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CRAMER-TYPE FORMULA FOR THE POLYNOMIAL SOLUTIONS OF COUPLED LINEAR EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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ABSTRACT

This paper derives a determinant form formula for the general solution of coupled linear equations with coefficients in $K[x_1,\dots,x_n]$, where K is a field of numbers, the number of unknowns is greater than the number of equations, and the solutions are in $K(x_1,\dots,x_{n-1})[x_n]$. The formula represents the general solution by the minimum number of generators, and it is a generalization of Cramer's formula for the solutions in $K(x_1,\dots,x_n)$. Compared with another formula which is obtained by a method typical in algebra, the generators in our formula are represented by determinants of quite small orders.

§1. Introduction

Let K be a field of numbers and let $x_1, \dots, x_n$ be indeterminates. In the following, we often represent x_n as x. Let S denote the field $K(x_1,\dots,x_{n-1})$. This paper considers the general solution of the following coupled linear equations

$$\left\{\begin{array}{l} P_{11}y_1 + \cdots + P_{1s}y_s = P_{1,s+1}, \\ \qquad \cdots \\ P_{r1}y_1 + \cdots + P_{rs}y_s = P_{r,s+1}, \end{array}\right. \tag{1}$$

with unknowns $y_1, \dots, y_s$ in $S[x]$, where $r<s$ and $P_{ij} \in K[x_1,\dots,x_n]$.

It is well known that coupled linear Diophantine equations with coefficients in Z are solved by the Euclidean algorithm and the general solution is represented by s−r generators if the rank of the coefficient matrix is r. Since the Euclidean algorithm applies also to polynomials in $S[x]$, we can solve (1) by applying the Euclidean algorithm successively and obtain s−r generators. However, the procedure is quite tedious and often causes severe coefficient growth (see, for example, ref. 2).

As for coupled linear equations over fields, Cramer's formula gives the general solution in a determinant form. We may, therefore, well expect the existence of determinant form formula which gives all the s−n generators of the solutions of (1). With such a formula, we can calculate the solutions easily without introducing unnecessary coefficient growth. The purpose of this paper is to derive such a formula.

§2. Basic lemmas

In this paper, the variable $x_n = x$ is treated as the main variable, and the degree and the leading coefficient in the main variable x of polynomial P are represented by deg(P) and lc(P), respectively. Furthermore, the resultant of polynomials F and G in x is represented by res(F,G).

[Lemma 1] Given polynomials F, G, H in $K[x_1,\dots,x_n]$ satisfying

$$\begin{cases} \mathrm{GCD}(F,G) = 1, \\ \deg(F) + \deg(G) > \deg(H), \end{cases} \tag{2.1}$$

there exist polynomials A and B in $K[x_1,\dots,x_n]$ such that

$$\begin{cases} AF + BG + RH = 0, \\ R \equiv \mathrm{res}(F,G), \\ \deg(A) < \deg(G), \quad \deg(B) < \deg(F), \end{cases} \tag{2.2}$$

where GCD over the field K is defined by omitting unit factor in K.

(Proof) Let F, G, H, and A, B, R be as follows:

$$\begin{cases} F = f_\ell x^\ell + f_{\ell-1}x^{\ell-1} + \cdots + f_0, \quad f_\ell \neq 0, \\ G = g_m x^m + g_{m-1}x^{m-1} + \cdots + g_0, \quad g_m \neq 0, \\ H = h_k x^k + h_{k-1}x^{k-1} + \cdots + h_0, \quad h_k \neq 0, \end{cases} \tag{2.3}$$

$$A = \begin{vmatrix} f_\ell & f_{\ell-1} & \cdots & f_0 & & & & x^{m-1} \\ & f_\ell & f_{\ell-1} & \cdots & f_0 & & & x^{m-2} \\ & & \cdot & \cdots & & \cdot & & \cdot \\ & & & f_\ell & f_{\ell-1} & \cdots & f_0 & x^0 \\ g_m & g_{m-1} & \cdots & g_0 & & & & 0 \\ & g_m & g_{m-1} & \cdots & g_0 & & & 0 \\ & & \cdot & \cdots & & \cdot & & \cdot \\ & & & g_m & g_{m-1} & \cdots & g_0 & 0 \\ & h_k & h_{k-1} & \cdots & \cdots & & h_0 & 0 \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} \ell \text{ rows} \\ \\ \end{array}, \tag{2.4}$$

$\longleftarrow$ ℓ+m+1 columns $\longrightarrow$

B = [replace the last column of A by $(0 \dots 0\ x^{\ell-1} \dots x^0\ 0)^T$], (2.5)

R = [replace the last column of A by $(0 \dots 0\ 0 \dots 0\ 1)^T$].

Then, we see that (2.2) is satisfied.//

Notes: Expanding R with respect to the rightmost column, we obtain famous Sylvester's determinant for the resultant. The above determinants and Lemma 1 were discovered in the process of generalizing the polynomial remainder sequence in ref. 1. See, also refs. 2 and 3. It is easy to prove that A and B in the above lemma are unique.

[Lemma 2] Let F, G, H, and R be the same as those in Lemma 1 except that $\deg(H) \geqq \deg(F) + \deg(G)$. Then, there exist polynomials A', B', A" and B" in $K[x_1,\dots,x_n]$ such that

$$\begin{cases} A'F + B'G + f_\ell^{k-\ell-m+1}RH = 0, \\ \deg(A') \leqq \deg(H)-\deg(F), \quad \deg(B') < \deg(F), \end{cases} \tag{2.6}$$

$$\begin{cases} A''F + B''G + g_m^{k-\ell-m+1}RH = 0, \\ \deg(A'') < \deg(G), \quad \deg(B'') \leqq \deg(H)-\deg(G). \end{cases} \tag{2.7}$$

(Hint to the proof) Extend the determinants in (2.4) and (2.5).

[Lemma 3] Let $D_{\xi\eta}$ be the following determinant of order r+2 with elements P_{ij}, $i=1,\dots,r,\xi,\eta$, $j=1,\dots,r+2$:

$$D_{\xi\eta} = -D_{\eta\xi} = \begin{vmatrix} P_{11} & \cdots & P_{1r} & P_{1\xi} & P_{1\eta} \\ \vdots & & \vdots & \vdots & \vdots \\ P_{r+2,1} & \cdots & P_{r+2,r} & P_{r+2,\xi} & P_{r+2,\eta} \end{vmatrix}, \tag{2.12}$$

where ξ and η are any two elements of $\{\alpha,\beta,\gamma,\delta\}$. Then, we have

$$D_{\alpha\beta}D_{\gamma\delta} + D_{\alpha\gamma}D_{\delta\beta} + D_{\alpha\delta}D_{\beta\gamma} = 0. \tag{2.13}$$

(Proof) For r = 0 and 1, we can easily prove (2.13) by direct expansion of determinants. Assume (2.13) is true for $r=0,1,\dots,t-1$, and consider the case of $r=t$. Defining

$$(a,b) = (b,a) \equiv P_{1a}P_{1b} + P_{2a}P_{2b} + \cdots + P_{t+2,a}P_{t+2,b},$$

we can represent $D_{\alpha\beta}D_{\gamma\delta}$, $D_{\alpha\gamma}D_{\delta\beta}$, and $D_{\alpha\delta}D_{\beta\gamma}$ as

$$D_{\alpha\beta}D_{\gamma\delta} = \begin{vmatrix} (1,1) & \cdots & (1,t) & (1,\gamma) & (1,\delta) \\ \vdots & & \vdots & \vdots & \vdots \\ (t,1) & \cdots & (t,t) & (t,\gamma) & (t,\delta) \\ (\alpha,1) & \cdots & (\alpha,t) & (\alpha,\gamma) & (\alpha,\delta) \\ (\beta,1) & \cdots & (\beta,t) & (\beta,\gamma) & (\beta,\delta) \end{vmatrix},$$

$D_{\alpha\gamma}D_{\delta\beta}$ = (similar to the above determinant),

$D_{\alpha\delta}D_{\beta\gamma}$ = (similar to the above determinant).

Let us expand these determinants with respect to the last two rows and columns, and consider the coefficient factor in the term proportional to $(\alpha,a)(\beta,b)(\gamma,c)(\delta,d)$, i.e., the factors other than $(\alpha,a)(\beta,b)(\gamma,c)(\delta,d)$ in the term.

Case 1: Terms proportional to $(\alpha,\beta)(\gamma,\delta)$, $(\alpha,\gamma)(\delta,\beta)$, and $(\alpha,\delta)(\beta,\gamma)$. We easily see that, except for the sign, the coefficient factors of these terms are the same, the top-left minor of order t. Hence, the sum of these terms is found to be zero.

Case 2: Terms proportional to $(\alpha,i)(\beta,j)(\gamma,\delta)$, $1\leq i\neq j\leq t$, or terms proportional to $(\alpha,i)(\beta,i)(\gamma,c)(\delta,d)$, $1\leq i\leq t$. There are only two terms which are proportional to $(\alpha,i)(\beta,j)(\gamma,\delta)$, and they come from $D_{\alpha\gamma}D_{\delta\beta}$ and $D_{\alpha\delta}D_{\beta\gamma}$. We easily see that, except for the sign, the coefficient factors of these terms are the same. (Note that the top-left $t\times t$ submatrix is symmetric.) Hence, terms proportional to $(\alpha,i)(\beta,j)(\gamma,\delta)$ cancel each other. The same is true for terms proportional to $(\alpha,i)(\beta,i)(\gamma,c)(\delta,d)$. Similarly, the terms proportional to $(\alpha,i)(\gamma,j)(\delta,\beta)$, etc. and terms proportional to $(\alpha,i)(\gamma,i)(\delta,d)(\beta,b)$, etc. disappear.

Case 3: Only the remaining terms are those proportional to $(\alpha,a)(\beta,b)(\gamma,c)(\delta,d)$, $1\leq a,b,c,d\leq t$, and a, b, c, d are different from each other, hence $t\geq 4$. There are three terms containing $(\alpha,a)(\beta,b)(\gamma,c)(\delta,d)$, and the coefficient factor of each term is the following determinant of order $t-2$:

$$\begin{vmatrix} (1,1) & \cdots & (1,t) \\ \vdots & & \vdots \\ (t,1) & \cdots & (t,t) \end{vmatrix} \leftarrow \text{no } (c',\cdot),\ (d',\cdot) \text{ rows}$$

$\uparrow$ no $(\cdot,a)$, $(\cdot,b')$ columns

where $\{b',c',d'\} = \{b,c,d\}$. If we call the last two columns of the determinant in (2.12) additional columns of types ξ and η, the above determinant is nothing but the product of two determinants of the form (2.12), where the order of the determinants is now $t-2$ and the additional columns are of types a and b' for one determinant and of types c' and d' for the other. Hence, the problem reduces to the case of $r=t-4$. //

§3. Solutions of single equation

We first investigate the following single equation:

$$P_1y_1 + \cdots + P_sy_s = P_{s+1}, \tag{3.1}$$

$$P_i \in K[x_1,\dots,x_n], \quad i=1,\dots,s+1.$$

Without loss of generality, we may assume that

$$\deg(D) = 0 \quad \text{where } D \equiv \mathrm{GCD}(P_1,\dots,P_n). \tag{3.2}$$

(If $\deg(D) \neq 0$, D must pseudo-divide P_{s+1}, i.e., P_{s+1}/D is a polynomial in $S[x]$ so far as (3.1) has solutions. Hence, we have only to divide (3.1) by D, satisfying (3.2).) Furthermore, without loss of generality, we may assume

$$\mathrm{GCD}(P_1,P_2) = 1. \tag{3.3}$$

The reason is as follows: If condition (3.3) is not satisfied, we may construct

$$P'_2 = P_2 + \lambda_3 P_3 + \cdots + \lambda_s P_s, \qquad \lambda_i \in K,$$

such that $\mathrm{GCD}(P_1,P'_2) = 1$ and consider the equation

$$P_1 y_1 + P'_2 y_2 + P_3(y_3 - \lambda_3 y_2) + \cdots + P_s(y_s - \lambda_s y_2) = P_{s+1}.$$

Suppose, for simplicity, that

$$\deg(P_i) < \deg(P_1) + \deg(P_2), \qquad i=3,\dots,s+1. \tag{3.4}$$

Then, Lemma 1 tells that there exist polynomials A_i and B_i in $K[x_1,\dots,x_n]$ such that

$$\begin{cases} A_i P_1 + B_i P_2 + R P_i = 0, \quad i=3,\dots,s+1, \\ R \equiv \mathrm{res}(P_1,P_2). \end{cases} \tag{3.5}$$

Multiplying R to (3.1) and using (3.5), we obtain

$$P_1(Ry_1 - A_3 y_3 - \cdots - A_s y_s + A_{s+1})$$
$$+ P_2(Ry_2 - B_3 y_3 - \cdots - B_s y_s + B_{s+1}) = 0.$$

Since $\mathrm{GCD}(P_1,P_2) = 1$, we have

$$\begin{cases} - Ry_1 + A_3 y_3 + \cdots + A_s y_s - A_{s+1} = - P_2 \tilde{y}, & (3.6) \\ - Ry_2 + B_3 y_3 + \cdots + B_s y_s - B_{s+1} = P_1 \tilde{y}, & (3.7) \end{cases}$$

where $\tilde{y} \in S[x]$. Since R is a unit in S, we can solve (3.6) and (3.7) for arbitrary $y_3, \dots, y_s$, and $\tilde{y}$ in $S[x]$. Therefore, representing the general solution of (3.1) as

$$\bar{y} = u_2 \bar{y}^{(2)} + \cdots + u_s \bar{y}^{(s)} + \bar{y}^{(s+1)}, \tag{3.8}$$

$$\bar{y} \equiv (y_1, y_2, \dots, y_s),$$

u_j, $j=2,\dots,s$, are arbitrary elements in $S[x]$,

we obtain the generators $\bar{y}^{(2)}, \dots, \bar{y}^{(s)}$ and a particular solution $\bar{y}^{(s+1)}$ as

$$\bar{y}^{(2)} = (P_2/R,\ -P_1/R,\ 0 \dots 0), \tag{3.9}$$

$$\bar{y}^{(i)} = (A_i/R,\ B_i/R,\ 0 \,..\, 0,\ 1,\ 0 \,..\, 0), \qquad i=3,\dots,s, \tag{3.10}$$

$$\bar{y}^{(s+1)} = (-A_{s+1}/R,\ -B_{s+1}/R,\ 0 \dots 0). \tag{3.11}$$

Since A_i, B_i and R are represented by determinants whose elements are coefficients of P_1, P_2, and P_i, the above generators are desired ones.

The only remaining task is to remove the restriction (3.4), which is quite easy if we use Lemma 2. That is, if

$$d \equiv \max\{\deg(P_i) \mid i=3,\dots,s+1\} - \deg(P_1 P_2) + 1 \geq 1, \tag{3.12}$$

we generate A_i and B_i, $i=3,\dots,s+1$, in $K[x_1,\dots,x_n]$ such that

$$\begin{cases} A_i P_1 + B_i P_2 + \mathrm{lc}(P_1)^d R P_i = 0, \\ \deg(A_i) \leq \max\{\deg(P_2)-1,\ \deg(P_i)-\deg(P_1)\}, \\ \deg(B_i) < \deg(P_1). \end{cases} \tag{3.13}$$

Then, (3.1) is reduced to the equations

$$\begin{cases} - \mathrm{lc}(P_1)^d R y_1 + A_3 y_3 + \cdots + A_s y_s - A_{s+1} = - P_2 \tilde{y}, & (3.6') \\ - \mathrm{lc}(P_1)^d R y_2 + B_3 y_3 + \cdots + B_s y_s - B_{s+1} = P_1 \tilde{y}, & (3.7') \end{cases}$$

which are directly solved to give the general solution.

§4. Solutions of coulped equations

Suppose equations in (1) are linearly independent in $K(x_1,\dots,x_n)$. Then, without loss of generality, we may assume

$$\Delta \equiv \begin{vmatrix} P_{11} & \cdots & P_{1r} \\ \vdots & & \vdots \\ P_{r1} & \cdots & P_{rr} \end{vmatrix} \neq 0. \tag{4.1}$$

With this assumption, we can rewrite (1) as

$$\begin{cases} \Delta y_1 = -\Delta_{1,r+1}y_{r+1} - \cdots - \Delta_{1s}y_s + \Delta_{1,s+1}, \\ \quad \cdot \qquad\qquad \cdots \\ \Delta y_r = -\Delta_{r,r+1}y_{r+1} - \cdots - \Delta_{rs}y_s + \Delta_{r,s+1}, \end{cases} \tag{4.2}$$

where Δ_{ik}, $1 \leqq i \leqq r$, $r+1 \leqq k \leqq s+1$, is the following determinant:

$$\Delta_{ik} = \begin{vmatrix} P_{11} & \cdots & P_{1,i-1} & P_{1k} & P_{1,i+1} & \cdots & P_{1r} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ P_{r1} & \cdots & P_{r,i-1} & P_{rk} & P_{r,i+1} & \cdots & P_{rr} \end{vmatrix}. \tag{4.3}$$

The following theorem is essential for proving the main theorem.

[Theorem 1] Let $\Delta \neq 0$, and for some i and j, $1 \leqq i \neq j \leqq r$, let $\Delta_{i,r+1} \neq 0$, $\Delta_{j,r+1} \neq 0$, and $GCD(\Delta,\Delta_{i,r+1}) = GCD(\Delta,\Delta_{j,r+1}) = 1$. For t=i and t=j, define R_t as

$$\begin{cases} R_t = lc(\Delta)^d res(\Delta,\Delta_{t,r+1}), \\ d = \max[0,\ \deg(\Delta_{tk}) - \deg(\Delta \cdot \Delta_{t,r+1}) + 1 \mid k=r+2,\dots,s+1]. \end{cases} \tag{4.4}$$

For $k=r+2,\dots,s+1$, construct A_{tk} and B_{tk} satisfying

$$\begin{cases} A_{tk}\Delta + B_{tk}\Delta_{t,r+1} + R_t\Delta_{tk} = 0, \\ \deg(B_{tk}) < \deg(\Delta). \end{cases} \tag{4.5}$$

Then, we have

$$R_jB_{ik} = R_iB_{jk}. \tag{4.6}$$

(Proof) We note that the R_t, A_{tk} and B_{tk} satisfying (4.4) and (4.5) can be calculated by using Lemma 1 or 2. Moving the i-th and j-th columns in Δ etc. to the rightmost, and representing Δ, $\Delta_{i,r+1}$, Δ_{ik}, $\Delta_{j,r+1}$, Δ_{jk} as determinants of the form given in (2.12) with additional columns of types i, j, r+1, k, we obtain the following relation by Lemma 3:

$$\Delta \cdot \tilde{\Delta}_{ijk} + \Delta_{j,r+1}\Delta_{ik} + \Delta_{jk}(-\Delta_{i,r+1}) = 0, \tag{4.7}$$

where $\tilde{\Delta}_{ijk}$ is the following determinant:

$$\tilde{\Delta}_{ijk} = \begin{vmatrix} P_{11} & \cdots & P_{1,r+1} & \cdots & P_{1k} & \cdots & P_{1r} \\ \vdots & & \vdots & & \vdots & & \vdots \\ P_{r1} & \cdots & P_{r,r+1} & \cdots & P_{rk} & \cdots & P_{rr} \end{vmatrix}.$$

$$\qquad\qquad (i) \qquad\qquad (j) \qquad\qquad \leftarrow \text{column number}$$

Let $GCD(\Delta,\Delta_{ik},\Delta_{jk}) = D$, then (4.5) implies $D \mid B_{ik}$ and $D \mid B_{jk}$. Hence, defining

$$\Delta = D\Delta', \quad \Delta_{ik} = D\Delta'_{ik}, \quad \Delta_{jk} = D\Delta'_{jk}, \quad B_{ik} = DB'_{ik}, \quad B_{jk} = DB'_{jk},$$

we can rewrite (4.5) and (4.7) as

$$A_{ik}\Delta' + B'_{ik}\Delta_{i,r+1} + R_i\Delta'_{ik} = 0, \quad \deg(B'_{ik}) < \deg(\Delta'), \tag{4.5'}$$

$$A_{jk}\Delta' + B'_{jk}\Delta_{j,r+1} + R_j\Delta'_{jk} = 0, \quad \deg(B'_{jk}) < \deg(\Delta'), \tag{4.5''}$$

$$\Delta' \cdot \tilde{\Delta}_{ijk} + \Delta'_{ik}\Delta_{j,r+1} - \Delta'_{jk}\Delta_{i,r+1} = 0. \tag{4.7'}$$

Eliminating Δ'_{ik} and Δ'_{jk} from (4.5') and (4.5''), we obtain

$$R_j\Delta'_{jk}(A_{ik}\Delta' + B'_{ik}\Delta_{i,r+1}) = R_i\Delta'_{ik}(A_{jk}\Delta' + B'_{jk}\Delta_{j,r+1}).$$

Eliminating $\Delta'_{jk}\Delta_{i,r+1}$ from (4.7') and the above equation, we obtain

$$\begin{aligned} &\Delta'_{ik}\Delta_{j,r+1}(R_iB'_{jk} - R_jB'_{ik}) \\ &\quad = \Delta'(R_j\Delta'_{jk}A_{ik} - R_i\Delta'_{ik}A_{jk} + R_iB'_{ik}\tilde{\Delta}_{ijk}). \end{aligned} \tag{4.8}$$

We consider only the case of $\Delta \neq D$, because if $\Delta = D$ then $B_{ik} = B_{jk} = 0$. Then, since $GCD(\Delta',\Delta_{j,r+1}) = 1$, (4.8) gives

$$\Delta' \mid \Delta'_{ik}(R_iB'_{jk} - R_jB'_{ik}).$$

Similarly, we obtain

$$\Delta' \mid \Delta'_{jk}(R_iB'_{jk} - R_jB'_{ik}).$$

Since $GCD(\Delta', \Delta'_{ik}, \Delta'_{jk}) = 1$, these relations imply

$$\Delta' \mid (R_iB'_{jk} - R_jB'_{ik}) \quad \text{or} \quad \Delta \mid (R_iB_{jk} - R_jB_{ik}).$$

Since $\deg(\Delta) > \deg(B_{ik}), \deg(B_{jk})$, the above relation leads to (4.6). //

Theorem 1 and (4.8) give the following corollary.

[Corollary] The A_{jk}, $j \neq i$, is calculated from A_{ik}, B_{ik}, R_i and R_j as

$$A_{jk} = (R_j\Delta_{jk}A_{ik} + R_iB_{ik}\tilde{\Delta}_{ijk})/R_i\Delta_{ik}. // \tag{4.9}$$

This relation is quite useful in actual calculations because calculation of R_i, A_{ik} and B_{ik} is quite time consuming.

Now, we prove the main theorem.

[Theorem 2] Let $\Delta \neq 0$, and for $i=1,\dots,r$ let $\Delta_{i,r+1} \neq 0$ and $GCD(\Delta, \Delta_{i,r+1}) = 1$. Representing the general solution of (1) as

$$\bar{y} = u_{r+1}\bar{y}^{(r+1)} + \cdots + u_s\bar{y}^{(s)} + \bar{y}^{(s+1)}, \tag{4.10}$$

u_j, $j=r+1,\dots,s$, are arbitrary elements in $S[x]$,

the generators $\bar{y}^{(r+1)}, \dots, \bar{y}^{(s)}$ and a particular solution $\bar{y}^{(s+1)}$ are given as

$$\left\{\begin{array}{l} \bar{y}^{(r+1)} = (\Delta_{1,r+1}/R_r, \ \dots, \ \Delta_{r,r+1}/R_r, \ -\Delta/R_r, \ 0 \ .. \ 0), \\ \bar{y}^{(r+2)} = (A_{1,r+2}/R_1, \ \dots, \ A_{r,r+2}/R_r, \ B_{r,r+2}/R_r, \ 1, \ 0 \ .. \ 0), \\ \cdot \qquad \cdot \quad \cdot \quad \cdot \\ \bar{y}^{(s)} = (A_{1s}/R_1, \ \dots, \ A_{rs}/R_r, \ B_{rs}/R_r, \ 0 \ .. \ 0, \ 1), \\ \bar{y}^{(s+1)} = (-A_{1,s+1}/R_1, \ \dots, \ -A_{r,s+1}/R_r, \ -B_{r,s+1}/R_r, \ 0 \ .. \ 0), \end{array}\right. \tag{4.11}$$

where R_t, A_{tk}, and B_{tk}, $t=1,\dots,r$, $k=r+2,\dots,s+1$, are defined by (4.4) and (4.5).

(Proof) We first solve the last equation in (4.2) by the method described in §3. Representing the general solution of the equation as

$$\bar{y}' = u_{r+1}\bar{y}'^{(r+1)} + \cdots + u_s\bar{y}'^{(s)} + \bar{y}'^{(s+1)},$$

$$\bar{y}' \equiv (y_r, \ y_{r+1}, \ \dots, \ y_s),$$

we obtain

$$\bar{y}'^{(r+1)} = (\Delta_{r,r+1}/R_r, \ -\Delta/R_r, \ 0 \ \dots \ 0),$$
$$\bar{y}'^{(r+2)} = (A_{r,r+2}/R_r, \ B_{r,r+2}/R_r, \ 1, \ 0 \ .. \ 0),$$
$$\cdot \quad \cdot \quad \cdot$$
$$\bar{y}'^{(s)} = (A_{rs}/R_r, \ B_{rs}/R_r, \ 0 \ .. \ 0, \ 1),$$
$$\bar{y}'^{(s+1)} = (-A_{r,s+1}/R_r, \ -B_{r,s+1}/R_r, \ 0 \ \dots \ 0).$$

Substituting the above solution $\bar{y}'$ into the i-th equation in (4.2), we have

$$\begin{aligned} \Delta y_i = &+ (\Delta_{i,r+1}\Delta)u_{r+1}/R_r \\ &- (\Delta_{i,r+1}B_{r,r+2} + R_r\Delta_{i,r+2})u_{r+2}/R_r \\ &- \quad \cdot \quad \cdot \quad \cdot \\ &- (\Delta_{i,r+1}B_{rs} + R_r\Delta_{is})u_s/R_r \\ &+ (\Delta_{i,r+1}B_{r,s+1} + R_r\Delta_{i,s+1})/R_r. \end{aligned}$$

Consider the coefficient of u_k and the last term of this equation:

$$(\Delta_{i,r+1}B_{rk} + R_r\Delta_{ik})/R_r, \quad k=r+2,\dots,s+1.$$

Using the relation (4.5), we can rewrite this expression as

$$\begin{aligned} &(\Delta_{i,r+1}R_iB_{rk} + R_rR_i\Delta_{ik})/R_iR_r \\ &\quad = - A_{ik}\Delta/R_i + \Delta_{i,r+1}(R_iB_{rk} - R_rB_{ik})/R_iR_r. \end{aligned}$$

Owing to Theorem 1, the last term of this expression vanishes and we obtain

$$y_i = + \Delta_{i,r+1} u_{r+1} / R_r + A_{i,r+2} u_{r+2} / R_i + \cdots + A_{is} u_s / R_i - A_{i,s+1} / R_i.$$

Therefore, $y_i \in S[x]$, $i=1,\ldots,r-1$, and combining the above solution and $\bar{y}'$, we obtain the generators (4.11).//

Notes: We can calculate the solutions of (1) by successively solving each equation in (1) and substituting the solution into the yet unsolved equations. This method introduces extremely large factors which exactly cancel each other between the numerator and denominator. However, proof of the cancellation is quite tedious.

[Corollary] Let $GCD(\Delta, \Delta_{i,r+1}, \ldots, \Delta_{is}) = D_i$. If $\deg(D_i) > 0$ and $D_i \nmid \Delta_{i,s+1}$ over S then (1) has no solution. If $D_i \mid \Delta_{i,s+1}$ over S and

$$GCD(\Delta/D_i, \Delta_{i,r+1}/D_i) = 1, \quad i=1,\ldots,s,$$

then we can construct R_i, A_{ik} and B_{ik} $k=r+2,\ldots,s+1$, such that

$$\begin{cases} R_i = lc(\Delta/D_i)^d res(\Delta/D_i, \Delta_{i,r+1}/D_i), \\ d = \max[0,\ \deg(\Delta_{ik}/D_i) - \deg(\Delta \cdot \Delta_{i,r+1}/D_i^2)+1 \mid k=r+2,\ldots,s+1], \end{cases} \tag{4.12}$$

$$\begin{cases} A_{ik}(\Delta/D_i) + B_{ik}(\Delta_{i,r+1}/D_i) + R_i(\Delta_{ik}/D_i) = 0, \\ \deg(B_{ik}) < \deg(\Delta/D_i), \end{cases} \tag{4.13}$$

and the formula (4.11) is still valid.

(Proof) Eq. (4.13) gives (4.5), and (4.7) is still valid. Hence, the proof of Theorem 2 also applies to this case.//

§5. Comparison with another formula

Using the idea of Hermann[4] (see, also Seidenberg[5]), we can easily represent the generators of the solutions of (1) in a determinant form. Equations in (4.1) shows that (1) has the following solutions

$$\begin{cases} \bar{y}_{ap}^{(r+1)} = (\Delta_{1,r+1}, \ldots, \Delta_{r,r+1}, -\Delta, 0 \ldots 0), \\ \cdot \qquad \cdots \\ \bar{y}_{ap}^{(s)} = (\Delta_{1s}, \ldots, \Delta_{rs}, 0 \ldots 0, -\Delta). \end{cases} \tag{5.1}$$

We call these solutions apparent solutions. Let

$$d = \max[\deg(P_{ij}) \mid i=1,\ldots,r,\ j=1,\ldots,s]. \tag{5.2}$$

Then $\deg(\Delta)$ and $\deg(\Delta_{ik})$ are less than or equal to rd. Following Hermann, it is easy to prove that every solution of (1) with $\deg(y_i) \geqq rd$ for some i can be represented by apparent solutions. The remaining solutions can be represented as

$$y_i = c_{i,rd-1} x^{rd-1} + c_{i,rd-2} x^{rd-2} + \cdots + c_{i0}, \quad i=1,\ldots,s, \tag{5.3}$$
$$c_{ij} \in S, \quad j=0,1,\ldots,rd-1.$$

Substituting (5.3) into (1) and equating coefficients of x^k terms, $k=0,\ldots,rd+d-1$, to zero, we obtain $r(rd+d)$ coupled equations for srd unknowns c_{ij}. Since $c_{ij} \in S$, the solutions of these equations are given by Cramer's formula.

The method described above is given in refs. 4 and 5, and it is a typical method in algebra. The degree in x of a generator obtained by this method is less than or equal to rd−1, which is the same as our formula. However, in this method, the order of numerator and denominator determinants is rd(r+1) which is considerably greater than 2rd−1, the order of determinants for A_{ik} and B_{ik}. Furthermore, the number of generators in the above method is as many as rd(s−r−1), which is very inconvenient in actual applications. On the other hand, the number of generators in our method is only s−r. Hence, our formula is much more beautiful and useful than the above method.

Finally, we present an example for the case of r=2 and s=5.

$$\begin{cases} (x^2+1)y_1 + (x^2+x+1)y_2 + (2x^2-x)y_3 + (2x^2+3)y_4 + (2x^2-3x+1)y_5 = 0, \\ (x^2+2x+2)y_1 + (3x^2-x+1)y_2 + (x^2+3x+5)y_3 + (2x^2-x+3)y_4 + (3x^2-1)y_5 = 0. \end{cases} \tag{5.4}$$

The Δ and Δ_{ik}, i=1,2, k=3,4,5, are calculated as

$$\Delta = 2x^4 - 4x^3 - x^2 - 5x - 1,$$
$$\Delta_{13} = 5x^4 - 9x^3 - 6x^2 - 9x - 5,$$
$$\Delta_{14} = 4x^4 - 3x^3 + 7x^2 - 5x,$$
$$\Delta_{15} = 3x^4 - 14x^3 + 6x^2 - 3x + 2,$$
$$\Delta_{23} = - x^4 + 4x^2 + 5x + 5,$$
$$\Delta_{24} = - 5x^3 - 2x^2 - 7x - 3,$$
$$\Delta_{25} = x^4 - x^3 + 3x^2 + 4x - 3.$$

These polynomials give the R_1 and R_2 as

$$R_1 = \mathrm{res}(\Delta,\Delta_{13}) = -396,$$
$$R_2 = \mathrm{res}(\Delta,\Delta_{23}) = -165.$$

Hence, the conditions in Theorem 2 are satisfied. Using determinant representations (2.4) and (2.5), we can calculate A_{ik} and B_{ik}, i=1,2, k=4,5, as

$$A_{14} = 12(- 1135x^3 - 932x^2 - 1078x - 410),$$
$$B_{14} = 12(+ 454x^3 + 282x^2 + 511x + 82),$$
$$A_{15} = 12(+ 470x^3 + 604x^2 + 341x + 199),$$
$$B_{15} = 12(- 188x^3 - 204x^2 - 152x - 53),$$
$$A_{24} = 5(+ 227x^3 + 595x^2 + 651x + 509),$$
$$B_{24} = 5(+ 454x^3 + 282x^2 + 511x + 82),$$
$$A_{25} = 5(- 94x^3 - 290x^2 - 327x - 166),$$
$$B_{25} = 5(- 188x^3 - 204x^2 - 152x - 53).$$

Formula (4.11) gives the generators of the general solution as

$$\bar{y}^{(3)} = (\Delta_{13}, \Delta_{23}, -\Delta, 0, 0),$$
$$\bar{y}^{(4)} = (A_{14}/R_1, A_{24}/R_2, B_{24}/R_2, 1, 0),$$
$$\bar{y}^{(5)} = (A_{15}/R_1, A_{25}/R_2, B_{25}/R_2, 0, 1).$$

That these generators satisfy (5.4) is easily checked. Furthermore, we can easily check the validity of relations (4.6) and (4.9).

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