点集合間の辺連結度を増大させる問題

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概 要

グラフ $G=(V,E)$，二つの点集合族 $\mathcal{W}_1, \mathcal{W}_2 \subseteq 2^V - \{\emptyset\}$，要求関数 $r: \mathcal{W}_1 \times \mathcal{W}_2 \rightarrow \mathbb{R}_+$ が与えられたとき，最小本数の辺を加えることで，$W_1 \in \mathcal{W}_1$ と $W_2 \in \mathcal{W}_2$ のすべての組において，W_1 から W_2 への間の辺連結度を $r(W_1, W_2)$ 以上に増大させる問題を考える．この問題は，辺連結度増大問題，局所辺連結度増大問題，節点領域辺連結度増大問題を特別な場合として含む．

本研究では，P=NP でなければ，有向グラフにおける節点領域辺連結度増大問題や無向グラフにおける局所節点領域辺連結度増大問題に限定しても，ある定数 c に対し，多項式時間で $c \log \alpha(\mathcal{W}_1, \mathcal{W}_2)$ 倍より良い近似ができないことを示す．ただし，$\alpha(\mathcal{W}_1, \mathcal{W}_2)$ は $r(W_1, W_2) > 0$ である $W_1 \in \mathcal{W}_1$ と $W_2 \in \mathcal{W}_2$ の組の数を表す．一方で，問題が $\mathrm{O}(\log \alpha(\mathcal{W}_1, \mathcal{W}_2))$ 倍近似可能であることも示す．つまり，P=NP でなければ，問題が $\Theta(\log \alpha(\mathcal{W}_1, \mathcal{W}_2))$ 倍近似可能であることを示す．

さらに，この問題に対し，模調関数を一般化した関数を用いた特徴づけを与える．

Augmenting Edge-Connectivity between Vertex Subsets

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Abstract

Given a graph $G=(V,E)$ and a requirement function $r: \mathcal{W}_1 \times \mathcal{W}_2 \rightarrow \mathbb{R}_+$ for two families $\mathcal{W}_1$, $\mathcal{W}_2 \subseteq 2^V - \{\emptyset\}$, we consider the problem (called *area-to-area edge-connectivity augmentation problem*) of augmenting G by a smallest number of new edges so that the resulting graph $\hat{G}$ satisfies $\delta_{\hat{G}}(X) \geq r(W_1, W_2)$ for all $X \subseteq V$, $W_1 \in \mathcal{W}_1$, and $W_2 \in \mathcal{W}_2$ with $W_1 \subseteq X \subseteq V - W_2$, where $\delta_G(X)$ denotes the degree of a vertex set X in G. This problem can be regarded as a natural generalization of the global, local, and node-to-area edge-connectivity augmentation problems.

In this paper, we show that there exists a constant c such that the problem is inapproximable within a ratio of $c \log \alpha(\mathcal{W}_1, \mathcal{W}_2)$, unless P=NP, even restricted to the directed global node-to-area edge-connectivity augmentation or undirected local node-to-area edge-connectivity augmentation, where $\alpha(\mathcal{W}_1, \mathcal{W}_2)$ denotes the number of pairs $W_1 \in \mathcal{W}_1$ and $W_2 \in \mathcal{W}_2$ with $r(W_1, W_2) > 0$. We also provide an $\mathrm{O}(\log \alpha(\mathcal{W}_1, \mathcal{W}_2))$-approximation algorithm for the area-to-area edge-connectivity augmentation problem. This together with the negative result implies that the problem is $\Theta(\log \alpha(\mathcal{W}_1, \mathcal{W}_2))$-approximable, unless P=NP, which solves open problems for node-to-area edge-connectivity augmentation in [9, 10, 12].

Furthermore, we characterize the node-to-area and area-to-area edge-connectivity augmentation problems as the augmentation problems with modulotone and extended modulotone functions.

1 Introduction

In communication networks, graph connectivity is one of the most fundamental parameters to measure the robustness and availability of the networks. Various kinds of connectivity augmentation problems have been extensively studied as an important subject in the network design, and many efficient algorithms have been developed so far (see [4, 7, 14] for surveys).

Let $G=(V,E)$ be a multigraph. For a graph G, let $\delta_G(X)$ denote the number of edges $e=(u,v)$ such that $u \in X$ and $v \notin X$. Two vertices s and t are *k-edge-connected* if there exists k-edge disjoint paths from s to t. By Menger's theorem, s and t are k-edge-connected if and only if every vertex set X with $s \in X \subseteq V - \{t\}$ satisfies $\delta_G(X) \geq k$.

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In this paper, we consider the following problem which deals with edge-connectivity not only between two vertices, but also between two vertex subsets. Here two vertex sets S and T are *k-edge-connected* if there exists k-edge disjoint paths from S to T, which is equivalent to the condition that every vertex set X with $S \subseteq X \subseteq V - T$ satisfies $\delta_G(X) \geq k$.

Problem 1.1 (AREA-TO-AREA EDGE-CONNECTIVITY AUGMENTATION)
Input: *A multigraph $G = (V, E)$ and a requirement function $r : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}_+$ for two families $\mathcal{W}_1, \mathcal{W}_2 \subseteq 2^V - \{\emptyset\}$.*
Output: *A minimum set F of new edges such that the resulting graph $\hat{G} = (V, E \cup F)$ satisfies $\delta_{\hat{G}}(X) \geq r(W_1, W_2)$ for all $X \subseteq V$, $W_1 \in \mathcal{W}_1$, and $W_2 \in \mathcal{W}_2$ with $W_1 \subseteq X \subseteq V - W_2$.*

Here $\mathbb{R}_+$ denotes the set of all nonnegative reals. We remark that multigraphs can be represented as simple graphs with edge capacity. Throughout the paper, we treat G as a multigraph just for simplicity of exposition.

This problem can be regarded as a natural generalization of the *global edge-connectivity augmentation problem* (GAP) [2], *local edge-connectivity augmentation problem* (LAP) [3], and *node-to-area edge-connectivity augmentation problem* (NAAP) [12].

Previous Work

Let us briefly survey the developments in the edge-connectivity augmentation problems. Let $\mathcal{V} = \{\{v\} \mid v \in V\}$.

LAP is equivalent to Problem 1.1 with $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{V}$, while GAP is a special case of LAP with uniform requirements $r(u, v) \equiv k$ for some integer k; namely, GAP (resp., LAP) asks to augment the global (resp., local) edge-connectivity of a given graph. GAP was first shown to be solvable in polynomial time by Watanabe and Nakamura [19] for undirected graphs and by Frank [3] for directed graphs. The fastest known algorithms achieve $O(mn + n^2 \log n)$ time due to Nagamochi [13] for undirected graphs and $O(mn^2 \log n^2/m)$ time due to Gabow [6] for directed graphs, where $n = |V|$ and m denotes the number of edges in the graph obtained from G by identifying multiple edges into a single one. As for LAP, Frank [3, 4] showed that it is polynomially solvable for undirected graphs, while it is NP-hard for directed graphs, even restricted to $r(u, v) \in \{0, 1\}$ for $u, v \in V$. Later it was shown by Nutov [15] and Kortsarz and Nutov [11] that it is $\Theta(\log n)$-approximable in polynomial time.

NAAP is equivalent to Problem 1.1 with $\mathcal{W}_1 = \mathcal{V}$, which includes LAP as a special case with $\mathcal{W}_2 = \mathcal{V}$. Miwa and Ito [12] showed that NAAP for undirected graphs is NP-hard, even if $r \equiv 1$, while it is known by Miwa and Ito [12] and Ishii, et al. [9] that it is solvable in polynomial time if $r \equiv k$ for some $k\,(\geq 2)$. Furthermore, Ishii and Hagiwara [10] showed that it is solvable in polynomial time, if a requirement function r depends only on $\mathcal{W}_2$ and $r \geq 2$. From the results of [10] and [15], we can observe that NAAP for undirected graphs with r depending only on $\mathcal{W}_2$ is $7/4$-approximable in polynomial time. However, no other approximation result is known for NAAP in undirected/directed graphs. For directed graphs with a uniform requirement r, no complexity result is even known; it is open whether the problem is NP-hard.

Tables 1 and 2 summarize the currently best known results on the complexity status for these problems.

We note that several extensions of the augmentation problems have been studied [1, 8, 11, 15, 18]. All the problems can be formulated as the problem of augmenting G so that the resulting graph $\hat{G}$ *covers* a given requirement function $r^* : 2^V \to \mathbb{R}_+$ (i.e., $\delta_{\hat{G}}(X) \geq r^*(X)$ for $\emptyset \neq X \subsetneq V$). Ishii [8] proved that the problem with a monotone r^* is equivalent to NAAP with r depending only on $\mathcal{W}_2$ and showed that it is polynomially solvable, if G is undirected and $r^*(X) \geq 2$ holds for all $X \subsetneq V$ with $r^*(X) > 0$, where r^* is called *monotone* if $r^*(X) \leq r^*(Y)$ holds for arbitrary two subsets $X \subseteq Y \subseteq V$.

Table 1: Currently best known results for GAP, LAP, and NAAP in undirected graphs

$\mathcal{W}_1$	$\mathcal{W}_2$	r: uniform	r: arbitrary
$\mathcal{V}$	$\mathcal{V}$	GAP $\mathrm{O}(mn + n^2 \log n)$ [13]	LAP $\mathrm{O}(mn^3 \log n^2/m)$ [6]
$\mathcal{V}$	arbitrary	NAAP NP-hard* [12]	NP-hard** [12]

*7/4-approximable [15] if $r \equiv 1$, solvable in $\mathrm{O}(|\mathcal{W}_2|n + m)$ time if $r \equiv 2$ [12], and in $\tilde{\mathrm{O}}(|\mathcal{W}_2|kn^3 + n(k^3 + n^2)(|\mathcal{W}_2| + kn)))$ time if $r \equiv k\,(\geq 3)$ [9].

**7/4-approximable [15] if r depends only on $\mathcal{W}_2$, while solvable in $\mathrm{O}(n^3|\mathcal{W}_2|(m + n \log n))$ time if in addition $r \geq 2$ holds [10].

Table 2: Currently best known results for GAP, LAP, and NAAP in directed graphs

$\mathcal{W}_1$	$\mathcal{W}_2$	r: uniform	r: arbitrary
$\mathcal{V}$	$\mathcal{V}$	GAP $\mathrm{O}(mn^2 \log n^2/m)$ [6]	LAP $\Theta(\log n)$-approximable [11, 15]
$\mathcal{V}$	arbitrary	NAAP –	$\Omega(\log n)$-approximable [15]

We further remark that the augmentation problem with r^* can also be represented by the problem of covering a given function $p : 2^V \to \mathbb{R}_+$ by a graph (V, F) with a minimum $|F|$; i.e., augmenting a graph $G = (V, \emptyset)$ by a new edge set F. The function p can be constructed from r^* by $p(X) = \max\{0, r^*(X) - \delta_G(X)\}$. Benczúr and Frank [1] showed that it is polynomially solvable, if p is symmetric supermodular, where p is *symmetric* if $p(X) = p(V - X)$ for every $X \subseteq V$, and *supermodular* if $p(X)+p(Y) \leq p(X\cap Y)+p(X\cup Y)$ for every $X, Y \subseteq V$ with $p(X), p(Y) > 0$. GAP in undirected graphs is a special case of the problem, since $-\delta_G$ is symmetric supermodular. Recently, Nutov [15] proved that if p is symmetric *skew-supermodular* (i.e., $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ or $p(X) + p(Y) \leq p(X - Y) + p(Y - X)$ for every $X, Y \subseteq V$ with $p(X), p(Y) > 0$), then it is APX-hard and 7/4-approximable in polynomial time under the *mild* assumption mentioned in [15] (note that the assumption holds for any supermodular function p [1]). The function p defined in LAP in undirected graphs and NAAP with r depending only on $\mathcal{W}_2$ in undirected graphs are not symmetric supermodular, but symmetric skew-supermodular, as observed in [3, 10]. Since the assumption holds for these cases, the result in [15] implies the 7/4-approximability for NAAP with r depending only on $\mathcal{W}_2$ in undirected graphs.

Our Contributions

The results obtained in this paper can be summarized as follows. Let $\alpha(\mathcal{W}_1, \mathcal{W}_2)$ be the number of pairs of sets $W_1 \in \mathcal{W}_1$ and $W_2 \in \mathcal{W}_2$ with $r(W_1, W_2) > 0$; by definition, we have $\alpha(\mathcal{W}_1, \mathcal{W}_2) \leq |\mathcal{W}_1||\mathcal{W}_2|$.

- We show that there exists a constant c such that NAAP is not approximable in polynomial time within a ratio of $c \log \alpha(\mathcal{V}, \mathcal{W}_2)$ unless P=NP, even in the case where (i) G is undirected and $r \in \{0, k\}^{\mathcal{V}\times\mathcal{W}_2}$ holds for any $k\,(\geq 1)$, or (ii) G is directed and $r \equiv k$ for any $k\,(\geq 1)$. This shows the inapproximability of NAAP, whose complexity status was left open [9, 10, 12]. Here we remark that the function p discussed in [1, 15] is *not* skew-supermodular for these cases, and hence Nutov's result is not applicable.

- For undirected/directed graphs, we propose $O(\log \alpha(\mathcal{W}_1, \mathcal{W}_2))$-approximation algorithms for Problem 1.1. By combining the hardness results described above, we can say that our approximation algorithms are *optimal*, i.e., Problem 1.1 in undirected/directed graphs are $\Theta(\log \alpha(\mathcal{W}_1, \mathcal{W}_2))$-approximable.

- Moreover, we characterize the node-to-area and area-to-area edge-connectivity augmentation problems by using k-modulotone functions, where a function $r^* : 2^V \to \mathbb{R}_+$ is called *k-modulotone* if each nonempty subset X of V has a subset W of X with $|W| \leq k$ such that $r^*(Y) \geq r^*(X)$ for all subsets Y of X with $W \subseteq Y$. Namely, we show that the area-to-area edge-connectivity augmentation problems with $r : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}_+$ can be regarded as the augmentation problems with k-modulotone functions with $k = \max\{|W| \mid W \in \mathcal{W}_1\}$; the node-to-area edge-connectivity augmentation problems can be regarded as the problems with 1-modulotone functions. 1-modulotone functions are first introduced in [17] for giving a generalized framework of the source location problem and extended network problem. The results give another application of 1-modulotone functions as well as its extension. By combining with the second results, we can see that the augmentation problems with k-modulotone functions are $O(\log \alpha(\mathcal{W}_1, \mathcal{W}_2))$-approximable in polynomial time, if the corresponding function $r : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}_+$ in Problem 1.1 can be constructed from r^* in polynomial time.

The rest of this paper is organized as follows. In Section 2, we present an $O(\log \alpha(\mathcal{W}_1, \mathcal{W}_2))$-approximation algorithm for Problem 1.1. Section 3 shows the inapproximability of NAAP. Section 4 discusses the augmentation problem with k-modulotone functions.

Due to space constraint, technical details of some proofs are omitted.

2 Approximation Algorithms for Area-to-Area Edge- Connectivity Augmentation

2.1 Transforming area-to-area edge-connectivity augmentation

In order to present an $O(\log \alpha(\mathcal{W}_1, \mathcal{W}_2))$-approximation algorithm for Problem 1.1, let us start with the following lemma which shows that the approximability of Problem 1.1 in directed graphs indicates the one in undirected graphs.

For two subsets $X, Y \subseteq V$ in G, we denote by $\lambda_G(X, Y)$ the *edge-connectivity from X to Y*, i.e., $\lambda_G(X, Y) = \min\{\delta_G(Z) \mid X \subseteq Z \subseteq V - Y\}$; here we define $\lambda_G(X, Y) = +\infty$ if $X \cap Y \neq \emptyset$. For a graph G and a function $r : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}_+$, we say that G *is r-edge-connected* if $\lambda_G(W_1, W_2) \geq r(W_1, W_2)$ holds for all sets $W_1 \in \mathcal{W}_1$ and $W_2 \in \mathcal{W}_2$.

Lemma 2.1 *If Problem 1.1 in directed graphs is β-approximable, then Problem 1.1 in undirected graphs is 2β-approximable.* □

Let us then consider the following problem which augments $G = (V, E)$ by adding a new vertex $s \notin V$ and new edges between s and V, defined as follows.

Problem 2.2 (s-BASED AREA-TO-AREA AUGMENTATION)
Input: *A multigraph $G = (V, E)$ and a requirement function $r : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}_+$ for two families $\mathcal{W}_1, \mathcal{W}_2 \subseteq 2^V - \{\emptyset\}$.*
Output: *A minimum set F of new edges between $s (\notin V)$ and V such that the resulting graph $\tilde{G} = (V \cup \{s\}, E \cup F)$ satisfies $\delta_{\tilde{G}}(X) \geq r(W_1, W_2)$ for all $X \subseteq V \cup \{s\}$, $W_1 \in \mathcal{W}_1$, and $W_2 \in \mathcal{W}_2$ with $W_1 \subseteq X \subseteq V \cup \{s\} - W_2$.*

The following lemma shows that the approximation ratios of Problems 1.1 and 2.2 differ only by a constant factor.

Lemma 2.3 (*i*) *If Problem 1.1 is β-approximable, then Problem 2.2 is 2β-approximable.* (*ii*) *If Problem 2.2 is β-approximable, then Problem 1.1 is 2β-approximable.* □

Lemmas 2.1 and 2.3 indicate that it suffices to prove that Problem 2.2 in directed graphs is approximable within a ratio of $O(\log \alpha(\mathcal{W}_1, \mathcal{W}_2))$. Below, we further transform Problem 2.2 in directed graphs into the problems below, where the technique is based on the one developed by Kortsarz and Nutov to construct an approximation algorithm for augmenting the mixed connectivity [11, Section 2].

Let $r_{max} = \max\{r(W_1, W_2) \mid W_1 \in \mathcal{W}_1, W_2 \in \mathcal{W}_2\}$, and let F_r^+ (resp., F_r^-) be the set of r_{max} multiple directed edges from s to each vertex $v \in V$ (resp., from each vertex $v \in V$ to s).

Problem 2.4 (s-BASED AREA-TO-AREA AUGMENTATION$^-$)
Input: *A directed multigraph $G = (V, E)$ and a requirement function $r : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}_+$ for two families $\mathcal{W}_1$, $\mathcal{W}_2 \subseteq 2^V - \{\emptyset\}$.*
Output: *A minimum set F^- of new edges from V to a vertex $s \notin V$, such that the resulting graph $\tilde{G} = (V \cup \{s\}, E \cup F^- \cup F_r^+)$ satisfies $\delta_{\tilde{G}}(X) \geq r(W_1, W_2)$ for all $X \subseteq V \cup \{s\}$, $W_1 \in \mathcal{W}_1$, and $W_2 \in \mathcal{W}_2$ with $W_1 \subseteq X \subseteq V \cup \{s\} - W_2$.*

Problem 2.5 (s-BASED AREA-TO-AREA AUGMENTATION$^+$)
Input: *A directed multigraph $G = (V, E)$ and a requirement function $r : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}_+$ for two families $\mathcal{W}_1$, $\mathcal{W}_2 \subseteq 2^V - \{\emptyset\}$.*
Output: *A minimum set F^+ of new edges from a vertex $s \notin V$ to V, such that the resulting graph $\tilde{G} = (V \cup \{s\}, E \cup F^+ \cup F_r^-)$ satisfies $\delta_{\tilde{G}}(X) \geq r(W_1, W_2)$ for all $X \subseteq V \cup \{s\}$, $W_1 \in \mathcal{W}_1$, and $W_2 \in \mathcal{W}_2$ with $W_1 \subseteq X \subseteq V \cup \{s\} - W_2$.*

For an instance $I = (G = (V, E), r)$ of Problem 2.2 in directed graphs, let I^- and I^+ be the instances of Problems 2.4 and 2.5 corresponding to I, respectively. Let F^- and F^+ be feasible solutions of Problems 2.4 and 2.5, respectively. It is easy to see that $F_r^- \cup F_r^+$ is feasible for I. The following lemma shows that $F^- \cup F^+$ is also feasible for I and keep the approximability.

Lemma 2.6 *If F^- and F^+ are β-approximate to I^- and I^+, respectively, then $F^- \cup F^+$ is β-approximate to I.* □

2.2 Submodular set cover problem

From discussion in the previous subsection, we have only to show that Problems 2.4 and 2.5 are both $O(\log(\alpha(\mathcal{W}_1, \mathcal{W}_2)))$-approximable. In this section, we show the $O(\log(\alpha(\mathcal{W}_1, \mathcal{W}_2)))$-approximability of Problem 2.4, since Problem 2.5 can be treated similarly. We shall prove this by showing that the problem can be formulated as the *submodular set cover problem*.

For a finite set U, a set function $f : 2^U \to \mathbb{R}_+$ is *submodular* if $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$ holds for arbitrary two subsets $X, Y \subseteq U$. Given a cost function $c : U \to \mathbb{R}_+$ and a monotone submodular function $f : 2^U \to \mathbb{R}_+$, the submodular set cover problem asks to find a subset F of U with the minimum cost satisfying $f(F) = f(U)$, i.e.,

$$\begin{array}{ll} \text{Minimize} & \sum_{i \in F} c(i) \\ \text{subject to} & f(F) = f(U),\ F \subseteq U. \end{array} \tag{2.1}$$

It is known [5, 20] that if f is integer-valued and $f(\emptyset) = 0$, then the problem is $(1 + \ln \max_{j \in U} f(\{j\}))$-approximable by a simple greedy algorithm.

Lemma 2.7 *Problem 2.4 can be formulated as the submodular set cover problem.*

Proof. We denote the graph $(V \cup \{s\}, E \cup F_r^+)$

by H^+. Let $U = F_r^-$,

$$f(F^-) = \sum_{\substack{(W_1,W_2)\in\mathcal{W}_1\times\mathcal{W}_2:\\ \lambda_{H^+}(W_1,W_2)<r(W_1,W_2)}} \Big(\min\{\lambda_{H^++F^-}(W_1,W_2), \lceil r(W_1,W_2)\rceil\} - \lambda_{H^+}(W_1,W_2)\Big) \quad (2.2)$$

for set $F^- \subseteq F_r^-$ where $H^+ + F^- = (V\cup\{s\}, E\cup F_r^+\cup F^-)$, and $c(e) = 1$ for each $e \in F_r^-$. Notice that Problem 2.4 can be formulated as (2.1). Clearly, f is integer-valued and monotone, and we have $f(\emptyset) = 0$. Furthermore, we can prove that f is submodular (we omit the details). These imply this lemma. □

Notice that for two subsets X and Y of V, $\lambda_H(X,Y)$ can be computed in polynomial time by max-flow computation. Hence, it is not difficult to see that all of the above construction can be done in polynomial time. Since $\max_{e\in F_r^-} f(\{e\}) \leq \alpha(\mathcal{W}_1,\mathcal{W}_2)$, we have the following theorem.

Theorem 2.8 *Problem* 1.1 *in undirected/directed graphs is* $O(\log\alpha(\mathcal{W}_1,\mathcal{W}_2))$*-approximable.* □

3 Inapproximability of NAAP

In this section, we show the inapproximability of NAAP, whose complexity status was left open [9, 10, 12]. Namely, we show the $\Omega(\log\alpha(\mathcal{V},\mathcal{W}_2))$-approximability of NAAP, even if (i) G is undirected and $r \in \{0,k\}^{\mathcal{V}\times\mathcal{W}_2}$ holds for any integer $k \geq 1$, (ii) G is directed and $r \equiv k$ holds for any integer $k \geq 1$.

We use a reduction from Hitting Set to prove the inapproximability of the problem.

Problem	Hitting Set
Input:	A finite set U of elements and a family $\mathcal{X} \subseteq 2^U$.
Output:	A minimum set $Z \subseteq U$ such that $Z\cap X \neq \emptyset$ for all $X \in \mathcal{X}$.

It was shown by Raz and Safra [16] that there is a constant c such that Hitting Set is not approximable within a ratio of $c\log|\mathcal{X}|$, unless P=NP, even restricted to the case where $|\mathcal{X}|$ and $|U|$ are polynomially related[1]. Throughout this section, we only consider such instances of Hitting Set.

3.1 Undirected Graphs

In this subsection, we consider the problems in undirected graphs. We first show the $\Omega(\log\alpha(\mathcal{V},\mathcal{W}_2))$-approximability of Problem 2.2 with $\mathcal{W}_1 = \mathcal{V}$ by a reduction from Hitting Set. Notice that this implies inapproximability of NAAP by Lemma 2.3.

Lemma 3.1 *For any integer $k \geq 1$, there exists a constant c such that Problem* 2.2 *with $\mathcal{W}_1 = \mathcal{V}$ and $r \in \{0,k\}^{\mathcal{V}\times\mathcal{W}_2}$ in undirected graphs is not approximable in polynomial time within a ratio of $c\log\alpha(\mathcal{V},\mathcal{W}_2)$, unless* P=NP.

Proof. Given an instance $I = (U, \mathcal{X} = \{X_1, X_2, \ldots, X_q\})$ of Hitting Set, we construct the corresponding instance $J = (G = (V,E), r)$ of Problem 2.2 with $\mathcal{W}_1 = \mathcal{V}$ as follows. Let $V = U\cup\{x_i \mid i = 1,2,\ldots,q\}\cup\{v^*\}$, where v^* and x_i $(i = 1,\ldots,q)$ are new vertices, and let E be the set of $k-1$ multiple undirected edges (v^*, x_i), $i = 1,2,\ldots,q$. Let $\mathcal{W}_2 = \{W_i = X_i\cup\{x_i\} \mid i = 1,2,\ldots,q\}$, and define r by $r(v, W_i) = k$ if $v = v^*$, and 0 otherwise. Then we can prove that if J is β-approximable, then I is 2β-approximable (we omit the details).

Thus, since we have $\alpha(\mathcal{V},\mathcal{W}_2) = |\mathcal{W}_2|$ and Hitting Set is $\Omega(\log|\mathcal{X}|)$-approximable, it follows that Problem 2.2 with $\mathcal{W}_1 = \mathcal{V}$ and $r \in \{0,k\}^{\mathcal{V}\times\mathcal{W}_2}$ is $\Omega(\log\alpha(\mathcal{V},\mathcal{W}_2))$-approximable for any integer $k \geq 1$. □

From Lemmas 2.1 and 2.3, we have the following theorem.

Theorem 3.2 *For any integer $k \geq 1$, there exists a constant c such that* NAAP *with $r \in \{0,k\}^{\mathcal{V}\times\mathcal{W}_2}$ in undirected/directed graphs is not approximable in polynomial time within a ratio of $c\log\alpha(\mathcal{V},\mathcal{W}_2)$, unless* P=NP. □

[1]Here we note that Hitting Set is equivalent to the set cover problem.

3.2 Directed Graphs

We consider NAAP with uniform requirements r in directed graphs, and show that even in this case, it is $\Omega(\log\alpha(\mathcal{V},\mathcal{W}_2))$-approximable. Similar to the arguments given in the previous subsection, we show the $\Omega(\log\alpha(\mathcal{V},\mathcal{W}_2))$-approximability of Problem 2.2 with $\mathcal{W}_1 = \mathcal{V}$ by a reduction from HITTING SET. We omit the details.

Theorem 3.3 *For any integer $k \geq 1$, there exists a constant c such that* NAAP *with uniform requirements $r \equiv k$ in directed graphs is not approximable in polynomial time within a ratio of $c\log\alpha(\mathcal{V},\mathcal{W}_2)$, unless* P=NP. □

4 Augmentation with Modulotone Requirements

In this section, we consider the following augmentation problem:

Problem 4.1 (r^*-AUGMENTATION PROBLEM)
Input: *A multigraph $G = (V, E)$ and a requirement r^*: $2^V \to \mathbb{R}_+$.*
Output: *A minimum set F of new edges such that the resulting graph $\hat{G} = (V, E \cup F)$ satisfies $\delta_{\hat{G}}(X) \geq r^*(X)$ for all $\emptyset \neq X \subsetneq V$.*

As mentioned in Introduction, the problem includes GAP, LAP, NAAP, and Problem 1.1, i.e., global, local, node-to-area, and area-to-area edge connectivity augmentation problems. For example, we can observe that Problem 1.1 can be represented by Problem 4.1 with r^* defined by

$$r^*(X) = \max\{r(W_1, W_2) \mid W_1 \in \mathcal{W}_1, W_2 \in \mathcal{W}_2, W_1 \subseteq X \subseteq V - W_2\}, \tag{4.1}$$

where we define $r^*(X) = 0$ if no $W_1 \in \mathcal{W}_1$ and $W_2 \in \mathcal{W}_2$ exist such that $W_1 \subseteq X \subseteq V - W_2$.

In this section, we give another formulations of NAAP and Problem 1.1. Namely, we show that NAAP and Problem 1.1 with $\max\{|W| \mid W \in \mathcal{W}_1\} = k$ can be characterized as Problem 4.1 with a 1-modulotone and a k-modulotone function r^*, respectively. Recall that r^* is *k-modulotone* if each nonempty subset X of V has a subset W of X with $|W| \leq k$ such that $r^*(Y) \geq r^*(X)$ for all subsets Y of X with $W \subseteq Y$. We remark that a 1-modulotone function is a modulotone function defined in [17], and that an arbitrary set function $r : 2^V \to \mathbb{R}_+$ is n-modulotone since for each nonempty subset X of V, the corresponding W can be chosen as $W = X$.

Lemma 4.2 (i) *Let $r : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}_+$ be a function for $\mathcal{W}_1, \mathcal{W}_2 \subseteq 2^V - \{\emptyset\}$. Then, the set function $r^* : 2^V \to \mathbb{R}_+$ given as* (4.1) *is k-modulotone with $k = \max\{|W| \mid W \in \mathcal{W}_1\}$.*
(ii) *Let $r^* : 2^V \to \mathbb{R}_+$ be a k-modulotone function with $r^*(\emptyset), r^*(V) = 0$ and $k \leq n$. Then, there exists a function $r : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}_+$ with $\max\{|W| \mid W \in \mathcal{W}_1\} = k$ that satisfies* (4.1), *where $\mathcal{W}_1, \mathcal{W}_2 \subseteq 2^V - \{\emptyset\}$.* □

This lemma shows that Problem 1.1 with $\max\{|W| \mid W \in \mathcal{W}_1\} = k$ (resp., NAAP) is equivalent to Problem 4.1 with k-modulotone functions r^* (resp., 1-modulotone functions r^*), since for NAAP, $\mathcal{W}_1 = \mathcal{V}$, i.e., the corresponding $k = 1$.

Corollary 4.3 (i) *Problem* 1.1 *with* $\max\{|W| \mid W \in \mathcal{W}_1\} = k$ *is equivalent to Problem* 4.1 *with k-modulotone functions r^*.*
(ii) NAAP *is equivalent to Problem* 4.1 *with 1-modulotone functions r^*.* □

For a k-modulotone $r^* : 2^V \to \mathbb{R}_+$, let $r : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}_+$ be the corresponding function given in the proof of Lemma 4.2, and let $\alpha(r^*) = \alpha(\mathcal{W}_1, \mathcal{W}_2)$. Similarly, for a 1-modulotone $r^* : 2^V \to \mathbb{R}_+$, let $r : \mathcal{V} \times \mathcal{W}_2 \to \mathbb{R}_+$ be the corresponding function, and let $\alpha(r^*) = \alpha(\mathcal{V}, \mathcal{W}_2)$.

Corollary 4.4 (i) *Problem* 4.1 *with k-modulotone functions r^* is* $O(\log \alpha(r^*))$*-approximable in polynomial time, if r can be constructed from r^* in polynomial time.* (ii) *Problem* 4.1 *with 1-modulotone functions r^* is* $O(\log\alpha(r^*))$*-approximable in polynomial time, if r can be constructed from r^* in polynomial time.*
□

We remark that the reduction used in Lemmas 4.2 is not polynomial; it is not easy to construct in polynomial time the corresponding function $r : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}_+$ from a given k-modulotone function r^*. Hence, Lemmas 4.2 does not indicate that the results in Section 2 lead directly to the approximability of the problem with k-modulotone requirements, since (2.2) cannot be computed efficiently. Thus, it is a future work to construct a nontrivial approximation algorithm for the problem with k-modulotone requirements.

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