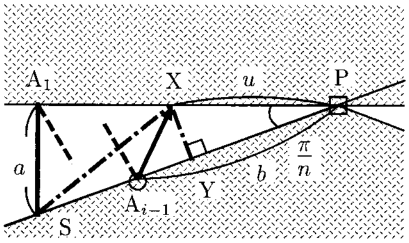

Figure 6: WFA$_\alpha$ for σ_3.

Secondly, we prove that $\{t_i\}$ approaches $\{s_i\}$ as $i \to \infty$. We can choose $\delta > 0$ such that $f(x,t) - f(\gamma,t) < \varepsilon$ for all x satisfying $|x-\gamma| < \delta$ and all t. Therefore $|t_{i_0+1} - s_{i_0+1}| < \delta$. Since $|\frac{\partial f}{\partial t}| < \exists r < 1$, we have $|t_{i_0+2} - s_{i_0+2}| < \delta + r\delta$ by a similar discussion as above. Thus for any $i \geq i_0$, $|t_i - s_i| < \delta + r\delta + r^2\delta + \cdots < \delta \cdot \frac{1}{1-r}$.

Finally, we calculate ΔWFA_α and ΔOPT when $i \to \infty$, i.e. the costs per one request, whose ratio approaches the ratio of the whole cost ratio $WFA_\alpha(\sigma_2)/OPT(\sigma_2)$. By formally substituting γ for both t_i and t_{i-1} in $\Delta WFA_\alpha = \overline{\mathrm{A}_{i-1}\mathrm{A}_i} = \sqrt{t_i^2 + (1-t_{i-1})^2 - 2t_i(1-t_{i-1}) \cdot \cos\frac{2\pi}{n}}$, we have

$$\Delta WFA_\alpha \to \frac{\sin\frac{\pi}{n}\sqrt{1-\alpha^2\sin^2\frac{\pi}{n}}}{\cos\frac{\pi}{n} - \alpha\sin^2\frac{\pi}{n}\left(\alpha\cos\frac{\pi}{n} + \sqrt{1-\alpha^2\sin^2\frac{\pi}{n}}\right)}. \tag{11}$$

On the other hand ΔOPT converges to $\sin\frac{\pi}{n}$ since the optimal algorithm moves from S directly to $\mathrm{P}_i(\frac{i-1}{2}\cdot(1-\cos\frac{2\pi}{n}), \frac{i-1}{2}\sin\frac{\pi}{n})$ for odd i or $\mathrm{P}_i(\frac{i-1}{2}\cdot(1-\cos\frac{2\pi}{n}) - \cos\frac{2\pi}{n}, \frac{i}{2}\sin\frac{\pi}{n})$ for even i. The proposition is thus proved for $0 \leq \alpha < \cos\frac{2\pi}{n}/\sin\frac{\pi}{n}$.

The remaining is the case that $\alpha \geq \cos\frac{2\pi}{n}/\sin\frac{\pi}{n}$. We focus on $i \geq i_0$ such that (8) holds and apply a similar analysis. Since $g(0,t) \geq 0$ and $g(x,t)$ is a monotonically increasing function, $f(x,t)$ turns out to be a non-decreasing function with respect to x. Therefore $f(x,t)$ attains minimum at $x=0$. By choosing $\delta > 0$ such that $f(x,t) - f(0,t) < \varepsilon$ for all x satisfying $0 < x < \delta$ and all t, we have $0 < t_i < \delta$ for all $i \geq i_0$. Therefore $\Delta WFA_\alpha = \overline{\mathrm{A}_{i-1}\mathrm{A}_i} \to 1$, which derives the proposition for $\alpha \geq \cos\frac{2\pi}{n}/\sin\frac{\pi}{n}$. □

Proposition 2. *Suppose that $n \geq 5$, $\alpha = k\cdot\cos\frac{2\pi}{n}/\sin\frac{\pi}{n}$, and $0 < k < 1$. Then,*

$$\frac{\sqrt{1-\alpha^2\sin^2\frac{\pi}{n}}}{\cos\frac{\pi}{n} - \alpha\sin^2\frac{\pi}{n}\left(\alpha\cos\frac{\pi}{n} + \sqrt{1-\alpha^2\sin^2\frac{\pi}{n}}\right)} < \frac{1}{\cos\frac{\pi}{n}\sqrt{1-k^2}}. \tag{12}$$

Proof. A simple calculation by applying $0 < \sin\theta < \theta$ and $1 - \theta^2/2 < \cos\theta < 1$ for all $0 < \theta \leq \pi/5$. □

Proposition 3. *For odd $n \geq 3$, consider the input sequence σ_3 and the server's initial location $\mathrm{S} = \mathrm{S}_3$. For even $n \geq 6$, consider the input sequence σ_3' and the server's initial location $\mathrm{S} = \mathrm{S}_3'$. The cost ratios $\frac{RTR(\sigma_3)}{OPT(\sigma_3)}$ and $\frac{RTR(\sigma_3')}{OPT(\sigma_3')}$ are no larger than $\frac{\pi}{2}$.*

Proof. We discuss odd n. Let $\{l_i\}$ be the family of lines l_i: $y = -\tan\frac{(i-1)\pi}{n}\cdot\left(x - \frac{1}{2}\right)$. The location of RTR's server immediately after serving the i-th request ($i \leq \frac{n-1}{2}$) is obtained as $\mathrm{A}_i(\frac{1}{2} - \overline{\mathrm{PH}}_i, 0)$ for odd i

and $A_i(\frac{1}{2} - \overline{PH}_i \cos\frac{\pi}{n}, -\overline{PH}_i \sin\frac{\pi}{n})$ for even i, where P is $P(\frac{1}{2}, 0)$ and H_i is the foot of the perpendicular from S to line l_i. The reason is as follows: Note that RTR moves the server to the terminal point of the (thus far) optimal tour to process the i-th request. Consider reflections of the plane through lines l_k $(1 \le k \le i-1)$ on the tour from S to H. One can observe that each H_i represents the image of A_i (see Figure 5). Thus RTR's server travels a distance of $\overline{H_{i-1}H_i} = a$ for each request until the $(\frac{n-1}{2})$-th request, since every H_i is on the circle whose diameter is SP. Finally, the server arrives on P for the $(\frac{n+1}{2})$-th request by traveling a distance of $a \cdot \cos\left(\frac{n-1}{2} \cdot \frac{\pi}{n}\right) / \sin\frac{\pi}{n}$. The optimal offline cost is $a/\sin\frac{\pi}{n}$ after the $(\frac{n+1}{2})$-th request. We have the cost ratio

$$\frac{RTR(\sigma_3)}{OPT(\sigma_3)} = \left(\frac{n-1}{2} + \frac{\cos\frac{n-1}{2}\cdot\frac{\pi}{n}}{\sin\frac{\pi}{n}}\right) \cdot \sin\frac{\pi}{n} < \frac{\pi}{2} \tag{13}$$

for all odd n. As for even n and σ_3' and S_3', RTR's server moves a total distance of $a \cdot \frac{n}{2}$, whereas the optimal offline cost is $a/\sin\frac{2\pi}{n}$ after the $(\frac{n}{2})$-th request. An upper bound of $\frac{\pi}{2}$ holds for $\frac{RTR(\sigma_3')}{OPT(\sigma_3')}$ as well. □

Proposition 4. *For odd $n \ge 5$, consider the input sequence σ_3 and the server's initial location $S = S_3$. For even $n \ge 6$, consider the input sequence σ_3' and the server's initial location $S = S_3'$. For any $\alpha > 0$ the cost ratios $\frac{WFA_\alpha(\sigma_3)}{OPT(\sigma_3)}$ and $\frac{WFA_\alpha(\sigma_3')}{OPT(\sigma_3')}$ are no larger than $1 + \frac{1}{\alpha}$.*

Proof. We focus on odd n since the case of even n can be shown by a similar discussion. Consider the optimal tour $S, B_1, B_2, \ldots, B_{i-1}, X$ that finally reaches X after serving requests $1, 2, \ldots, i-1$. Suppose that i is odd and X is somewhere on A_1P (see Figure 6). Y is the foot of the perpendicular from X to line SP. By applying the discussion in the proof of Proposition 3, one can observe that there exists i_1 such that $B_1 = B_3 = \cdots = B_{i-2} = X$ and $B_2 = B_4 = \cdots = B_{i-1} = Y$ for all $i \ge i_1$. Hence, the work function is written as

$$w_i(X) = \overline{SX} + (i-1)\cdot\overline{XY} = \sqrt{a^2 + \left(\frac{a}{\tan\frac{\pi}{n}} - u\right)^2} + (i-1)\cdot u \cdot \sin\frac{\pi}{n}, \tag{14}$$

where u is $\overline{XP}$ and a is $\overline{SA_1}$. WFA_α chooses the next position X as A_i that minimizes

$$f(u) := \overline{A_{i-1}X} + \alpha \cdot w_i(X) = \sqrt{u^2 + b^2 - 2ub\cos\frac{2\pi}{n}} + \alpha \cdot w_i(X), \tag{15}$$

where b is $\overline{A_{i-1}P}$. We have

$$f'(u) := \frac{u - b\cos\frac{2\pi}{n}}{\sqrt{u^2 + b^2 - 2ub\cos\frac{2\pi}{n}}} - \alpha \cdot \frac{\frac{a}{\tan\frac{\pi}{n}} - u}{\sqrt{a^2 + \left(\frac{a}{\tan\frac{\pi}{n}} - u\right)^2}} + \alpha\cdot(i-1)\cdot\sin\frac{\pi}{n}. \tag{16}$$

Since $f'(u)$ increases monotonically, $f(u)$ is a non-decreasing function if $f'(0) \ge 0$, for which it suffices that $i \ge 1 + (1+\alpha)/(\alpha\tan\frac{\pi}{n}) =: i_2$. $f(u)$ achieves minimum at $u = 0$ and therefore $A_i = P$ for $i \ge i_2$. Thus $WFA_\alpha(\sigma_3) < a \cdot (i_2 - 1)$. On the other hand $OPT = a/\sin\frac{\pi}{n}$ and it concludes the proof. □

Proposition 5. *For odd $n \ge 3$, consider the input sequence σ_3 and the server's initial location $S = S_3$. For even $n \ge 6$, consider the input sequence σ_3' and the server's initial location $S = S_3'$. The cost ratios $\frac{GRD(\sigma_3)}{OPT(\sigma_3)}$ and $\frac{GRD(\sigma_3')}{OPT(\sigma_3')}$ approach $1/\tan\frac{\pi}{2n}$ and $1/\tan\frac{\pi}{n}$, respectively.*

Proof. Apply a similar analysis to the proof of Lemma 3. □

The performance results of WFA_α are summarized in Table 2. Proposition 2 implies that the cost ratio has an upper bound for σ_2 when $0 \le \alpha < \cos\frac{2\pi}{n}/\sin\frac{\pi}{n}$. We conjecture that by setting $0 \le \alpha < \cos\frac{2\pi}{n}/\sin\frac{\pi}{n}$, WFA_α achieves a constant competitive ratio. However, the analysis for general input sequences seems much more difficult.

Table 2: Competitive ratio of WFA_α for some input sequences.

	σ_2	σ_3	general
$\alpha = 0$ (i.e. GRD)	$O(1)$	$O(n)$	$O(n)$
$0 \leq \alpha < \cos\frac{2\pi}{n} / \sin\frac{\pi}{n}$	$O(1)$	$O(1)$	?
$\alpha \geq \cos\frac{2\pi}{n} / \sin\frac{\pi}{n}$	$O(n)$	$O(1)$	$\Omega(n)$
$\alpha \to \infty$ (i.e. RTR)	$O(n)$	$O(1)$	$\Omega(n)$

4 Concluding Remarks

Apparently many problems remain to be attacked, including (i) more formal analysis of WFA_α for general inputs and (ii) investigation of other shapes than regular polygons, especially a circle and some simple shape which is not convex.

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