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項書き換え理論における順序構造と証明論的順序数の理論

―― 項書き換えシステムによる記号処理への論理学の応用にむけて ――

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本論の目的は項書き換えシステムの理論で使用されている順序構造と論理学における証明論的順序数の理論との密接な関係を明らかにすることにある。我々はアッケルマン順序数の体系を半順序の理論に一般化し、その上でこの一般化されたアッケルマン順序数の体系と recursive path ordering, lexicographic path ordering, semantic path ordering 等の項書き換え理論に現われる順序構造との関係を明らかにする。証明論的順序数は論理学において帰納的関数全体のある部分クラスに対する計算の複雑さの指標として用いられて来たので、我々の与える関係から項書き換えシステムによる計算の複雑さについての情報が得られる。

Ordering Structures of Term Rewriting Theory and Theory of Proof Theoretic Ordinals

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The purpose of this paper is to show a close relationship between proof theoretic ordinals in logic and ordering structures used in term rewriting theory. We generalize the system of Ackermann's ordinals as a theory of partial ordering, to elucidate thereby its relationship with several orderings of term rewriting theory. Since proof theoretic ordinals have been used as a measure of computational complexity for a subclass of the recursive functions, as a corollary we have much information on the computational complexity of term rewrite systems.

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Introduction.

A term rewrite system R over a set T of terms is a set of rewrite rules of the form $s \to t$, where s and t are terms of T which may contain variables ranging over T (cf. Huet-Oppen[14]). $s \to {}^*t$ denotes that s is obyained by a successive number of rewritings from t. A rewrite system is called terminate if every successi rewriting sequence stops in finite steps. The question whether or not a given rewrite system terminates is called termination problem. The termination problem is often reduced to the well foundedness problem for a known ordering structure. For example, for a partially ordered structure $\langle D,<\rangle$, if we find a mapping $f:T \to D$, and if $s \to t \Rightarrow f(s)<f(t)$, we say "the term rewrite system is embedable into $\langle D,<\rangle$. Then if a term rewite system is embedable into $\langle D,<\rangle$, the well foundedness for $\langle D,<\rangle$ implies the termination for the rewrite system. There are many important ordering structures from this point of view. In this paper we mainly consider the recursive path ordering, lexicographic path ordering, semantic path ordering, and their variants.

The purpose of this paper is to show a close relationship between proof theoretic ordinals in logic and some ordering structures used in term rewrite theory. We generalize the system of Ackermann's ordinals [1] which is one of proof theoretic ordinals, to the theory of partial ordering, to elucidate thereby its relationship with the recursive path ordering of Dershowitz [6], the lexicographic path ordering of Kamin-Levy [16], the semantic path ordering of Plaisted (cf. [9]), and others. Since proof theoretic ordinals have been used in logic as a measure of computational complexity for a subclass of the recursive functions, as a corollary we have information on computational complexity of term rewrite systems. For this purpose we introduce from logic two notions; ordinal recursive functions and provably recursive functions. We show that fairly simple term rewrite systems may have very strong computational power, which is much stronger than the computational power of the primitive recursive functions or the provably recursive functions in Peano Arithmetic. We also give an example of the use of a stronger proof theoretic ordinals than Ackermann's ordinals, which is called 'ordinal diagrams' (cf. Okada-Takeuti [23]). Such stronger proof theoretic ordinals seem very useful for further development of term rewriting theory, like the cases of non-simplification rewrite systems or conditional rewrite systems.

Kruskal theorem (on homeomorphic embedding) and its variations are commonly used for well foundedness poof of a given ordering structure. This method was firstly introduced by Dershowitz[6]. There have been several attempts to extend Kruskal theorem,(independently Puel, Leeb[20], and Friedman [33]). We show that Friedman's extremely strong form of Kruskal theorem is applicable for well foundedness proof of the system of ordinal diagrams. We also sketch an axiomatic approach for proving well foundedness of Ackermann type orderings. The usual proof of Kruskal theorems is based on the "minimal bad sequence" argument, which has strongly non-constructive character. On the other hand, our axiomatic approach provides not only more constructive proof but also a nice logical framework for well foundedness proof of a given ordering structure.

§1. Basic Notations.

Now we give some basic notations and basic facts.

Definition. (1) $\leq$ is a quasi ordering on a set D if (i) $a \leq a$ for all $a \in D$ and (ii) for $a,b,c \in D$, $a \leq b$ and $b \leq c \Rightarrow a \leq c$.

(2) $\leq$ is a partial ordering on D if $\leq$ is a quasi ordering on D and (iii) for $a,b \in D$, $a \leq b$ and $b \leq a \Rightarrow a=b$.

(3) $\leq$ is a linear ordering on D if $\leq$ is a partial ordering on D and (iv) for $a,b \in D$, $a \leq b$ or $b \leq a$.

(4) A quasi ordering $\leq$ on D is called a well quasi ordering if for any sequence $a_1, a_2, \ldots$ from D, there exist i and j such that $i<j$ and $a_i \leq a_j$.

(5) A quasi ordering $\leq$ on D is called a well founded ordering if for any descendeng sequence $a_1 \geq a_2 \ldots$ from D there exist i and j such that $i<j$ and $a_i \leq a_j$.

(6) A linear ordering $\leq$ on D is called well ordering if it is well founded.

Fact. (1) If a quasi ordering is a well quasi ordering, then it is a well founded ordering.

(2) If $\leq$ is a linear ordering, then the notions of well quasi ordering, well founded ordering, and well ordering are equivalent.

(3) If $\leq$ is a partial ordering, then the notion of well founded is the same as the following condition: Any strictly descending sequence $a_1 > a_2 \ldots$ from D stops in a finite steps.

In this paper we mainly consider so called precedence orderings. These are ordering systems based on a given precedence (ordering) on operators and constants: Such precedence orderings are orderings on a set T(F,C) of terms, where each term in T(F,C) is constructed from constants from C and operators from F. We are interested in the following two types of theorems.

(1) For any well quasi ordered sets C and F, T(F,C) is well quasi ordered by a given ordering.

(2) For any well founded ordered sets C and F, T(F,C) is well-founded by a given ordering.

Practically speaking, from the view point of termination problem of term rewrite system, (2) is enough. However, for the most of term rewrite orderings in literature (including all the orderings in this paper), (1) implies (2) (by using Zorn's Lemma (cf.[6])).So in this paper we often consider theorems of the form (1).

Examples. Let F= (*,...,*) , where (*,...,*) is the n-tuple. Let C be a given well quasi ordered set of constants. Consider the set $T_n=T(\{()\},C)$ of terms by the following constructing rules: If $c\in C$ then $c\in T_n$; if $t_1,\dots,t_n\in T_n$ then $(t_1,\dots t_n)\in T_n$.

The n-ordering $\leq_n$. $(t_1,\dots,t_n)\leq_n(s_1,\dots,s_n)$ iff for some permutation of $(1,\dots,n)$, say $(j_1,\dots,j_n)$, $t_k\leq s_{j_k}$ for all $k\leq n$.

The lexicographic ordering $\leq_\ell$. $(t_1,\dots,t_n)\leq_\ell(s_1,\dots,s_n)$ iff $t_1=s_1,\dots,t_k=s_k$ and $t_{k+1}<_\ell s_{k+1}$ for some k n, or $t_i=s_i$ for all $i\leq n$.

For both of the above orderings, $c\leq d$ for $c,d\in C$ is defined in the sense of the precedence C, and for $c\in C$, $c\leq(t_1,\dots t_n)$ always holds, but $(t_1,\dots,t_n)\leq c$ never holds.

Then for any well quasi ordered set C, T_n is well quasi ordered by both of the above orderings.

Now we introduce the notions of direct system and direct limit. Consider a sequence $\{S_n\}_{n\in\omega}$ of sets of terms. If there exists $H_{ij}:S_i\to S_j$ for all i,j where $i\leq j$, such that (1) H_{ii} is an identity function, (2) $H_{jk}\circ H_{ij}=H_{ik}$, then $(\{S_n\}_n\ \{H_{ij}\}_{i,j\in\omega})$ is called a direct system. Then we introduce $\approx$ on $\cup S_n$ and D as follows.

(1) $s\approx t$ iff $i\leq k$, $j\leq k\Rightarrow H_{ik}(t)=H_{jk}(s)$.

(2) Take $D=\cup S_n/\approx$.

Then the following $(D,\{f_i\}_i)$ is called a direct limit for the direct system above; $f_i:S_i\to D$ satisfies

(1) $f_j\circ H_{ij}=f_i\ (i\leq j)$.

(2) For any set X and any $g_i:S_i\to X$ such that $g_j\circ H_{ij}=g_i\ (i\leq j)$, there exists a unique $f:D\to X$ such that $f\circ f_i=g_i$.

Examples. We give some examples of direct system and direct limit.

(1) Consider the ordering $\leq_n$ on T_n in the above example. Define $H_{ij}:T_i\to T_j$ (where $i\leq j$) by

$$H_{ij}((t_1,\dots,t_i))=(t_1,\dots,t_i,\underbrace{0,\dots,0}_{j-i}).$$

Here 0 is a minimal element of C. Then $\{T_n\}_n,\{H_{i,j}\}_{i,j}$ is a direct system. Define $T=\cup T_n$. Define the ordering $\leq$ as follows: $(t_1,\dots,t_n)\ll(s_1,\dots,s_m)$ iff $n\leq m$ and for some permutation of $(1,\dots,m)$, say $(j_1,\dots,j_m)$, $t_k\ll s_j$ for all $k\leq n$. Then $\langle T,\ll\rangle$ is a direct limit (of $(\{T_n\},\{H_{i,j}\})$).

(2) Consider the lexicographic ordering $\leq_\ell$ on T_n. Then again $(\{T_n\},\{H_{ij}\})$ is a direct system. The natural extension of the lexicographic ordering to $T=\cup T_n$ is the direct limit.

Category theoretic interpretation.

Quasi ordered structure $\langle D,\leq\rangle$ can be naturally interpreted in category theory. D is considered a set of objects. $s\leq t$ is interpreted as " there exists a morphism $f:s\to t$". Then the first condition of quasi orderedness "$s\leq s$ for all s in D" corresponds to the condition of category "there exists an identity morphism I_s for all s in D", and the second condition " if $s\leq t$ and $t\leq r$ then $s\leq r$" corresponds to the condition of category "for any $f:s\to t$ and $g:t\to r$, there exists a morphism $h:s\to r$". A category D is called "well" if for any sequence $s_1,s_2,\dots$ of objects there exists a morphism f and i,j such that $i<j$ and $f:s_i\to s_j$. Then the notion of well quasi orderedness for a quasi ordering corresponds to the notion of "well" for the corresponding category. Then our notion of "direct limit" above corresponds to the usual notion of (the dual of) direct limit of a direct system $\{H_{ij}\}$ of functors.

§2. A relationship between Ackermann's ordering and term rewriting orderings

Definition. (The set of generalized Ackermann terms $A_n(F,C)$). Let F be a set of operators, C a set of constants. Then

(1) If $c\in C$ then $c\in A_n(F,C)$.

(2) If $t_1,\dots,t_n\in A_n(F,C)$, $f\in F$, then $f(t_1,\dots,t_m)\in A_n(F,C)$.

(3) If $t_1,\dots,t_m\in A_n(F,C)$, then $t_1\#\dots\#t_m\in A_n(F,C)$.

Each element of $A_n(F,C)$ is called an Ackermann term. An Ackermann term of the form $f(t_1,\dots t_n)$ is called a connected term.

Definition. (The Ackermann ordering on $A_n(F,C)$) Let F and C be partially ordered.

Case 1. Let $s=f(s_1,\dots,s_n)$, $t=g(t_1,\dots,t_n)$. Then $s>t$ if

(1) $s_i\geq t$ for some i $(1\leq i\leq n)$, or

(2) $f=g$, $s_1=t_1,\dots,s_{i-1}=t_{i-1}$, $s_i>t_i$, $s>t_{i+1},\dots,s>t_n$, for some $i(1\leq i\leq n)$.

(3) $f>g$, $s>t_i$ for all $i\ (1\leq i\leq n)$.

Case 2. If $s,t \in C$, then $s>t$ is $s \succ t$ in C. If $s \in C$, $t \notin C$, then $s<t$ holds, but $s \geq t$ does not hold, for connected t.

Case 3. Let $s = s_1 \# \ldots \# s_m$, $t_1 \# \ldots \# t_1$. Then $s>t$ if $\{s_1, \ldots s_m\} \gg \{t_1, \ldots, t_k\}$, where $\gg$ is the multiset ordering induced by $>$, in the sense of Dershowitz-Manna[7]. More precisely,

$s=\{s_1,\ldots,s_m\} \gg \{t_1,\ldots,t_k\}=t$ if

$s_i > t_{j1}, t_{j2}, \ldots, t_{jh}$ for some i

and $s-\{s_i\} \gg t-\{t_{j1}, t_{j2}, \ldots, t_{jk}\}$.

Theorem. (1) For any f and c, the system $A_n(\{f\},\{c\})$ is the same as Ackermann's system of ordinal notation; more precisely, Ackermann's original system [1] is $A_3(\{f\},\{0\})$ and its generalization $A_n(\{f\},\{0\})$. $A_2(\{f\},\{0\})$ is Feferman-Schütte's system (cf.[31]§14) of ordinal notations less than Γ_0. For any finite total ordered set F and C, the ordertype of $A_2(F,C)$ is also Γ_0. Moreover, even if C is any well ordered set of order type less than Γ_0, the order type of $A_2(F,C)$ is Γ_0 for any finite F.

(2) $A_1(F,C)$ is the same as the multiset extension of Dershowitz's recursive path ordering over the set of terms $\tau(FUC)$. In other words, if we pay attention only to the connected terms of $A_1(F,C)$, the Ackermann's ordering is exactly the same as the recursive path ordering. Here a term of the form $f(t_1,\ldots,t_m) \in \tau(F,C)$ in Dershowitz's system of recursive path ordering is interpreted as $f(t_1 \# \ldots \# t_m)$ in $A_1(F,C)$.

(3) For any finite set C or for any well-ordered set C of order type less than ε_0, $A_1(\{f\},C)$ is a well ordered set of order type ε_0. For any well ordered set F of order type α and C above, $A_1(F,C)$ is a well ordered set of order type $\phi_\alpha(0)$ (of Feferman-Schütte's ordinal[31]).

Dershowitz[9] noticed the ordering of Γ_0 as an extention of his recursive path ordering. Hence we call $A_2(F,C)$ the extended recursive path ordering.

Definition. The system $A_n^*(F,C)$ is obtained from $A_n(F,C)$ by avoiding any use of #. More precisely, the set $A_n(F,C)$ of terms is obtained using only (1) and (2) of the definition of $A_n(F,C)$, (i.e., by deleting (3)).

Remark. The set $A_n{}^*(F,C)$ is identified with the set of n-branching trees, and the set $A_1(F,C)$ is identified with the set of finitely branching forests, ignoring the order of branchings of each node.

From now on we assume F and C are well-quasi ordered. As Dershowitz proved, for any $s,t \in \tau(FUC)$ (or $A_1(F,C)$),

(*) $\qquad s \lesssim_{ho} t \Rightarrow s \lesssim_{rpo} t,$

where $\lesssim_{ho}$ is homeomorphic embedding between trees (or forests), in the sense of Kruskal[19](cf.[9]). Then the well quai orderedness for $\lesssim_{ho}$ on $\tau(F,C)$ (or $A_1(F,C)$) (which is called Kruskal Theorem (cf.[19])) implies the well quasi orderedness of $\lesssim_{rpo}$ on $\tau(F,C)$ (or on $A_1(F,C)$). This argument can be extended to the case of $A_n(F,C)$, by naturally extending the notion of homeomorphic embedding and the Kruskal theorem. Hence we have

Theorem. (1) For any well quai ordered sets F and C, $A_n(F,C)$ is well quasi ordered for each n.

(2) For any well founded sets F and C, $A_n(F,C)$ is well founded for each n.

Another commonly used method to prove well quasi orderedness for a given precedence ordering is the so called "minimal bad sequence argument", which was originally introduced for a simple proof of the Kruskal theorem in[21]. However, the proof by "bad sequence argument" or by the Kruskal theorem has a strongly non-constructive character. In fact we cannot obtain any fairly small formal system in which these proofs are formulated. Now we introduce an axiomatic method for proving well foundedness, to provide a fairly small formal system in which the proof is formulated. This method would be useful when one would like to consider an upper bound of the computational power of a given term rewrite system (cf. §3).

We consider the following system S. S is based on PA (Peano Arithmetic). Here we assume that PA has a suitable form of mathematical induction rule (like a form of induction on the construction of terms (cf. [22]). Moreover, S has the new predicate W(t), which means "t is well founded", in other words, "any descending sequence begining with t terminates". We have the following inductive definition schemata for W, as axioms of S.

(1) $\forall x(x<t \supset W(x)) \supset W(t)$.

(2) For any formula F(x) of S,

$$\forall y(\forall x(x<y \supset F(x)) \supset F(y)) \supset \forall z(W(Z) \supset F(z)).$$

S has the definition of $<$ on $A_n(F,C)$, as axioms. We aslo assume that F, C and their orderings are primitive recursive. Then we have the following theorem.

Theorem. The following formula is provable in S.

$\forall x(F(x) \supset W(x)) \wedge \forall x(C(x) \supset W(x)) \supset \forall x W(x)$.

(1),(2) above are considered a special case of the inductive definition in the sense of [11]. Hence S is a subsystem ID_1 of (non-iterated) inductive definition. On the other hand, these schemata of inductive definition are provable in the full second order arithmetic. For example, the predicate W above can be defined in the second order arithmetic as $W(a) = \forall P(\forall y(\forall x(x<y \supset P(x)) \supset P(y)) \supset P(a))$.

Now we extend the Ackermann's ordering of $A_n(F,C)$ to a direct limit $A_\omega(F,C)$. The set $A_\omega(F,C)$ is defined in the same way as $A_n(F,C)$, except that for each $f \in F$, f may have m-argument place for any m, in other words, we have a term of the form $f(t_1,\ldots,t_m)$ for any m. The Ackermann's ordering $<'$ for A (F,C) is defined in the same way as before, where, when we compare $f(t_1,\ldots,t_n)$ with $g(s_1,\ldots,s_m)$ and $n<m$, we interprete $f(t_1,\ldots,t_n)$ as $f(\underbrace{0,\ldots,0}_{m-n},t_1,\ldots t_n)$, then follow the definition of the ordering on $A_n(F,C)$ as before. $<$ is the same as $<'$ but we interprete $f(t_1,\ldots,t_n)$ as $f(t_1,\ldots,t_n, 0,\ldots,0)$. Here 0 is a minimal element of C. In other words, the direct limit $\langle A(F,C),<'\rangle$ is defined by $H_{nm}(f)(t_1,\ldots,t_n)=f(0,\ldots,0,t_1,\ldots,t_n)$, and the direct limit $\langle A(F,C),<\rangle$ is defined by $H_{nm}(f)(t_1,\ldots,t_n)=f(t_1,\ldots,t_n,0,\ldots,0)$. $A^*(F,C)$ and $<', <$ for $A^*(F,C)$ are defined in the same way as before.

Theorem. (1) $\langle A(\{f\},\{c\}),<'\rangle$ is the same as Schutte's system of ordinals of §11 in [30].

(2) $\langle A_\omega^*(F,C),<\rangle$ is the same as the lexicographic path ordering of Kamin-Levy [16].

(3) $\langle A_2^*(\{f\},\{0\}),<\rangle$ (and $\langle A_2^*(F,C),<\rangle$ for any finite sets F,C has the order type ε_0.

Theorem.(1) For any well quasi ordered sets F and C, $A_\omega(F,C)$ is well quasi ordered by $<'$.

(2) For any well founded sets F and C, $A_\omega(F,C)$ is well founded by $<'$.

(3) The above theorems do not hold for $<$. In fact, A ($\{f,g\},\{0\}$) is not well founded by $<$. On the other hand, each $A_n^*(F,C)$ is well quasi ordered by $<$ for any well quasi ordered sets F and C.

In the rest of this section, we give a relationship between the Ackermann's ordering and the semantic path ordering of Plaisted. The following is a formulation of the semantic path ordering by Dershowitz[8], as the theory of quasi-ordering.

Definition. (The semantic path ordering) Let $\succsim$ be a quasi ordering on $A_n(F,C)$.

<u>Case 1.</u> Let $s=f(s_1,\ldots,s_n)$ and $t=g(t_1,\ldots t_n)$. Then $s \gtrsim_{SPO} t$ if

(1) $s_i \gtrsim_{SPO} t$ for some i $(1 \le i \le n)$, or

(2) $s \succ t$ and $s >_{SPO} t_j$ for all $j(1 \le j \le n)$, or

$s \approx t$ and $\{s_1,\ldots,s_n\} \ggcurly_{SPO} \{t_1,\ldots,t_n\}$.

<u>Case 2</u> and <u>Case 3</u> are the same as those of the definition of the Ackermann's ordering.

Definition. Consider the following orderings.

(1) $f(s_1,\ldots,s_n) <_0 g(t_1,\ldots,t_n)$ on $A_n(F,C)$ iff $f<g$ in the precedence F.

(2) $f(s_1\#\ldots\#s_n) <_1 g(t_1\#\ldots\#t_m)$ on $A_1(F,C)$ iff

(i) $f<g$ in F, or

(ii) f=g and $\{s_1,\ldots,s_n\} \ll_{SPO} \{t_1,\ldots,t_m\}$, where $\ll_{spo}$ is the multisets extension of $\lesssim_{SPO}$.

(3) $f(s_1,\ldots s_n) <_{lo} g(t_1,\ldots,t_m)$ on $A_n(F,C)$ iff

(i) $f<g$ in F, or

(ii) f=g, $s_1=t_1,\ldots,s_{i-1}=t_{i-1}$, $s_i <_{SPO} t_i$, for some i $(1 \le i \le n)$, or

(iii) f=g, $s_1=t_1,\ldots,s_n=t_n$, $n<m$.

Theorem. (1) (Kamin-Levy [16]) If we take $<_0$ for $\prec$, $<_{SPO}$ on $A_\omega^*(F,C)$ is the same as the recursive path ordering on A_1 (F,C).

(ii) If we take $<_1$ for $\prec$, the semantic path ordering on A_1 (F,C) is the same as the recursive path ordering on $A_1(F,C)$.

(3) If we take $<_{lo}$ for $\prec$,

(i) $<_{SPO}$ on A (F,C) is the same as the Ackermann's ordering, therefore

(ii) $<_{SPO}$ on A *(F,C) is the same as the lexicographic path ordering, and

(iii) $<_{SPO}$ on $A_1(F,C)$ is the same as the recursive path ordering.

§3. Computational Complexity of rewrite systems based on precedence orderings

In this section we only consider primitive recursive rewrite systems i.e., rewite systems whose set of rewrite rules is primitive recursive under a suitable coding (Gödel numbering). This assumption is strong enough for any practical rewrite systems in the literature.

Establishing a relationship between the rewrite structure of a given rewrite system and a proof theoretic ordinals is very useful because the notion of proof theoretic ordinals is related to two important notions of computational complexity; (1) ordinal recursiveness, and (2) provably recursiveness. The both notions are considered an extension of the notion of primitive recursive functions, and characterize subclasses of the (total) recursive functions.

Definition. (ordinal recursive functions) Let α be an ordinal number, and $<\cdot$ a primitive recursive well ordering on the natural numbers of order type α. We are assuming the structure $\langle\alpha, <\cdot\rangle$ is coded in the natural numbers. The class of α-recursive functions is defined by the following schemata:

(i) $S(a)=a+1$ (Successor)

(ii) $Z(a_1,\ldots,a_n)=0$ (Zero)

(iii) $P_{jn}(a_1,\ldots,a_n)=a_j$ $(1\leq j\leq n)$ (Projection)

(iv) $f(a_1,\ldots,a_n)=g(h_1(a_1,\ldots a_n),\ldots,h_m(a_1,\ldots a_n))$, where g and h_j are already known to be α-recursive.

(v) (α-recursion)

$$f(a_1,\ldots a_n)=\begin{cases} h(f(k(a_1,\ldots,a_n),a_2,\ldots,a_n),a_1,\ldots,a_n), \\ \quad \text{if } k(a_1,\ldots a_n)<\cdot a_1, \\ g(a_1,\ldots,a_n), \text{ otherwise} \end{cases}$$

where g,h and k are already defined α-recursive functions.

The idea of (v) is that $f(a,a_2,\ldots,a_n)$ is defined either outright or in terms of $f(b,a_2,\ldots,a_n)$ for certain $b<\cdot a$.

If we take ω for α and the natural ordering of ω for $<\cdot$, the class of α-recursive functions is exactly the same as the class of primitive recursive functions.

Definition. (provably recursive functions) For a given mathematical system G, the class of provably recursive function in G is the subclass of the (total) recursive functions for which termination of the calculation procedure of each input (i.e., each natural number) is uniformly proved in the system G. In other words, a recursive function f(x) is provably recursive in G iff $f(x)=U(\mu yT(e,x,y))$ for some e in the sense of Kleene, and $G\vdash \forall x\exists yT(e,x,y)$, where U is a fixed primitive recursive function and T is a fixed primitive recursive predicate (called Kleene's predicate)(cf.[27]).

Definition. A rewrite system R is computable by a recursive function *f* if (under a suitable coding (Gödel numbering) in the natural numbers) we have a recursive function *f* such that for any ground term t, there exists a normal form s of t (i.e., $t\rightarrow^* s$ and no rewrite rule is applicable to s in the rewrite system) and $f(\ulcorner t\urcorner)=\ulcorner s\urcorner$, where $\ulcorner t\urcorner$ and $\ulcorner s\urcorner$ are Gödel numbers of t and s respectively.

From the result of §2, we have the following proposition.

Proposition. (1) For any finite sets F and C, any term rewrite system embedable into the extended recursive path ordering on A (F,C) is computable by a Γ_0-recursive function.
(2) For any finite sets F and C, any term rewrite system embedable to the lexicpgraphic path ordering on $A_2(F,C)$ is computable by an ε_0-recursive function.
(3) For any well founded set C whose rank is less than ε_0, any term rewrite system embedable into $A_1(\{f\},C)$ is computable by an ε_0-recursive function.

As well known in proof theory, the class of $<\varepsilon_0$-recursive functions (i.e., α-recursive functions for any $\alpha<\varepsilon_0$) is exactly the class of provably recursive functions in the system PA (Peano Arithmetic), and the class of $<\Gamma_0$-recursive functions is exactly the class of provably recursive functions in Predicative Analysis (of Feferman cf. [31]). On the other hand, Gentzen type reduction rules of proof trees for the consistency proof of Peano Arithmetic (Predicative Analysis) (cf. [30], [34]) can be interpreted as term rewrite systems when proof trees are regarded as terms. It is known in proof theory that this rewrite system is embedable into an ordering of order type ε_0 (order type Γ_0, respectively), but not embedable into any ordering of less than ε_0 (less than Γ_0, respectively). Hence we have

Theorem. (1) There exists term rewrite system embedable to the extended recursive path ordering on $A_2(\{f\},\{0\})$ such that it is not computable by any provably recursive function of Predicative Analysis.
(2) There exists a term rewrite system embedable into the lexicographic path ordering on $A_2(\{f\},\{0\})$ such that it is not computable by any provably recursive function in Peano Arithmetic.
(3) Ther exists a term rewrite system embedable into the recursive path ordering on $A_1(\{f\},\{0\})$ such that it is not computable by any provably recursive function in Peano Arithmetic.

Since the recursive path ordering embedable into the (partial order version of) path of subterm ordering of

Plaisted and also embedable into the recursive decomposition ordering of Lescanne and into Kapur-Narendran-Sivakumar's ordering (see Rusinowitch [28] for these facts), "the recursive path ordering" in (3) above can be replaced by any of thses orderings.

An example of (2) can also be found in Kirby-Paris [18]. We extend this system in the next section. On the other hand, as mentioned in §1, since the well foundedness of the extended Ackermann ordering for $A_\omega(F,C)$ is provable in ID_1 (the system of (non-iterated) inductive definition), we have an upper bound for the computational complexity.

Theorem. If F and C are finite (or if the well foundedness of F and C is provable in ID_1), any rewrite system embedable into a segment of $A_\omega(F,C)$, is computable by a provably recursive function of ID_1.

§4. Application of other proof theoretic ordinals, and concluding remarks.

In the case of more complicated systems of term rewriting, like a rewrite system for conditional equations or a system of non-simplification rewriting (cf. [9] for the notion of simplification), more complicated well founded ordering structures than the Ackermann ordering should be required. For such cases, more stronger systems of proof theoretic ordinals, like the Howard ordinals, the ordinal notations of Feferman-Schütte, or Takeuti's ordinal diagrams seem very useful and promising.

Here we only sketch one example of the use of the system of ordinal diagrams, (for the precise definition of ordinal diagrams, see Takeuti [34], Okada-Takeuti [23] or Okada [24]).

Let F and C be quasi ordered. $\lesssim_d$ on $A_1(F,C)$ denote an ordering $\lll_0$ in the sense of ordinal diagrams in [34], [23], or [24]. Then (1) for any well founded set C and any well ordered set F, $A_1(F,C)$ is well founded by $\lesssim_d$; (2) for any finite set F and any well quasi ordered set C, $A_1(F,C)$ is well quasi ordered by $\lesssim_d$.

(1) is proved by a combination of the "minimal bad sequence" argument (of [21]) and the transfinite induction on F (cf. [23]). (2) is proved by establishing a similar relation (*)' to (*) of §2, using the extended Kruskal theorem in [33].

(*)' For any s, $t \in A_1(F,C)$, $s \lesssim_F t \Rightarrow s \lesssim_d t$.

Extended Kruskal Theorem (Friedman). For any finite set F and any well quasi ordered set C, $A_1(F,C)$ is well quasi ordered by $\lesssim_F$.

Here $s \lesssim_F t$ is the homeomorphic embedding with the following Gap Condition. (We recall that any term of $A_1(F,C)$ can be identified with a forest whose end nodes (leaves) have labels from the set C and whose inner nodes have labels from the set F.)

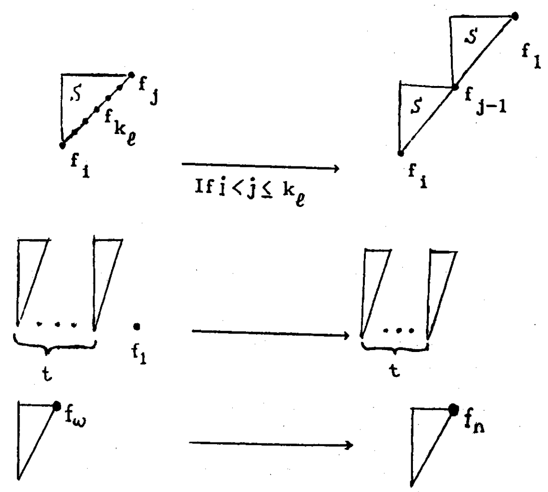
(Gap Condition). Let $f: s \to t$ be a homeomorphic embedding. For any inner node a of a forest s and its immediate predecessor b in s, if $f(b) < c < f(a)$ then $l(c) \geq l(f(a))$, where $l(d)$ is the label of the node d in the forest t, i.e., the element of F which is placed at the node d. And for any root a of s, if $c < f(a)$ then $l(c) \geq l(f(a))$.

Now we give an example of conditional non-simplification rewrite rules. This is essentially an extension of the rewrite system of Kirby-Paris [18].
Consider $T = A_1(\{f_1,\ldots,f_\omega\},\{0\})$, where f_i is ordered by $f_1 < f_2 < \ldots < f_\omega$. We consider the following rewrite rules R on T.

(1) $f_i(t_1 \# f_{k_1}(t_2 \# f_{k_2}(\ldots(t_m \# f_{k_m}(t_{m+1} \# f_j(0)))\ldots))) \longrightarrow$
$f_i(t_1 \# f_{k_1}(\ldots t_{k_m}(t_{m+1} \# f_{j-1}(t_1 \# f_{k_1}(\ldots f_{k_m}(t_{m+1} \# f_1(0))..)))..))$
...if $i < j \leq k_1,\ldots,k_m$ (m is an arbitrary number).

(2) $t \# f_1(0) \longrightarrow t$

(3) $t \# f_\omega(0) \longrightarrow t \# f_n$ (n is an arbitrary number).



Then $s \to^* t \Rightarrow s >_d t$, therefore this rewrite system is embedable into $\langle T, <_d \rangle$. Since $\langle T, <_d \rangle$ is well founded, we can see that the rewrite system R on $T = A_1(\{f_1, \ldots, f_\omega\}, \{0\})$ terminates. However this type of rewrite system is very powerful. In fact, by using the method of Buchholz (cf.[2]) we can see that termination of the rewrite system R is unprovable even in Π^1_1-CA+BI of impredicative analysis. By modifying the system R, we can easily see

Theorem. There exists a rewrite system embedable into $\langle T, <_d \rangle$ which is not computable by any provably recursive function in Π^1_1-CA+BI.

Conclusion.

From the results in §2 we conclude that proof theoretic ordinals in logic provide a very general basis for several important ordering structures used in term rewriting theory. Since proof theoretic ordinals are considered a measure of computational complexity and logical complexity, as a corollary, we can get information on computational power of term rewrite systems and on logical complexity of termination problem of term rewrite systems. The results of §3 and §4 show that fairly simple rewrite systems may have strong computational power.

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