On Orthogonal Ray Graphs

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Abstract. This paper introduces the orthogonal ray graphs, a new class of intersection graphs. An orthogonal ray graph is an intersection graph of horizontal and vertical rays (half-lines) in the plane. We first show that the class of orthogonal ray graphs is a proper subset of the class of unit grid intersection graphs. We next show a characterization of 2-directional orthogonal ray graphs. We also show a characterization of 2-directional orthogonal ray trees, which implies a linear time algorithm to recognize such trees. We finally show that the class of convex bipartite graphs is a proper subset of the class of 2-directional orthogonal ray graphs.

1 Introduction

Motivated by defect tolerance schemes for nanotechnology circuits, the orthogonal ray graphs, a new class of grid intersection graphs, are introduced and investigated.

A bipartite graph G with a bipartition (U, V) is called a *grid intersection graph* if there exist a family of non-intersecting line segments $L_u, u \in U$, parallel to the x-axis in the xy-plane, and a family of non-intersecting line segments $L_v, v \in V$, parallel to the y-axis such that for any $u \in U$ and $v \in V, (u, v) \in E(G)$ if and only if L_u and L_v intersect. Hartman, Newman, and Ziv [5] and de Fraysseix, de Mendez, and Pach [4] independently showed that every planar bipartite graph is a grid intersection graph. Kratochvil [8] showed that the recognition problem for grid intersection graphs is NP-complete.

Let G be a bipartite graph with a bipartition (U, V). A $(0, 1)$-matrix $M = [m_{ij}]$ is called a *bipartite adjacency matrix* of G if the rows of M are corresponding to the vertices of U, the columns of M are corresponding to the vertices of V, and $m_{ij} = 1$ if and only if $(u_i, v_j) \in E(G)$, where $u_i \in U$ is a vertex corresponding to row i and $v_j \in V$ is a vertex corresponding to column j. Let A and B be matrices. A is said to be *B-free* if A does not contain B as a submatrix. For a set S of matrices, A is said to be *S-free* if A is M-free for every $M \in S$. A is said to be *S-freeable* if there exist a permutation of rows of A and a permutation of columns of A such that the permuted matrix is S-free. Let

$$\Gamma = \left\{ \begin{bmatrix} w & 1 & x \\ 1 & 0 & 1 \\ y & 1 & z \end{bmatrix} \middle| w, x, y, z \in \{0, 1\} \right\},$$

and let

$$\gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

It is shown in [5] that a bipartite graph G is a grid intersection graph if and only if a bipartite adjacency matrix of G is Γ-freeable.

A grid intersection graph is said to be *unit* if all the line segments corresponding to the vertices have the same length. Otachi, Okamoto, and Yamazaki [10] showed that a bipartite graph G is a unit grid intersection graph if a bipartite adjacency matrix of G is γ-freeable.

This paper introduces *orthogonal ray graphs* which are a special kind of grid intersection graphs. A bipartite graph G with a bipartition (U, V) is called an orthogonal ray graph if there exist a family of non-intersecting rays (half-lines) $R_u, u \in U$, parallel to the x-axis in the xy-plane, and a family of non-intersecting rays $R_v, v \in V$, parallel to the y-axis such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if R_u and R_v intersect. We show in Section 3 that the class of orthogonal ray graphs is a proper subset of the class of unit grid intersection graphs.

We also introduce *2-directional orthogonal ray graphs* defined as follows. Let G be an orthogonal ray graph with a bipartition (U, V). G is called a 2-directional orthogonal ray graph if $R_u = \{(x, b_u) \mid x \geq a_u\}$ for each $u \in U$, and $R_v = \{(a_v, y) \mid y \geq b_v\}$ for each $v \in V$, where a_w and b_w are real numbers for any $w \in U \cup V$.

We show in Section 4 that a bipartite graph G is a 2-directional orthogonal ray graph if and only if a bipartite adjacency matrix of G is γ-freeable. We also show that the class of convex bipartite graphs is a proper subset of 2-directional orthogonal ray graphs.

The *3-claw* is a tree obtained from a complete bipartite graph $K_{1,3}$ by replacing each edge with a path of length 3. We show in Section 5 that a tree T is a 2-directional orthogonal ray graph if and only if T does not contain 3-claw as a subtree. It follows that we can decide in linear time whether a given tree is a 2-directional orthogonal ray graph.

2 Preliminaries

A bipartite graph G with bipartition (U, V) is said to be *convex* if there exists an ordering $(v_1, v_2, \ldots, v_{|V|})$ of V such that, for every $u \in U$ and integers i, j $(1 \leq i < j \leq |V|)$, $(u, v_i) \in E(G)$ and $(u, v_j) \in E(G)$ imply that $(u, v_k) \in E(G)$ for every integer k $(i \leq k \leq j)$. It is mentioned in [3] that convex bipartite graphs can be recognized in linear time using PQ-trees.

A graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is called a *permutation graph* if there exists a pair of permutations π_1 and π_2 on $N = \{1, 2, \ldots, n\}$ such that for all $i, j \in N$, $(v_i, v_j) \in E(G)$ if and only if

$$(\pi_1^{-1}(i) - \pi_1^{-1}(j))(\pi_2^{-1}(i) - \pi_2^{-1}(j)) < 0.$$

A bipartite graph G with bipartition (U, V) is said to be *strongly orderable* if there exist an ordering $(u_1, u_2, \ldots, u_{|U|})$ of U and an ordering $(v_1, v_2, \ldots, v_{|V|})$ of V such that for any integers i, i', j, j' $(1 \leq i < i' \leq |U|, 1 \leq j < j' \leq |V|)$, $(u_i, v_{j'}) \in E(G)$ and $(u_{i'}, v_j) \in E(G)$ imply $(u_i, v_j) \in E(G)$ and $(u_{i'}, v_{j'}) \in E(G)$. Spinrad, Brandstadt, and Stewart [11] showed that a bipartite graph G is a permutation graph if and only if G is strongly orderable, and gave a linear time recognition algorithm for bipartite permutation graphs based on this characterization. It follows from the characterization that the class of bipartite permutation graphs is a proper subset of the class of convex bipartite graphs as mentioned by Haiko [9]. Let

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

It also follows from the characterization that a bipartite graph G is a permutation graph if and only if a bipartite adjacency matrix of G is β-freeable as shown by Chen and Yesha [2].

3 Orthogonal Ray Graphs

The following theorem is implicit in [6], and can be seen without difficulty.

Theorem 1 *A cycle C_{2n} of length $2n$ is an orthogonal ray graph if and only if $2 \leq n \leq 6$.*

Theorem 2 *The class of orthogonal ray graphs is a proper subset of the class of unit grid intersection graphs.*

Proof. Let G be an orthogonal ray graph with bipartition (U, V). Let S be a square on the xy-plane such that all the cross points of rays R_w, $w \in U \cup V$, lie in S. Let l be the length of a side of S. Each ray intersects with at least one side of S. If a ray R_w intersects with both the opposite sides of S, let L_w be the line segment such that the endpoints of L_w are the two crossing points where R_w intersects with the sides of S. If a ray R_w intersects with only one side of S, let L_w be the line segment such that the endpoints of L_w are the endpoint of R_w and the point on R_w at a distance l from the endpoint of R_w. Since all the line segments have length l, and L_u and L_v intersect if and only if R_u and R_v intersect, G is a unit grid intersection graph for line segments $\{L_u \mid u \in U\} \cup \{L_v \mid v \in V\}$. Thus the class of orthogonal ray graphs is a subset of the class of unit grid intersection graphs.

It is easy to see that C_{2n} is a unit grid intersection graph for any $n \geq 2$. Thus we conclude by Theorem 1 that the class of orthogonal ray graphs is a proper subset of the class of unit grid intersecion graphs. □

4 Two-Directional Orthogonal Ray Graphs

Theorem 3 *C_{2m} is a 2-directional orthogonal ray graph if and only if $m = 2$.*

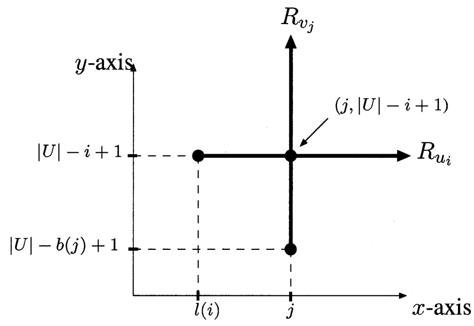


Figure 1: Rays R_{u_i} and R_{v_j} intersect at $(j, |U| - b(j) + 1)$.

Proof. It is easy to see that C_4 is a 2-directional orthogonal ray graph.

We show that C_{2m} is not a 2-directional orthogonal ray graph for any $m \geq 3$. Suppose to the contrary that C_{2m} is a 2-directional orthogonal ray graph for some $m \geq 3$. Let $V(C_{2m}) = \{0, 1, \ldots, 2m-1\}$ and $E(C_{2m}) = \{(i, i+1 \pmod{2m}) \mid 0 \leq i \leq 2m-1\}$. Suppose without loss of generality that $R_0 = \{(a_0, y) \mid y \geq b_0\}$, for some real numbers a_0 and b_0. Since $(0, 1) \in E(C_{2m})$, R_1 intersects with R_0 at some point. Similarly, R_2 intersects with R_1 at some other point. We distinguish two cases.

Case 1 When R_2 intersects with R_1 such that R_2 is to the left of R_0: Then R_3 must intersect with R_2 such that R_3 lies below the endpoint of R_0. Similarly R_4 must intersect with R_3 such that R_4 lies to the left of the endpoint of R_1. Continuing in this manner, R_i $(5 \leq i \leq 2m-1)$ must lie below(to the left of) the endpoint of R_{i-3} for odd(even) i. Therefore R_{2m-1} lies in the region below the endpoint of R_4. However, R_0 is in the region right of R_2 and above R_3, making it impossible for R_0 to intersect with R_{2m-1} without intersecting with R_3,$R_5, \ldots, R_{2m-3}$, a contradiction.

Case 2 When R_2 intersects with R_1 such that R_2 is to the right of R_0: We further distinguish two cases.

Case 2-1 When R_3 intersects with R_2 such that R_3 is below R_1: Then R_4 must lie to the left of the endpoint of R_1. This confines R_0 within the region left of R_2 and above R_3, making it impossible for ray R_{2m-1} to intersect with R_0 without intersecting with R_2, a contradiction.

Case 2-2 When R_3 intersects with R_2 such that R_3 is above R_1: This case may be further broken down into two cases depending on whether R_4 is to the left of R_2 or right of R_2. In the former case, R_4 gets confined within the region left of R_2 and above R_1 making it impossible for R_5 to intersect with R_4 without intersecting with R_2, a contradiction. In the latter case, $R_5, \ldots, R_{2m-1}$ must lie in the region right of R_2 and above R_3, making it impossible for R_{2m-1} to intersect with R_0 without intersecting with R_2,R_4, $\ldots$,R_{2m-2}, a contradiction.

Thus we conclude that C_{2m} is not a 2-directional orthogonal ray graph for any $m \geq 3$. □ □

The following is immediate from Theorems 1 and 3.

Theorem 4 *The class of* 2*-directional orthogonal ray graphs is a proper subset of the class of orthogonal ray graphs.*

The following is obvious from the definition of γ.

Lemma 1 *An* $m \times n$ *matrix* $M = [m_{ij}]$ *is* γ*-free if and only if for any integers* i, j, i', j' $(1 \leq i < i' \leq m,\ 1 \leq j < j' \leq n)$, $m_{ij} = 1$ *and* $m_{i'j'} = 1$ *imply* $m_{ij'} = 1$.

We can characterize the 2-directional orthogonal ray graphs as follows.

Theorem 5 *A bipartite graph* G *is a* 2*-directional orthogonal ray graph if and only if a bipartite adjacency matrix of* G *is* γ*-freeable.*

Proof.

Let G be a bipartite graph with a bipartition (U, V). Suppose that a bipartite adjacency matrix of G is γ-freeable, and let $M = [m_{ij}]$ be a bipartite adjacency matrix of G which is γ-free. We denote by $u_i \in U$ the vertex corresponding to row i, and by $v_j \in V$ the vertex corresponding to column j. For each row i, suppose the leftmost 1 is at column $l(i)$. Then define ray $R_{u_i} = \{(x, |U| - i + 1) \mid x \geq l(i)\}$.

Similarly for each column j, suppose the bottommost 1 is at row $b(j)$, then define ray $R_{v_j} = \{(j, y) \mid y \geq |U| - b(j) + 1\}$. Note that from this definition, if two rays R_{u_i} and R_{v_j} intersect, they must do so at $(j, |U| - i + 1)$. It is also obvious that two rays R_{u_i} and R_{v_j} intersect if and only if $l(i) \leq j$ and $b(j) \geq i$. (See Figure 1.) We are now ready to show that R_{u_i} and R_{v_j} intersect if and only if $(u_i, v_j) \in E(G)$. Suppose first that $(u_i, v_j) \in E(G)$. Then $m_{ij} = 1$, which means that $l(i) \leq j$ and $b(j) \geq i$. Therefore, rays R_{u_i} and R_{v_j} intersect. Suppose next that $(u_i, v_j) \notin E(G)$. Then $m_{ij} = 0$. Since M is γ-free, we have $m_{i'j} = 0$ for every $i' < i$ or $m_{ij'} = 0$ for every $j' > j$, by Lemma 1. This means that $l(i) > j$ or $b(j) < i$, which implies that R_{u_i} and R_{v_j} do not intersect. Thus we conclude that G is a 2-directional orthogonal ray graph for rays $\{R_{u_i} | u_i \in U\} \cup \{R_{v_j} | v_j \in V\}$.

Conversely, suppose that G is a 2-directional orthogonal ray graph, and $\{R_u \mid u \in U\} \cup \{R_v \mid v \in V\}$ is the set of rays corresponding to the vertices. Let $(u_1, u_2, \ldots, u_{|U|})$ be the ordering of U such that for any integers i, i' $(1 \leq i < i' \leq |U|)$, R_{u_i} is above $R_{u_{i'}}$ in the xy-plane. Similarly, let $(v_1, v_2, \ldots, v_{|V|})$ be the ordering of V such that for any integers j, j' $(1 \leq j < j' \leq |V|)$, R_{v_j} is to the left of $R_{v_{j'}}$. Construct a bipartite adjancency matrix $M = [m_{ij}]$ of G such that $m_{ij} = 1$ if and only if $(u_i, v_j) \in E(G)$. We shall show that M is γ-free. Let i, i', j, j' be any integers such that $1 \leq i < i' \leq |U|$ and $1 \leq j < j' \leq |V|$. Suppose $m_{ij} = 1$ and $m_{i'j'} = 1$. Since ray R_{u_i} is above ray $R_{u_{i'}}$ and R_{v_j} is to the left of $R_{v_{j'}}$, R_{u_i} must intersect with $R_{v_{j'}}$ implying that $m_{ij'} = 1$. Thus from Lemma 1, M is γ-free. □

A bipartite graph G with bipartition (U, V) is said to be *weakly orderable* if there exist an ordering $(v_1, v_2, \ldots, v_{|V|})$ of V and an ordering $(u_1, u_2, \ldots, u_{|U|})$ of U such that for every i, i', j, j' $(1 \leq i < i' \leq |U|, 1 \leq j < j' \leq |V|)$, $(u_i, v_{j'}) \in E(G)$ and $(u_{i'}, v_j) \in E(G)$ imply $(u_i, v_j) \in E(G)$. The following corollary is immediate.

Corollary 1 *A bipartite graph G is a 2-directional orthogonal ray graph if and only if G is weakly orderable.*

Theorem 6 *The class of convex bipartite graphs is a proper subset of the class of 2-directional orthogonal ray graphs.*

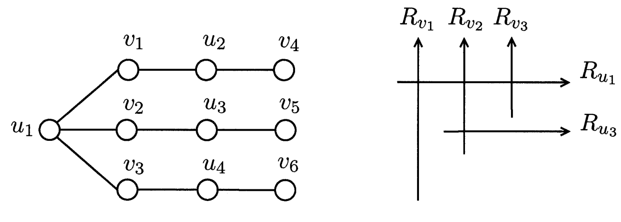
Proof. Let G be a convex bipartite graph with bipartition (U, V). Let $(v_1, v_2, \ldots, v_{|V|})$ be an ordering of V such that for every $u \in U$ and integers i, j $(1 \leq i < j \leq |V|)$, $(u, v_i) \in E(G)$ and $(u, v_j) \in E(G)$ imply that $(u, v_k) \in E(G)$ for every integer k $(i \leq k \leq j)$. Let $U = \{u_1, u_2, \ldots, u_{|U|}\}$, and let $M = [m_{ij}]$ be a bipartite adjacency matrix of G such that row i corresponds to u_i and column j corresponds v_j. It is easy to see that the 1's in each row of M are consecutive. We will show that M is γ-freeable. Let $B = [b_{ij}]$ be a matrix obtained by permuting the rows of M such that for any two rows i, i' $(i < i')$ of B, $r_B(i) \geq r_B(i')$, where $r_B(i)$ is defined as the column containing the rightmost 1 in row i of B. Note that the 1's in each row of B are also consecutive. We claim that B is γ-free. Let i, i', j, j' be any integers such that $1 \leq i < i' \leq |U|$ and $1 \leq j < j' \leq |V|$. Suppose $b_{ij} = 1$ and $b_{i'j'} = 1$. Since $i < i'$, we have $r_B(i) \geq j'$. Since the 1's in row i are consecutive, $b_{ik} = 1$ for every integer k $(j \leq k \leq r(i))$. This means that $b_{ij'} = 1$. Thus we conclude from Lemma 1 that B is γ-free, and so M is γ-freeable. Hence G is a 2-directional orthogonal ray graph by Theorem 5. Therefore the class of convex bipartite graphs is a subset of the class of 2-directional orthogonal ray graphs.

Let G be a graph obtained from a complete bipartite graph $K_{3,3}$ by adding a new vertex v' and an edge (v, v'), for each vertex v of $K_{3,3}$. It is easy to see that G is a 2-directional orthogonal ray graph, but not a convex bipartite graph. Thus we conclude that the class of convex bipartite graphs is a proper subset of the class of 2-directional orthogonal ray graphs. □

5 Two-Directional Orthogonal Ray Trees

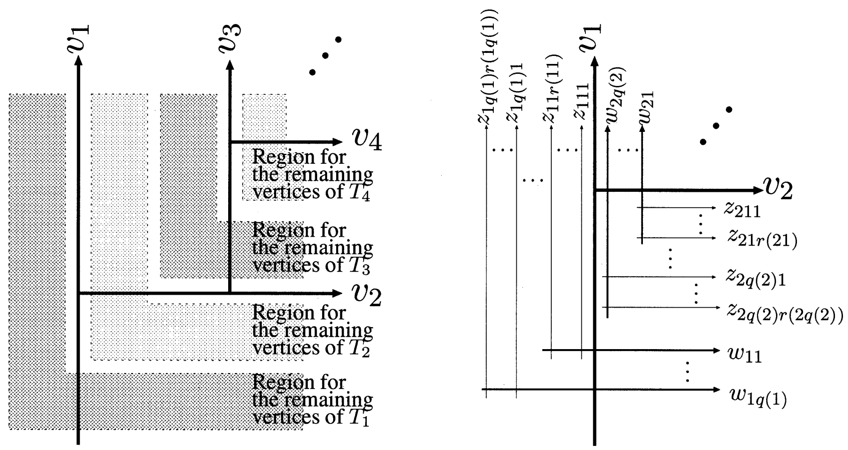
Lemma 2 *The 3-claw is not a 2-directional orthogonal ray graph.*

Proof. Assume to the contrary that the 3-claw is a 2-directional orthogonal ray graph. Let the vertices of the *3-claw* be named as in Figure 2(a). We shall refer to the endpoint of the ray corresponding to a vertex v as (a_v, b_v). Without loss of generality, suppose $R_{u_1} = \{(x, b_{u_1}) \mid x \geq a_{u_1}\}$ for arbitrary real numbers a_{u_1} and b_{v_1}. Also, without loss of generality, suppose R_{v_1}, R_{v_2}, R_{v_3} intersect with R_{u_1} such that R_{v_2} lies to the right of R_{v_1} and to the left of R_{v_3} as shown in Figure 2(b). It is easy to observe that $b_{v_3} > b_{v_2} > b_{v_1}$, or else it is not possible to define R_{u_3} and R_{u_4}. Since R_{u_3} has to be defined such that



(a) 3-claw (b)

Figure 2:



(a) (b)

Figure 3: Rays corresponding to the vertices of T. Each ray is labelled with the vertex it corresponds to. (a) shows the rays corresponding to the vertices in P and the region in which rays corresponding to the remanining vertices of T_i $(1 \leq i \leq p)$ will be placed. The actual placement of rays corresponding to the vertices of T_1 and T_2 is shown in (b).

$a_{u_3} > a_{v_1}$ and $b_{u_3} < b_{u_1}$, it is not possible to define R_{v_5} such that it intersects with R_{u_3} but not with R_{u_1}, a contradiction. □

A path P in a tree T is called a *spine* of T if every vertex of T is within distance two from at least one vertex of P.

Theorem 7 *A tree T has a spine if and only if T does not contain 3-claw as a subtree.*

Proof. The necessity is obvious. To prove the sufficiency, assume T does not contain a 3-claw. Let P be a longest path in T. We claim that P is a spine. Assume it is not. Let $V(P) = \{v_1, v_2, \ldots, v_p\}$, and $(v_i, v_{i+1}) \in E(P)$, $1 \leq i \leq p-1$. Let F be a forest obtained from T by deleting the edges in $E(P)$. Let T_i be a tree in F containing v_i, $1 \leq i \leq p$. Since P is a longest path in T, T_1 consists of only one vertex, v_1, and T_p consists of only one vertex, v_p. Also all vertices in T_2 and T_{p-1} are within distance one from v_2 and v_{p-1}, respectively; and all vertices in T_3 and T_{p-2} are within distance two from v_3 and v_{p-2}, respectively. Since we assumed that P is not a spine, there exists an integer j $(4 \leq j \leq p-3)$ such that T_j contains a vertex w_j whose distance from v_j is three. Let P' be the path from v_j to w_j. Then the subgraph of T induced by the vertices in $\{v_i \mid j-3 \leq i \leq j+3\} \cup V(P')$ is a 3-claw. This contradicts the assumption that T does not contain 3-claw as a subtree, and therefore P is a spine. □

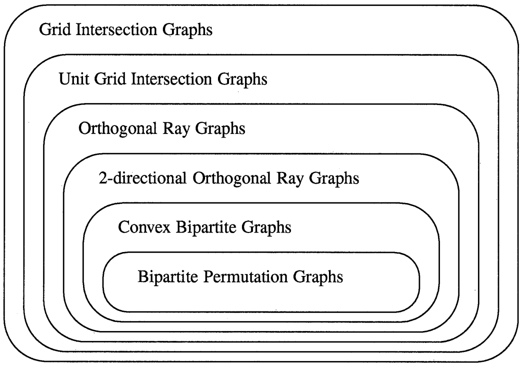


Figure 4: Relationship between different grid intersection graphs considered in this paper.

We can characterize the 2-directional orthogonal ray trees as follows.

Theorem 8 *A tree T is a 2-directional orthogonal ray tree if and only if T does not contain 3-claw as a subtree.*

Proof. The necessity follows from Lemma 2. We will show the sufficiency. Assume T does not contain 3-claw as a subtree. Then from Theorem 7, T contains a spine P. Let $V(P) = \{v_1, v_2, \dots, v_p\}$, and $(v_i, v_{i+1}) \in E(P)$, $1 \le i \le p-1$. Corresponding to each vertex v_i in P, define ray $R_{v_i} = \{(i, y) \mid y \ge i-1\}$ if i is odd, and define ray $R_{v_i} = \{(x, i) \mid x \ge i-1\}$ if i is even. Let F be a forest obtained from T by deleting the edges in $E(P)$. Let T_i be a tree in T containing v_i, $1 \le i \le p$. Consider T_i to be rooted at v_i. Let $w_{i1}, w_{i2}, \dots, w_{iq(i)}$ be the children of v_i in T_i, where $q(i)$ is the number of children of v_i in T_i. Let $z_{ij1}, z_{ij2}, \dots, z_{ijr(ij)}$ be the children of w_{ij} in T_i, where $r(ij)$ is the number of children of w_{ij} in T_i. The rays corresponding to w_{ij} and z_{ijk}, $(1 \le i \le p, 1 \le j \le q(i), 1 \le k \le r(ij))$ can be added as shown in Figure 3. Thus T is a 2-directional orthogonal ray graph. □

From Theorem 8 and the proof of Theorem 7, we can see that in order to decide if a given tree T is a 2-directional orthogonal ray graph, we need only to verify whether or not a longest path in T is a spine of T. Since a longest path in a tree can be obtained in linear time (see [1], for example), we can recognize 2-directional orthogonal ray trees in linear time.

6 Concluding Remarks

Figure 4 shows the relationship between the classes of grid intersection graphs considered here.

It is open to characterize the orthogonal ray graphs. The complexity of the recognition for the 2-directional orthogonal ray graphs is also open. In this connection, it is interesting to note that the complexity of the problem of deciding if a matrix is γ-freeable is an open problem posed by Klinz, Rudolf, and Woeginger [7] more than ten years ago. It should be noted that we can decide in polynomial time if a matrix is M-freeable for every matrix $M \in \gamma$ [7], and if a matrix is β-freeable [2, 11].

References

[1] Bulterman, R., van der Sommen, F., Zwaan, G., Verhoeff, T., van Gasteren, A., Feijen, W.: On computing a longest path in a tree. Information Processing Letters **81** 93–96 (2002)

[2] Chen, L., Yesha, Y.: Efficient parallel algorithms for bipartite permutation graphs. Networks **23** 29–39 (1993)

[3] Damaschke, P., Muller, H., Kratsch, D.: Domination in convex and chordal bipartite graphs. Information Processing Letters **36** 231–236 (1990)

[4] de Fraysseix, H., de Mendez, P., Pach, J.: Representation of planar graphs by segments. Intuitive Geometry Coll. Math. Soc. J. Bolyai **63** 109–117 (1994)

[5] Hartman, I., Newman, I., Ziv, R.: On grid inresection graphs. Discrete Mathematics **87** 41–52 (1991)

[6] Kostochka, A., Nesetril, J.: Coloring relatives of intervals on the plane, i: Chromatic number versus girth. Europ. J. Combinatorics **19** 103–110 (1998)

[7] Klinz, B., Rudolf, K., Woeginger, G.: Permuting matrices to avoid forbidden submatrices. Discrete Applied Mathematics **60** 223–248 (1995)

[8] Kratochvil, J.: A special planar satisfiability problem and a consequence of its np-completeness. Discrete Applied Mathematics **52** 233–252 (1994)

[9] Muller, H.: Hamiltonian circuits in chordal bipartite graphs. Discrete Mathematics **156** 291–298 (1996)

[10] Otachi, Y., Okamoto, Y., Yamazaki, K.: Relationships between the class of unit grid intersection graphs and other classes of bipartite graphs. Discrete Applied Mathematics **155** 2383–2390 (2007)

[11] Spinrad, J., Brandstadt, A., Stewart, L.: Bipartite permutation graphs. Discrete Applied Mathematics **18** 279–292 (1987)