対称二部グラフのマッチング構造

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概要

対称二部グラフとは，二つの頂点集合を分ける軸で線対称性を持つ二部グラフである．本発表では，対称な二部グラフのマッチング構造を調べることを目的とする．まず対称な二部グラフに対し Dulmage-Mendelsohn 分解を行なうと，得られた成分 (マッチング被覆グラフ) は対称性を有することが分かる．マッチング被覆二部グラフは，耳分解により，ある辺に長さ奇数のパスを繰り返し付け加えることで構成できる．本研究では，マッチング被覆二部グラフが対称ならば高々２本のパスを付け加えることで対称性を保持したまま耳分解できることを示す．さらに，対称二部グラフの組合せ的行列理論への応用として，Pólya の問題を一般化した問題を提案し，その問題が多項式可解であることを述べる．

Matching Structure of Symmetric Bipartite Graphs

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Abstract

A bipartite graph is said to be *symmetric* if it has symmetry of reflecting two vertex sets. This paper investigates matching structure of symmetric bipartite graphs. We first apply the *Dulmage-Mendelsohn decomposition* to a symmetric bipartite graph. The resulting components, which are matching-covered, turn out to have symmetry. We then decompose a matching-covered bipartite graph via an *ear decomposition*, which is a sequence of subgraphs obtained by adding an odd-length path repeatedly. We show that, if a matching-covered bipartite graph is symmetric, an ear decomposition can retain symmetry by adding no more than two paths.

As an application of these decompositions to combinatorial matrix theory, we present a natural generalization of Pólya's problem. We introduce the problem of deciding whether a rectangular matrix has a signing that is totally sign-nonsingular or not, where a rectangular matrix is *totally sign-nonsingular* if the sign of the determinant of each submatrix with row size is uniquely determined by the signs of the nonzero entries. We show that this problem can be solved in polynomial time with the aid of the matching structure of symmetric bipartite graphs.

1 Introduction

Let $G = (U, V; E)$ be a simple bipartite graph with two disjoint vertex sets $U = \{u_1, \ldots, u_m\}$, $V = \{v_1, \ldots, v_n\}$, and edge set $E \subseteq U \times V$. A bipartite graph $G = (U, V; E)$ with $|U| = |V|$ is said to be *symmetric* if $(u_j, v_i) \in E$ holds for any $(u_i, v_j) \in E$. A symmetric bipartite graph is associated with a combinatorially symmetric matrix [16], where a square matrix $A = (a_{ij})$ of order n is said to be *combinatorially symmetric* if $a_{ij} \neq 0$ implies $a_{ji} \neq 0$ for any two distinct indices i, j. Combinatorially symmetric matrices were studied in the contexts of matrix completion problems [7] and qualitative matrix theory [8, 10, 24, 26]. Another work related to symmetric bipartite graphs is given by Gabow [5]. He discussed an upper degree-constrained partial orientation of graphs, which can be viewed as the problem of finding a maximum subgraph G' with degree constraints in a symmetric bipartite graph such that G' has at most one edge of (u_i, v_j) and (u_j, v_i) for any indices i, j.

For a bipartite graph $G = (U, V; E)$, an edge subset $M \subseteq E$ is a *matching* if no two edges in M share a common vertex incident to them. A matching is *perfect* if $|M| = |U| = |V|$. For an edge subset $F \subseteq E$, we denote by $F^\top = \{(u_j, v_i) \mid (u_i, v_j) \in F\}$ the *transpose* of F. The matching structure of a symmetric bipartite graph has symmetry, since M is a matching if and only if so is $M^\top$. This paper aims at investigating decompositions related to the matching structure of symmetric bipartite graphs.

We first deal with the *Dulmage-Mendelsohn decomposition* (*DM-decomposition*) [3, 4]. We say that a connected graph is *matching-covered* if every edge is contained in some perfect matching. The DM-decomposition is a unique decomposition of a bipartite graph with respect to the maximum matchings, which yields the matching-covered subgraphs and the remaining subgraphs. The subgraphs obtained by the DM-decomposition are called the *DM-components*. We show that, if a bipartite graph is symmetric, then each DM-component is the transpose of some DM-component, where the *transpose* of a subgraph $H = (U, V; F)$ is the subgraph $H^\top = (U, V; F^\top)$. A subgraph $H = (U, V; F)$ is called *symmetric* if $F = F^\top$. Our result means that a symmetric bipartite graph can be assembled from symmetric matching-covered subgraphs and pairs of subgraphs whose union is symmetric.

Each of DM-components, i.e., a matching-covered bipartite graph, is characterized by the *ear decomposition* [15]. An elementary path P of odd length is an *ear* of a subgraph G' if G' contains both of the end vertices of P, but no interior vertices and no edges. We denote by $G' + P$ the subgraph obtained from G' by adding an ear P. For a subgraph G' of a graph G, an *ear decomposition starting from* G' is a sequence $G_0, G_1, \dots, G_k$ of subgraphs such that $G_0 = G'$, $G_k = G$, and $G_i = G_{i-1} + P_i$ for some ear P_i of G_{i-1} for $i = 1, \dots, k$. It is known that a bipartite graph has an ear decomposition starting from an edge if and only if it is matching-covered.

Assume that a matching-covered bipartite graph G is symmetric. The symmetry of G motivates us to find an ear decomposition having symmetry. Unfortunately, G does not always have an ear decomposition in which every subgraph is itself symmetric. In fact, the complete bipartite graph with two vertex sets of size three has no such ear decomposition. Thus we may have to add more than one ears to maintain symmetry in an ear decomposition. We will see, however, that we can retain symmetry by adding no more than two ears. An ear decomposition $G_0, G_1, \dots, G_k$ starting from G_0 is said to be *symmetric* if one of two consecutive subgraphs is symmetric, i.e., G_{l-1} or G_l is symmetric for $l = 1, \dots, k$. We show that, if G is symmetric, G has a symmetric ear decomposition starting from an edge or a crossing pair, where a *crossing pair* is a pair of edges $(u_i, v_j) \in E$ and $(u_j, v_i) \in E$ for some distinct $i, j \in N$. In addition, we describe a linear-time algorithm for finding a symmetric ear decomposition of a matching-covered symmetric graph with a perfect matching.

As an application of these decompositions to combinatorial matrix theory, we discuss a generalization of Pólya's problem. A square matrix is said to be *term-nonsingular* if the determinant has a nonzero expansion term. A term-nonsingular matrix is *sign-nonsingular* if all nonzero expansion terms of the determinant have the same sign. For a $\{0, 1\}$-matrix A, a *signing* of A is a $\{0, \pm 1\}$-matrix obtained from A by replacing some ones with minus ones. *Pólya's problem* is the problem of deciding whether a given square $\{0, 1\}$-matrix has a sign-nonsingular signing or not. Such a sign-nonsingular signing is called a *Pólya matrix*. Pólya's problem has a plenty of polynomial-time equivalent problems [1, 11, 15, 17, 21]. Robertson, Seymour, and Thomas [20] devised a polynomial-time algorithm for Pólya's problem. Excellent surveys on Pólya's problem can be found in [18, 25].

An $m \times n$ matrix with $m \leq n$ is said to be *totally sign-nonsingular* if each term-nonsingular submatrix of order m is sign-nonsingular. Totally sign-nonsingular matrices play an important role in the sign-solvability of linear systems of equations [2, 12, 13, 23], linear programming [6], and linear complementarity problems [9]. Total sign-nonsingularity can be recognized in polynomial time by testing sign-nonsingularity of the related symmetric matrix [6].

In this paper, we introduce the problem of deciding whether a rectangular $\{0, 1\}$-matrix has a totally sign-nonsingular signing or not. If a matrix is term-nonsingular, this problem is in fact Pólya's problem. It follows from [6] that this problem can be reduced to the problem of deciding whether the related symmetric matrix has a symmetric Pólya matrix with positive diagonals or not. We show that a symmetric Pólya matrix with a nonzero diagonal entry can be obtained in polynomial time with the aid of the DM-decomposition and ear decomposition for symmetric bipartite graphs. Thus a totally sign-nonsingular signing can be found in polynomial time.

Before closing this section, we give some definitions and notations. For an $m \times n$ matrix $A = (a_{ij})$, we define the associated bipartite graph $G(A) = (U, V; E)$ with vertex sets $U = \{u_1, \ldots, u_m\}$, $V = \{v_1, \ldots, v_n\}$, and edge set $E = \{(u_i, v_j) \mid a_{ij} \neq 0, u_i \in U, v_j \in V\}$. Then A is combinatorially symmetric if and only if $G(A)$ is symmetric. A matrix A is term-nonsingular if and only if $G(A)$ has a perfect matching.

Let $G = (U, V; E)$ be a bipartite graph. For vertex subsets $I \subseteq U$ and $J \subseteq V$, we denote by $G[I, J]$ the subgraph induced by vertex subsets I and J. For a subgraph H, we denote by $U(H)$ and $V(H)$ the sets of vertices in H belonging to U and V, respectively, and by $E(H)$ the set of edges in H. Let $G \setminus H$ be the graph obtained from G by deleting $U(H)$ and $V(H)$ together with edges incident to them. For an edge subset $F \subseteq E$, we denote by $U(F)$ and $V(F)$ the set of the end vertices of F which belong to U and V, respectively. For a matching M, we say that a path P of G is *M-alternating* if the elements of P alternate between elements of M and $E \setminus M$ along P. For two edge subsets F_1 and F_2, the *symmetric difference* $(F_1 \setminus F_2) \cup (F_2 \setminus F_1)$ is denoted by $F_1 \triangle F_2$. Notice that, for an M-alternating path P with a matching M, the symmetric difference $M \triangle E(P)$ is also a matching.

This paper is organized as follows. Section 2 discusses the DM-decomposition of symmetric bipartite graphs. In Section 3, we present the ear decomposition of matching-covered symmetric bipartite graphs. Sections 4 and 5 describe applications of results in Sections 2 and 3. Section 4 discusses Pólya matrices of combinatorially symmetric matrices. In Section 5, we introduce the problem of a totally sign-nonsingular signing of a rectangular matrix.

2 DM-Decomposition of Symmetric Bipartite Graphs

In this section, we discuss the symmetry of the DM-components of a symmetric bipartite graph.

We first review the Dulmage-Mendelsohn decomposition of a bipartite graph following the exposition in [19]. Let $G = (U, V; E)$ be a bipartite graph with $W = U \cup V$. A pair (I, J) of $I \subseteq U$ and $J \subseteq V$ is said to be a *cover* if no edges exist between $U \setminus I$ and $V \setminus J$. The *size* of a cover (I, J) is defined to be $|I| + |J|$. It is well-known that the maximum size of matchings is equal to the minimum size of covers. For convenience, we define the *cut function* $\kappa : 2^W \to \mathbb{Z} \cup \{+\infty\}$ as $\kappa(X) = |U \setminus X| + |V \cap X|$ if $(U \setminus X, V \cap X)$ is a cover, and $\kappa(X) = +\infty$ otherwise. Note that $\kappa(X)$ is finite if and only if $(U \setminus X, V \cap X)$ is a cover. The function κ satisfies submodularity, i.e.,

$$\kappa(X) + \kappa(Y) \geq \kappa(X \cap Y) + \kappa(X \cup Y), \quad \forall X, Y \subseteq W.$$

The set of minimizers of a submodular function forms a distributive lattice. Hence there exist unique minimal and maximal minimizers.

Let $\mathcal{L}$ be the set of minimizers of κ. Take a maximal ascending chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k$ in $\mathcal{L}$, where k is a nonnegative integer, and X_0 and X_k are the unique minimal and maximal minimizers, respectively. We put

$$\begin{aligned} W_0 &= X_0, \\ W_l &= X_l \setminus X_{l-1}, \quad l = 1, \ldots, k, \\ W_\infty &= W \setminus X_k. \end{aligned} \tag{1}$$

The family of the difference sets $\{W_l \mid l = 0, 1, \ldots, k, \infty\}$ is uniquely determined independently of the choice of the chain by a Jordan-Hölder type theorem. Define a partial order $\preceq$ on $\{W_l \mid l = 1, \ldots, k\}$ by

$$W_h \preceq W_l \iff [W_l \subseteq X \in \mathcal{L} \Rightarrow W_h \subseteq X].$$

Moreover, we extend this partial order on $\{W_0, W_\infty\} \cup \{W_l \mid l = 1, \ldots, k\}$ by defining

$$W_0 \preceq W_l \preceq W_\infty, \quad l = 1, \ldots, k.$$

The pair of $\{W_l \mid l = 0, 1, \ldots, k, \infty\}$ and $\preceq$ defined above is called the *Dulmage-Mendelsohn decomposition* of G. Let $U_l = W_l \cap U$ and $V_l = W_l \cap V$ for $l = 0, 1, \ldots, k, \infty$. The subgraphs $G[U_l, V_l]$ ($l = 0, 1, \ldots, k, \infty$) are called the *DM-components*. Note that the subgraph $G[U_h, V_l]$ has no edges for $0 \leq l < h \leq \infty$.

We say that a bipartite graph with nonempty vertex set is *DM-irreducible* if it cannot be decomposed into more than one nonempty component via the DM-decomposition. Suppose that a bipartite graph with no vertices is DM-irreducible. Assume that $|U| \leq |V|$. Since the DM-irreducibility means that $\mathcal{L}$ contains no proper subsets of W, the graph G is DM-irreducible if and only if $\kappa(X) \geq |U| + 1$ for any nonempty proper subset $X \subset W$. Thus a bipartite graph $G = (U, V; E)$ with $|U| = |V|$ is DM-irreducible if and only if it is matching-covered.

We now obtain the following theorem for a symmetric bipartite graph. For a vertex subset $X \subseteq W$, we denote $X^\top = \{v_i \in V \mid u_i \in X \cap U\} \cup \{u_i \in U \mid v_i \in X \cap V\}$.

Theorem 2.1. *Let $G = (U, V; E)$ be a symmetric bipartite graph, and $(\{W_l \mid l = 0, 1, \ldots, k, \infty\}, \preceq)$ be the DM-decomposition obtained by a maximal ascending chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k$ in $\mathcal{L}$. Then the DM-decomposition satisfies the followings.*

(1) *For each of DM-component* $G[U_l, V_l]$ $(l = 0, 1, \ldots, k, \infty)$*, there exists a DM-component* $G[U_h, V_h]$ *which is the transpose of* $G[U_l, V_l]$.

(2) *If* $W_l = W_l^\top$ *and* $W_h = W_h^\top$*, then there exists no partial order between* W_l *and* W_h.

Proof. Since G is symmetric, $(U \setminus X, V \cap X)$ is a cover if and only if so is $(U \cap X^\top, V \setminus X^\top)$. Hence $\kappa(X) = \kappa(W \setminus X^\top)$ holds for any $X \subseteq W$. This implies that $X \in \mathcal{L}$ if and only if $W \setminus X^\top \in \mathcal{L}$. Hence $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k$ is a maximal ascending chain in $\mathcal{L}$ if and only if $W \setminus X_k^\top \subsetneq W^\top \setminus X_{k-1}^\top \subsetneq \cdots \subsetneq W^\top \setminus X_0^\top$ is that in $\mathcal{L}$. As (1), this ascending chain in $\mathcal{L}$ yields the partition $\{W_l' \mid l = 0, 1, \ldots, k, \infty\}$ of W:

$$
\begin{aligned}
W_0' &= W_\infty^\top, \\
W_l' &= (W \setminus X_{l-1}^\top) \setminus (W \setminus X_l^\top) = W_l^\top, \quad l = 1, \ldots, k, \\
W_\infty' &= W \setminus (W \setminus X_0^\top) = W_0^\top.
\end{aligned}
$$

By a Jordan-Hölder type theorem, this coincides with $\{W_l \mid l = 0, 1, \ldots, k, \infty\}$. Therefore, for each DM-component $G[U_l, V_l]$ $(l = 0, 1, \ldots, k, \infty)$, the subgraph $G[V_l^\top, U_l^\top]$ is also a DM-component of G, where $V_l^\top = W_l^\top \cap U$ and $U_l^\top = W_l^\top \cap V$. Thus the statement (1) holds.

To prove (2), we may assume that $l \leq h$. The set X_l in the ascending chain satisfies that $W_l \subseteq X_l$ and $X_l \cap W_h = \emptyset$ by (1). By $W_l = W_l^\top$ and $W_h = W_h^\top$, it holds that the minimizer $W \setminus X_l^\top$ does not include W_l, but includes W_h. Hence there exists no partial order between W_l and W_h by the definition of the partial order. □

The concept of the DM-decomposition is applied to matrices. Let A be a matrix and $G(A)$ be the associated bipartite graph. The *DM-decomposition of a matrix* A is the partition of rows and columns obtained by the DM-decomposition of $G(A)$. For $I \subseteq U$ and $J \subseteq V$, the submatrix corresponding to $G[I, J]$ is denoted by $A[I, J]$. Since $A[U_h, V_l] = O$ for $0 \leq l < h \leq \infty$, the matrix A can be rearranged into the block triangular matrix by row and column permutations. Thus the DM-decomposition of a matrix can be depicted as in Fig. 1.

Let A be a combinatorially symmetric matrix. It follows from Theorem 2.1 that the DM-decomposition of A can maintain symmetry. That is, for each DM-component $A[U_l, V_l]$ $(l = 0, 1, \ldots, k, \infty)$, the block submatrix $A[U_l, V_l]$ is symmetric, or $A[U_l \cup U_h, V_l \cup V_h]$ is symmetric for some $h \in \{0, 1, \ldots, k, \infty\}$. Thus a combinatorially symmetric matrix A has a permutation matrix S such that $S^\top A S$ is a block triangular matrix depicted

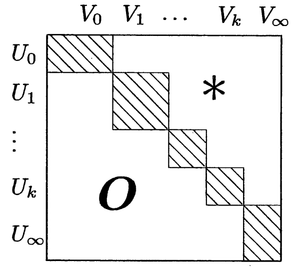

Figure 1: The DM-decomposition of a matrix

as in Fig. 2. The DM-decomposition can be computed efficiently with the aid of bipartite matching algorithms. Hence such a block triangular form of a combinatorially symmetric matrix can be obtained in polynomial time.

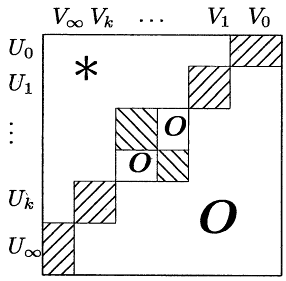

Figure 2: The DM-decomposition of a combinatorially symmetric matrix

3 Ear Structure of Matching-Covered Symmetric Graphs

In this section, we discuss ear decomposition of a matching-covered symmetric bipartite graph. Let $G = (U, V; E)$ be a matching-covered symmetric bipartite graph. Recall that an ear decomposition $G_0, G_1, \ldots, G_k$ is *symmetric* if G_l or G_{l+1} is symmetric for $l = 0, 1, \ldots, k-1$. A *diagonal edge* is an edge $(u_i, v_i) \in E$ for some $i \in N$. The main purpose of this section is to prove the following theorem.

Theorem 3.1. *Let* $G = (U, V; E)$ *be a matching-covered symmetric bipartite graph. Then* G *has a symmetric ear decomposition starting from an edge or a crossing pair. In particular, if* G *has a diagonal edge,* G *has a symmetric one starting from the diagonal edge.*

We say that a subgraph G' is *central* if $G \setminus G'$ has a perfect matching. In order to prove Theorem 3.1, we first show that, for any central symmetric subgraph G', there exist an ear P of G' and an ear Q of $G'+P$ such that $G'+P+Q$ is symmetric and central, where Q may be empty.

Let $G'=(U',V';E')$ be a central symmetric subgraph. If $U'=U$ and $V'=V$, then any diagonal edge and any crossing pair are the desired ears. Hence we may assume that $U' \subset U$ and $V' \subset V$. Let $\bar{G}'=G[U \setminus U', V \setminus V']$ be the remaining symmetric subgraph. Since G' is central, $\bar{G}'$ has a perfect matching M.

We first assume that $M=M^{\top}$ holds. Note that $M=M^{\top}$ implies that, if a path P is M-alternating, then so is $P^{\top}$. The graph G has an edge (u_i,v_j) for some $u_i \in U'$ and $v_j \notin V'$. Since G is matching-covered, G has a perfect matching M' with $(u_i,v_j) \in M'$. By taking $M \cup M'$, we obtain an M-alternating ear $\hat{P}$ of G' from u_i. If the inner vertices in $\hat{P}$ and $\hat{P}^{\top}$ are disjoint, then $\hat{P}^{\top}$ is an ear of $G'+\hat{P}$ and $G'+\hat{P}+\hat{P}^{\top}$ is symmetric. Hence we may assume that $\hat{P}$ and $\hat{P}^{\top}$ have a common inner vertex. This implies that $\hat{P}$ has an index $s \in N$ with $u_s \in U(\hat{P})$ and $v_s \in V(\hat{P})$ such that all vertices in P_{ss} have different indices, where P_{ss} is the path between u_s and v_s along $\hat{P}$. Among such s, we choose s such that the length of P_{is} is minimum, where P_{is} is the shorter one of the path from u_i to u_s along $\hat{P}$ and the path from u_i to v_s along $\hat{P}$. Define $P=P_{is} \cup P_{ss} \cup P_{is}^{\top}$, and Q to be empty if P_{ss} is a diagonal edge and $Q=P_{ss}^{\top}$ otherwise. Then P is an M-alternating ear of G', and, if Q is nonempty, Q is an M-alternating ear of $G'+P$. The subgraph $G'+P+Q$ has the edge set $E' \cup E(P_{is} \cup P_{ss}) \cup E((P_{is} \cup P_{ss})^{\top})$, and hence $G'+P+Q$ is symmetric. Moreover, since P and Q are M-alternating paths of odd length, $G'+P+Q$ is central.

Therefore, the following lemma holds. Note that, if Q is empty, then P has exactly one diagonal edge; and, otherwise, P and Q have no diagonal edges.

Lemma 3.2. *Let $G=(U,V;E)$ be a matching-covered symmetric bipartite graph, and $G'=(U',V';E')$ be a central symmetric subgraph. Assume that the remaining subgraph $\bar{G}'=G[U \setminus U', V \setminus V']$ has a perfect matching M with $M^{\top}=M$. Then there exist an ear P of G' and an ear Q of $G'+P$ such that $G'+P+Q$ is central and symmetric, where Q may be empty.*

We now discuss the case where M may not coincide with $M^{\top}$. For a bipartite graph $G=(U,V;E)$ with a matching M, we define *contracting an M-alternating circuit C to an edge (x,y)* as contracting $U(C)$ and $V(C)$ to vertices x and y, respectively, deleting resulting multiple edges, and replacing M with $M \setminus C \cup \{(x,y)\}$. The converse procedure is *expanding an edge to a circuit.* Note that, if G is matching-covered and M is a perfect matching of G, then the graph obtained by contracting an M-alternating circuit is also matching-covered.

Assume that $M \neq M^{\top}$. Then consider $M \cup M^{\top}$, which consists of diagonal edges, crossing pairs, pairs of asymmetric circuits, and symmetric circuits. By $M \neq M^{\top}$, the union $M \cup M^{\top}$ has pairs of asymmetric circuits, or symmetric circuits. For each pair of asymmetric circuits C and $C^{\top}$ in $M \cup M^{\top}$, replace M with $M \triangle E(C)$. Moreover, for each symmetric circuit C in $M \cup M^{\top}$, contract C to a diagonal edge e_C. Let F be the set of diagonal edges obtained by the contraction of all symmetric circuits in $M \cup M^{\top}$. The resulting graph G_* is symmetric and matching-covered, and G' is a central symmetric subgraph of G_*. Moreover, M is a perfect matching in $G_* \setminus G'$ with $M=M^{\top}$.

Therefore, it follows from Lemma 3.2 that G_* has an ear P_* of G' and an ear Q_* of $G'+P_*$ such that $G'+P_*+Q_*$ is symmetric and central, where Q_* may be empty. If P_* and Q_* have no edges in F, then $G'+P_*+Q_*$ is also a central symmetric subgraph of G. Assume that P_* has a diagonal edge e in F. Then Q_* is empty. We denote by C the contracted circuit corresponding to e. Since P_* has exactly one edge in F, the edge subset $P_* \setminus \{e\} \cup C$ forms an ear P of G' and an ear Q of $G'+P$ such that $G'+P+Q$ is symmetric and central.

By the above discussion, we obtain the following theorem.

Theorem 3.3. *Let G be a matching-covered symmetric bipartite graph, and G' be a central symmetric subgraph. Then there exist an ear P of G' and an ear Q of $G'+P$ such that $G'+P+Q$ is central and symmetric, where Q may be empty.*

For a symmetric bipartite graph with perfect matchings, Kakimura and Iwata [10] showed the following proposition.

Proposition 3.4 (Kakimura and Iwata [10]). *Let G be a symmetric bipartite graph with perfect matchings. If G is not a disjoint union of symmetric circuits, then G satisfies the following* (a) *or* (b).

(a) *The graph G has a perfect matching with a diagonal edge (u_i,v_i) for some $i \in N$.*
(b) *The graph G has a perfect matching with a crossing pair (u_i,v_j) and (u_j,v_i) for some distinct $i,j \in N$.*

Theorem 3.3, together with Proposition 3.4, implies Theorem 3.1.

Proof of Theorem 3.1. It is not difficult to see that a symmetric graph consisting of one circuit has a symmetric ear decomposition starting from an edge. Hence assume that G is not a circuit. It follows from Proposition 3.4 that G has a perfect matching with a diagonal edge or a crossing pair. Hence G has a central subgraph G_0 consisting of a diagonal edge or a crossing pair. By applying Theorem 3.3 repeatedly, we obtain an ear decomposition $G_0, G_1, \ldots, G_k = G$ such that G_l or G_{l+1} is symmetric for $l = 0, 1, \ldots, k-1$. □

The proof of Theorem 3.1 also leads to a linear-time algorithm for finding a symmetric ear decomposition.

Theorem 3.5. *Let $G = (U, V; E)$ be a matching-covered symmetric bipartite graph, and M' be a perfect matching in G. Then we can find a symmetric ear decomposition starting from an edge or a crossing pair in $\mathrm{O}(|E|)$ time.*

4 Symmetric Pólya Matrices with a Nonzero Diagonal Entry

In this section, we discuss Pólya matrices of combinatorially symmetric matrices as an application of the decompositions described in Sections 2 and 3.

Pólya's problem is equivalent to the problem of deciding whether a given bipartite graph has an orientation called *Pfaffian*. Let $G = (W, E)$ be a graph. An *orientation* $\vec{G}$ of G is a directed graph obtained from G by orienting its edges. For an orientation $\vec{G}$ of G, a circuit C of even length in G is said to be *oddly* (*evenly*) *oriented* in $\vec{G}$ if an odd (even) number of its edges are directed in the same direction along C. For a graph $G = (W, E)$, we say that an orientation of G is *Pfaffian* if every central circuit of even length is oddly oriented. For a square matrix A, it is known that A has a Pólya matrix if and only if $G(A)$ has a Pfaffian orientation. Robertson, Seymour, and Thomas [20] devised a polynomial-time algorithm to decide whether a given bipartite graph has a Pfaffian orientation (cf. McCuaig [18]).

Suppose that a bipartite graph $G = (U, V; E)$ with perfect matchings has Pfaffian orientations. We discuss constructing a Pfaffian orientation of G. We may assume that a bipartite graph $G = (U, V; E)$ is matching-covered, because G has a Pfaffian orientation if and only if so does each DM-component. Since G is matching-covered, G has an ear decomposition starting from an edge [15]. It is known that the following theorem holds, which implies a polynomial-time algorithm for finding a Pfaffian orientation.

Theorem 4.1 (Little [14], Seymour and Thomassen [22]). *Let G be a matching-covered bipartite graph which has Pfaffian orientations, and $G_0, G_1, \ldots, G_k = G$ be an ear decomposition starting from an edge with $G_l = G_{l-1} + P_l$ for $l = 1, \ldots, k$. Then an orientation is Pfaffian if and only if $C_1, \ldots, C_k$ are oddly oriented, where C_l is a central circuit of G_l which uses P_l for $l = 1, \ldots, k$.*

Let $G = (U, V; E)$ be a symmetric bipartite graph with perfect matchings. Suppose that G has a Pfaffian orientation. We discuss finding a symmetric Pfaffian orientation in G, where an orientation of a bipartite graph is *symmetric* if the two edges of any crossing pair are oriented in the same direction. Again, we may assume that G is matching-covered, because it follows from Theorem 2.1 that G has a symmetric Pfaffian orientation if and only if so does each symmetric DM-component and each non-symmetric DM-component has a Pfaffian orientation. Using Theorem 3.1, we have the following theorem.

Theorem 4.2. *Let $G = (U, V; E)$ be a matching-covered symmetric bipartite graph with a diagonal edge. If G has a Pfaffian orientation, then G has a symmetric one.*

Since a symmetric ear decomposition can be obtained in linear time, we can find a symmetric Pfaffian orientation in $\mathrm{O}(|E|)$ time.

Theorem 4.2 can be written as the following corollary in terms of a Pólya matrix. Recall that a square matrix A is DM-irreducible if $G(A)$ is matching-covered.

Corollary 4.3. *Let A be a DM-irreducible symmetric $\{0,1\}$-matrix with a nonzero diagonal entry. If A has a Pólya matrix, then A has a symmetric one.*

If A has no diagonal entries, then it is not necessarily true that A has a Pólya matrix which is symmetric. For example, consider the symmetric matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Then A has a Pólya matrix

$$\begin{pmatrix} 0 & +1 & +1 & +1 \\ +1 & 0 & -1 & +1 \\ -1 & -1 & 0 & +1 \\ +1 & -1 & +1 & 0 \end{pmatrix}.$$

However, A has no Pólya matrix which is symmetric. Indeed, if A has a Pólya matrix in the form of

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ a_1 & 0 & a_4 & a_5 \\ a_2 & a_4 & 0 & a_6 \\ a_3 & a_5 & a_6 & 0 \end{pmatrix},$$

where $a_1, \dots, a_6 \in \{1, -1\}$, then the determinant has nonzero expansion terms $a_1^2 a_6^2$, $-a_1a_3a_4a_6$, $-a_2a_3a_4a_5$, and $-a_1a_2a_5a_6$. Since these terms have the same sign, $a_1^2a_6^2 = -a_1a_3a_4a_6$ and $-a_2a_3a_4a_5 = -a_1a_2a_5a_6$ hold, which is a contradiction.

5 Totally Sign-Nonsingular Signing

Recall that an $m \times n$ rectangular matrix is totally sign-nonsingular if each term-nonsingular submatrix of order m is sign-nonsingular. In this section, we consider the problem of deciding whether a given rectangular $\{0, 1\}$-matrix has a totally sign-nonsingular signing or not. If a matrix is term-nonsingular, this problem is equivalent to Pólya's problem. We show the following theorem.

Theorem 5.1. *The following two problems are polynomially equivalent.*

(1) *Deciding whether a given square $\{0, 1\}$-matrix has a Pólya matrix or not (Pólya's problem).*
(2) *Deciding whether a given rectangular $\{0, 1\}$-matrix has a totally sign-nonsingular signing or not.*

For an $m \times n$ matrix A, we denote by A^* the square matrix having the form

$$A^* = \begin{pmatrix} O & A \\ A^\top & I \end{pmatrix},$$

where I is the identity matrix of order n. We call A^* the *augmented matrix of A*.

The following proposition asserts the equivalence between the total sign-nonsingularity of a matrix A and the sign-nonsingularity of A^*. A matrix A has *row-full term-rank* if A has a term-nonsingular submatrix with row size. If A does not have row-full term-rank A is clearly totally sign-nonsingular.

Proposition 5.2 (Iwata and Kakimura [6]). *Let A be a matrix with row-full term-rank. Then A is totally sign-nonsingular if and only if the augmented matrix A^* is sign-nonsingular.*

We say that two matrices A and A' with same size are *equivalent* if A' can be obtained from A by multiplying -1 to some rows and columns, that is, if there exist two $\{1, -1\}$-diagonal matrices D_r and D_c with $A' = D_r A D_c$. It is known in [14] that, if a square $\{0, 1\}$-matrix has a Pólya matrix, then all of Pólya matrices are equivalent. For totally sign-nonsingular signings, a similar statement holds.

Lemma 5.3. *If a $\{0, 1\}$-matrix A has a totally sign-nonsingular signing, then all of totally sign-nonsingular signings are equivalent.*

Using Corollary 4.3 and the following lemma, we prove Theorem 5.1.

Lemma 5.4. *Let A be an $m \times n$ matrix which is not block diagonal ($m < n$), and A^* be its augmented matrix. If two symmetric signings $\tilde{A}^*$ and $\hat{A}^*$ of A^* are equivalent, then $\tilde{A}^* = \hat{A}^*$ or $\tilde{A}^* = -\hat{A}^*$.*

Pólya's problem can be solved in polynomial time [20]. A symmetric Pólya matrix of a symmetric matrix can be found in linear time by Theorem 3.5. Thus we can obtain a totally sign-nonsingular signing in the same complexity as Pólya's problem.

Corollary 5.5. *Given a rectangular $\{0, 1\}$-matrix A, we can test whether A has a totally sign-nonsingular signing or not in polynomial time. Moreover, if A has a totally sign-nonsingular signing, we can find such signing in polynomial time.*

Testing sign-nonsingularity is polynomially equivalent to Pólya's problem [14, 22] (see also [27]). Theorem 5.1, together with Proposition 5.2, is summarized as the following corollary.

Corollary 5.6. *The following problems are polynomially equivalent.*

(1) *Deciding whether a given square matrix has a Pólya matrix or not (Pólya's problem).*
(2) *Deciding whether a given square matrix is sign-nonsingular or not.*
(3) *Deciding whether a given rectangular matrix has a totally sign-nonsingular signing or not.*
(4) *Deciding whether a given rectangular matrix is totally sign-nonsingular or not.*

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