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メンバシップ条件付きTRSの合流性について
Confluence of Membership Conditional TRS

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Abstract We propose a sufficient condition for confluence of noetherian membership conditional term rewriting systems and its application to a completion algorithm of such systems. We introduce contextual rewriting preserving contexts and critical pairs with contexts. It is shown that noetherian membership conditional term rewriting system is confluent if all such critical pairs are convergent.

1. Introduction

Equality has been occupied a very special position in our computation, most of our computation are carried out by the usage of this relation. To execute calculi based on equality automatically and to treat computation on machines formally, we regard equalities as rewriting rules by introducing order. In this direction, classical unconditional TRS has been studied, stressed on its two principle characteristics, termination and confluence. The former guarantees existence of results of computation in TRS and the latter does uniqueness of those respectively. When we try to apply the results for automated theorem proving, algebraic specification, verification and transformation of programs, we face to a difficulty. In real program, for example, the application of equalities is usually restricted by some conditions, then we come to a natural extension, conditional TRS in which rewriting rules have conditions for their usage. Such systems have already been investigated well, and we can find also results on confluence of such systems. But there is another approach for conditional TRS, membership conditional TRS whose rewriting rules are restricted by membership condition for its variables in lefthand sides of rules. Such systems seem to realize restrictions on types and values for variables in real programs naturally, and will enable us to discuss automated theorem proving, specification, verification and transformation based on them. Discussion on confluence of conditional TRS can be classified roughly in two, one assumes some kind of normality for their conditions [4] and another noetherian property for systems whole [8]. As for membership conditional TRS, the results in the former category are already known [11] and we are going to show that in the latter. and we propose a completion algorithm for membership conditional TRS.

2. Term Rewriting Systems

In this section we briefly explain about term rewriting system, TRS in short and prepare necessary notions for the following sections. We assume that the reader is familiar with TRS and he can consult with, for example, [2],[3],[5],[7], if necessary.

A term set $T = T(F, V)$ is the set of first order terms composed of the elements in a denumerable set of variables V, and a set of function symbols F graded by arities such that $F \cap V = \phi$. We use $Var(t)$ for a term $t \in T$ the set of all the variables in t.

For any term $t \in T$ we can define its *occurrences* $\mathcal{O}(t) \subset \mathcal{N}^*$ the set of sequences of positive integers and *subterm* t/u of t at *occurrence* $u \in \mathcal{O}(t)$.

$$\mathcal{O}(t) = \Lambda \text{ and } t/\Lambda = t, \text{ for } t = x \in V$$

$$\mathcal{O}(t) = \{\Lambda\} \cup \{iu | i = 1, \cdots, n, u \in \mathcal{O}(t_i)\}, t/\Lambda = t \text{ and } t/iu = t_i/u$$
$$\text{where } f \in F, t_i \in T \text{ for } t = ft_1 \cdots t_n.$$

Λ is the empty sequence in $\mathcal{N}^*$.

Next for $t, s \in T$ and $u \in \mathcal{O}(t)$, we define $t[u \leftarrow s]$ or simply $t[s]$ by:

$$t[u \leftarrow s] = s, \quad ft_1 \cdots t_n[iu \leftarrow s] = ft_i \cdots t_{i-1} s t_{i+1} \cdots t_n.$$

A *substitution* θ is a map from V to $T(F, V)$ such that $\theta(x) = x$ almost everywhere.

A *rewriting rule* on T is a pair of two terms (l, r) with $Var(l) \supset Var(r)$ and $l \notin V$. We denote a set of rewriting rules by $\triangleright$, and write $l \triangleright r$ if $(l, r) \in \triangleright$. A term t *reduce* t' at occurrence u of term t, $t \rightarrow t'$ by a rewriting rule $l \triangleright r$ is defined as follows:

$$t \rightarrow t' \text{ if and only if } t = s[u \leftarrow l\theta], t = s[u \leftarrow r\theta] \text{ for some } s \in T, \theta.$$

We call t/u a *redex* of the rule.

We define term rewriting system:

Definition 2.1. (Term Rewriting System)
A TRS is a structure $(T, \rightarrow)$, with object set T and a binary relation $\rightarrow$ defined by a set $\triangleright$ of rewriting rules on T.

We express by $\rightarrow^*$ the transitive reflexive closure of $\rightarrow$. A term t is said to be a *normal form* when there is no t' such that $t \rightarrow t'$, and t' called a *normal form* of t when $t \rightarrow^* t'$

and t' is a normal form. Two terms t and t' *converges*, if t and t' have an identical term as their normal forms.

When two rules of $l_i \triangleright r_i$ for $i = 1, 2$ with no common variables in TRS are *overlapping*, if

$$l_i\theta_i/u = l_j\theta_j \text{ for some } \theta_i, \theta_j, u \in \mathcal{O}(t) \text{ such that } l_i/u \notin V.$$

We can define a critical pair of two overlapping rules.

Definition 2.2. (Critical Pair)
A critical pair $\langle P, Q\rangle$ of two overlapping rules $l_i \triangleright r_i$ for $i = 1, 2$ is:

$$P = l_1\theta[u \leftarrow r_2\theta], \quad Q = r_2\theta$$

where θ is the most general unifier of l_1/u and l_2.

The following two notions characterize TRS.

Definition 2.3. (Noetherian)
A TRS $R = (T, \rightarrow)$ is *noetherian*, if every reduction in R terminates, i.e., there is no infinite reduction sequence as

$$t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow \cdots \text{ where } t_i \in T.$$

Definition 2.4. (confluence and Local Confluence)
A TRS $R = (T, \rightarrow)$ is *confluent*, if

$$\forall u, v, w \in T[u \rightarrow^* v, u \rightarrow^* w \Rightarrow \exists u' \text{ such that } v \rightarrow^* u', w \rightarrow^* u']$$

and *locally confluent*, if

$$\forall u, v, w \in T[u \rightarrow v, u \rightarrow w \Rightarrow \exists u' \text{ such that } v \rightarrow^* u', w \rightarrow^* u'].$$

These two properties have been of our chief concern, because noetherian property guarantees the existence of normal forms and confluence does uniqueness of normal forms provided existence of them. If a TRS is equipped with both properties, the system has necessarily an unique term reduced from every term. For noetherian classical unconditional TRS the following results on confluence are well-known:

Lemma 2.5.
A noetherian relation is confluent if and only if it is locally confluent.

Lemma 2.6. (Critical Pair Lemma)

A TRS R is locally confluent if and only if every critical pair of R converges.

Combining these two lemmas, the next theorem on confluence of noetherian TRS holds.

Theorem 2.7.
Let R be a noetherian term rewriting system. R is confluent if and only if every critical pair of R converges.

3. Membership Conditional Term Rewriting Systems

We introduce a kind of conditional TRS, membership conditional TRS.

Definition 3.1. (c-Term, MC-Rule)
A *c-term* is a term with a membership condition on the variables in the term:

$$t : (x_1, \cdots, x_n) \in S_1 \times \cdots \times S_n$$

where $\{x_1, \cdots, x_n\} = Var(t)$ and $S_i \subset T$ for all i, and write simply as $t : c$. A *MC-rule* is a reduction/rewriting rule with a membership condition on the lefthand side of the rule:

$$l \triangleright r : c$$

where $l \triangleright r$ is a reduction/rewriting rule of TRS, and $l : c$ is a MC-term.

Definition 3.2. (Membership Conditional TRS)
A *membership conditional TRS*, is a term rewriting system defined by a set of MC-rules.

We say a term t reduce t' by a MC-rule $l \triangleright r : (x_1, \cdots, x_n) \in S_1, \cdots, S_n$ in a membership conditional TRS, when

$$t = s[l\theta], \quad t' = s[r\theta] \text{ for some } s \in T, \theta$$

and

$$x_1\theta \in S_1, \cdots, x_n\theta \in S_n.$$

Below is an example of membership conditional TRS and its reductions.

Example 3.3.
Let $F = \{eq, d, +, s, 0\}$ and $F' = \{+, s, 0\}$. We can define a membership conditional TRS R which represents addition in the set of natural numbers $\mathcal{N} = T(\{s, 0\})$.

$$R : \begin{cases} x + 0 \triangleright x \\ x + s(r) \triangleright s(x + y) \\ eq(x, x) \triangleright x \text{ if } x \in T(F') \\ d(x) \triangleright x + x \text{ if } x \in T(F') \end{cases}$$

In this system, we have the following reduction sequence:

$$eq(d(0), d(0)) \to eq(0+0, d(0)) \to eq(0+0, 0+0) \to 0+0 \to 0.$$

Note that a direct reduction $eq(d(0), d(0)) \to d(0)$ is impossible by the third rule in R since $d(0) \notin T(f')$.

On the confluence of UN-noetherian membership conditional TRS, there are some results in [11]. And there seems to be some criteria on confluence providing noetherian property as in classical unconditional TRS and some other conditional TRS (cf. [1], [4], [8]).

4. Contextual Rewriting

Here we introduce contextual rewriting which does not modify context parts of terms and differs the one studied in [12]. This is prerequisite to discuss critical pairs of MC-rules.

but ours does not modify context parts of terms, therefore, differs from it.

Definition 4.1. (c-Reduction)
A c-term $t : c$ is *c-reducible* by a MC-rule $l \triangleright r : (x_1, \cdots, x_n) \in S_1 \times \cdots S_n$, if some subterm t' at occurrence u of t is $l\theta'$ for some substitution θ' and $x_i\theta' \in S_i$ for $i = i, \cdots, n$ hold under c. Then $t : c$ *c-reduces* $s : c = t[u \leftarrow r\theta'] : c$, and we denote $t : c \to_c s : c$. When we have $s : c$ from $t : c$ by c-reduction of zero or more times, we denote $t : c \to_c^* s : c$. In the above definition, because each $x_i\theta'$ includes variables restricted by context c, we have to verify that $x_i\theta' \in S_i$. A c-term $t : c$ is a *c-normal form* when there is no $t' : c$ such that $t : c \to_c t' : c$, and $t' : c$ is a c-normal-form of $t : c$ if $t : c \to_c^* t' : c$, and $t' : c$ is a c-normal-form.

This is a kind of contextual rewriting that preserves contexts, c in the definition, and we show an example.

Example 4.2.
By a MC-rule $f(x) \triangleright g(x)$ if $x \in \mathcal{N}$,

$$h(f(s^2(y))) : y \in \mathcal{N} \to_c h(g(s^2(y))) : y \in \mathcal{N}$$

where $\mathcal{N}$ is a set of natural numbers. In this case, we have to check $s^2(y) \in \mathcal{N}$ under $y \in \mathcal{N}$ and succeed.

We formalize the relation between terms and c-terms.

Definition 4.3. (Instance of c-Term, Associated c-Term)
A term $t\theta$ is called an *instance* of a c-term $t : c$, if $x\theta$ satisfies the condition c for any variable $x \in Var(t)$. Conversely, we call $t : c$ an *associated c-term* of $t\theta$. We call a set of

c-terms $T_c = \{t : c | t \in T, c$ is a membership condition on variables in $V(t)\}$ as *associated c-term set* of T.

We have to establish a correspondence between TRS and c-TRS and one between terms and c-terms.

Lemma 4.4. (Existence of Associated c-Term)

For any term t there is some associated c-term $t' : c'$ and $t = t'\theta$.

Proof.

Clear by the following inclusion map:

$$t \longmapsto t : x_1 \in \{x_1\}, \cdots, x_n \in \{x_n\}. \qquad \text{Q.E.D.}$$

Lemma 4.5.

If $t : c$ c-reduces $s : c$ and $t\theta$ is an instance of $t : c$, then there is an instance $s\theta$ of $s : c$ such that $t \rightarrow s$. That is, the diagram below is commutative:

$$\begin{array}{ccc} t : c & \longrightarrow_c & s : c \\ \downarrow & & \downarrow \\ t\theta & \longrightarrow & s\theta \end{array}$$

Proof.

Let $t : c \rightarrow_c s : c$ by a c-rule $l \triangleright r : \tilde{c}$ applying on a redex $t/u : c$ of $t : c$. If the rule is applicable also for $t\theta/u$, then we have the below commutative diagram:

$$\begin{array}{cccc} t \equiv & t[l\theta'] : c & \rightarrow_c & s[r\theta'] : c \\ & \downarrow & & \downarrow \\ t \equiv & t\theta[l\theta'\theta] & \rightarrow & s\theta[r\theta'\theta] \end{array}$$

Then it remains only to show the rule is applicable for $t\theta/u$. Let $y \in S$ be one of the conditions in $\tilde{c}$, then $y\theta' \in S$ under $c = (x_1, \cdots, x_n) \in S_1 \times \cdots \times S_n$. From the definition of instance $(x_1\theta, \cdots, x_n\theta) \in S_1 \times \cdots \times S_n$, so we have $y\theta'\theta \in S$, i.e., $t\theta/u$ is also a redex of the rule. This concludes this proof. Q.E.D.

By lemmas 4.4 and 4.5, we can define an *associated cTRS* $(T_c, \rightarrow_c)$ of $(T, \rightarrow)$:

Definition 4.6.

For a TRS $R = (T, \rightarrow)$, we have a set of associated c-terms T_c and c-reduction relations $\rightarrow_c$, and a TRS $(T_c, \rightarrow_c)$ called an *associated cTRS* of R. Moreover we can necessarily define an associated cTRS for a TRS.

Based on the notion of c-reduction, we can define critical pairs of membership conditional rewriting rules in cTRS:

Defintion 4.7. (c-Critical Pairs)

A critical pair of two MC-rules (ri) $l_i \triangleright r_i : c_i$ for $i = 1, 2$ with no common variables is defined as follows. Let $\langle P, Q \rangle$ be a critical pair of (r1) and (r2) in classical unconditional TRSby ignoring their conditions c_i, that is,

$$P = l_1\theta[u \leftarrow r_2\theta], \quad Q = r_2\theta.$$

When there is the most general membership condition c on variables in $Var(l_1\theta)$ such that both $c_1\theta$ and $c_2\theta$ hold under c, we say (r1) and (r2) *c-overlapping*. We call a triple $\langle P, Q, c \rangle$ a *c-critical pair* of two MC-rules (r1) and (r2).

5. Confluence of Membership Conditional Term Rewriting Systems

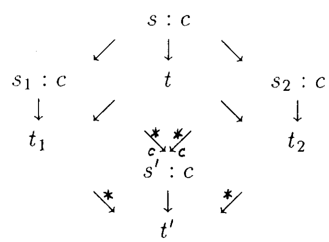
The following is our key lemma:

Lemma 5.1.

A TRS $R = (T, \rightarrow)$ is locally confluent, if its associated cTRS $R_c = (T_c, \rightarrow_c)$ is locally confluent.

Proof.

Let t_1, t_2 be two terms reduced from a single term t in $(T, \rightarrow)$, then there are three associated c-terms $s : c$, $s_1 : c$ and $s_2 : c$ of t, t_1 and t_2 respectively such that $s_1 : c$, $s_2 : c$ are c-reduced from $s : c$ in $(T_c, \rightarrow_c)$.



By hypothesis there is a c-term $s' : c$ such that $s_1 : c \rightarrow_c^* s' : c$ and $s_2 : c \rightarrow_c^* s' : c$. Then we have an instance t' of $s' : c$ and $t_1 \rightarrow^* t'$ and $t_2 \rightarrow^* t'$ by Lemma 4.5. Q.E.D.

This a critical pair lemma for cTRS.

Lemma 5.2.

If all the c-critical pairs of a cTRS $R_c = (T_c, \rightarrow_c)$ are convergent, then R_c is locally confluent.

Proof.

Let a c-term $t : c$ c-reduces two distinct c-terms $t' : c$ and $t'' : c$, by two MC-rules (r1) $l_1 \triangleright r_1 : c_1$ and (r2) $l_2 \triangleright r_2 : c_2$ on redexes t/u and t/v respectively, where u and v are two occurrences of t.

There are three cases by the relative position of u and v.

Case1: u and v are disjoint.

Reductions at t'/v by (r2) and t''/u by (r1) result both in an identical term.

Case2:

When u and v are not disjoint, we may assume t/v is a subterm of t/u and we have only to consider t/u a subterm of t.

Case2a: $v = u \cdot w$ and $l_1/w = x$ (variable).

Let

$$\{w, w'_1, \cdots, w'_m\} := \{\text{all the occurrences of } x \text{ in } t'/u\} \quad w \text{ might not be contained,}$$

$$\{w, w''_1, \cdots, w''_n\} := \{\text{all the occurrences of } x \text{ in } t''/u\}.$$

Then we have the same term by applications of (r2) on redexes $t'/w, t'/w'_1, \cdots, t'/w'_m$ of t'/u, and by those of (r2) on redexes $t''/w''_1, \cdots, t''/w''_n$ of t''/u followed by that of (r1) on t''/u whole.

Case2b: $v = u \cdot w$ and $l_1/w \neq$ variable.

Only in this case, $\langle t', t'', c\rangle$ is a c-critical pair and converges by the hypothesis, namely, R_c is locally confluent. Q.E.D.

Now we can give a criterion of confluence similar to one for noetherian TRS.

Theorem 5.3. (Main Theorem)

Let R be a noetherian membership conditional TRS. If every c-critical pairs of R converges, R is confluent.

Proof.

We have an associated cTRS R_c of R, and R_c is locally confluent as its all the critical pairs are convergent by lemma 5.2. Then also R is locally confluent by lemma 5.1, moreover confluent for its noetherian property and lemma 2.3. Q.E.D.

We can design a completion algorithm as in classical unconditional case ([6], [10]) and in one of conditional cases ([9]).

Let a set of membership conditional equations E and some reduction ordering $\gg$ be given, and if the following completion algorithm stops then we have a complete set of MC-rules R.

Completion Algorithm

E : a set of MC-equalities (given)

R : a set of MC-rules (initially $= \phi$)

Loop while $E \neq \phi$ do

if $E = \phi$ then $return(R)$;;; Stops with success, R is complete.

$f := m = n : c$;;; a candidate of a new rule, chosen from E

$r := l \triangleright r : c$;;; a new rule, l, r are c-normal forms of $m : c$, $n : c$

;;; by the current rule set R and $l \gg r$

;;; If c-normal forms of $m : c$ and $r : c$ are IN-comparable by $\gg$ then stops with failure.

$R' := \{l' \triangleright r' : c' \in R | l' : c' \text{ or } r' : c' \text{ c-reducible by } f\}$

$R'_{eq} := \{l' = r' : c' | l' \triangleright r' : c' \in R'\}$

$R := R + \{r\} - R'$

$E := E - \{f\} + R'_{eq} + CP\langle R, \{r\}\rangle$;;; $CP\langle R, \{r\}\rangle$ is all the c-critical pairs

;;; between the rules in new R and r.

Now we show a theorem on the completeness of above algorithm and its proof.

Theorem 5.4.

For a given set of MC-equations E, when above algorithm stops and we have a MC-TRS R, then R is complete, that is, confluent and noetherian and moreover $=_E \equiv \sim_R$. Here $=_E$ and $\sim_R$ denotes equivalence relations generated by $=$ of E and $\rightarrow$ of R respectively.

Proof.

We express E, R, etc. in i-th loop as E_i, R_i, and so on. As $=_{E_{i+1}} \cup \sim_{R_{i+1}} \supset =_{E_i} \cup \sim_{R_i}$, clearly and its converse is also true by $=_{CP\langle R_i, R_i^* \rangle} \subset =_{R_i} \cup =_{f_i}$, we have $=_E \equiv \sim_R$. When $E = \phi$, from R is locally confluent because there is no critical pair, and noetherian for the ordering used in the algorithm. Q.E.D.

6. Conclusion

We investigated the confluence of noetherian membership conditional TRS. For that purpose we introduced contextual terms, contextual rewritings which preserve contexts of them, and contextual critical pairs. Using such notions, it was shown that a noetherian membership conditional TRS is confluent if every contextual critical pair converge. Based on this criterion, we proposed a completion procedure for membership conditional TRS.

As membership conditional TRS is a natural model of equational logic and programs expressed by equations, so our algorithm is applicable to automated theorem proving, verification and transformation of programs.

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