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# BCK-論理式の証明の個数*

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**Abstract**

BCK-論理式とは次の公理 $B = (a \to b) \to (c \to a) - c \to b$, $C = (a \to b \to c) \to b \to a \to c$, $K = a \to b \to a$ から代入と modus ponens を用いて導かれる論理式のことである．BCK-論理式の中で，自明でない代入では他の BCK-論理式から導けないものを最小論理式という．$\alpha$ が最小論理式でかつ，$\beta$ が $\alpha$ から代入で得られるとき $\alpha$ は $\beta$ の 最小論理式という．本稿では，与えられた BCK-論理式 $\alpha$ に対し，その正規形証明図の個数は $\alpha$ の最小論理式の個数と一致することを示す．系として BCK-論理式 $\alpha$ に対し，正規形の証明図が唯一となる必要十分条件は $\alpha$ の最小論理式が唯一であることを示す．

## Number of proofs for BCK-formulas

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**Abstract**

A formula is a BCK-formula iff it is derivable from axioms $B = (a \to b) \to (c \to a) - c \to b$, $C = (a \to b \to c) \to b \to a \to c$ and $K = a \to b \to a$ by substitution and modus ponens. A formula is BCK-minimal iff it is BCK-provable and it is not a non-trivial substitution instance of any BCK-formula. A BCK-minimal formula $\alpha$ is a minimal formula of $\beta$ iff $\beta$ is a substitution instance of $\alpha$. It is shown that given a BCK-formula $\alpha$, the number of normal proof figures for $\alpha$ is identical to the number of minimal formulas of $\alpha$. As a corollary, it is shown that a BCK-formula $\alpha$ has the unique normal form proof iff $\alpha$ has the unique minimal form.

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BCK-formulas are the formulas which are derivable from axioms $B = (a \to b) \to (c \to a) \to c \to b$, $C = (a \to b \to c) \to b \to a \to c$ and $K = a \to b \to a$ by substitution and modus ponens. They are identical to the set of provable formulas in the natural deduction system with the following two inference rules.

$$\dfrac{\begin{matrix}[\gamma]\\ \vdots \\ \delta\end{matrix}}{\gamma \to \delta}\,(\to I) \qquad \dfrac{\gamma \to \delta \quad \gamma}{\delta}\,(\to E)$$

Here $\gamma$ occurs at most once in $(\to I)$. By formulae-as-types correspondence [7], the set is identical to the type-schemes of closed BCK-$\lambda$-terms. (See [3].) A BCK-$\lambda$-term is a $\lambda$-term in which no variable occurs twice. Basics notions are referred to [2] concerning to type assignment system.

By coherence theorem in cartesian closed categories [1, 9], balanced formulas has unique proofs. It was shown in [5] that the proofs for balanced formulas are BCK-proofs. Relevantly balanced formulas were defined in [6], and it was proved that such formulas have unique normal form proofs. Balanced formulas and one-two-formulas are included in the set of relevantly balanced formulas. In this note, we show a necessary and sufficient condition for a BCK-formula to have the unique normal form proof.

The notion of BCK-minimality was introduced by Komori [8]. A formula is *BCK-minimal* iff it is a BCK-formula and it is not a non-trivial substitution instance of other BCK-formula. A BCK-formula $\alpha^*$ is a *minimal formula of* $\alpha$ iff $\alpha^*$ is BCK-minimal and $\alpha$ is a substitution instance of $\alpha^*$. A principal type-scheme of a $\lambda$-term is a most general type-scheme of the term with respect to substitution. It is clear that a BCK-minimal formula is a principal type-scheme of a closed BCK-$\lambda$-term.

**Lemma 1 ([4])** *If two closed BCK-$\lambda$-terms have the same principal type, then they are identical.* ∎

**Lemma 2 ([5])** *A BCK-formula is BCK-minimal iff it is a principal type-scheme of a closed BCK-$\lambda$-term in $\eta$-normal form.* ∎

**Theorem 1** *Given a BCK-formula $\alpha$, the number of closed BCK-$\lambda$-terms in $\beta$-$\eta$-normal form which is stratifiable to $\alpha$ is identical to the number of minimal formulas of $\alpha$.*

**Proof.** Let $\alpha$ be a BCK-formula. Note that every BCK-$\lambda$-terms have types. (See Hindley [3].) So $\alpha$ is a principal type-scheme of some closed

BCK-$\lambda$-term $M$ in $\beta$-$\eta$-normal form. By principal type-scheme theorem (Theorem 15.26 of [2]), $M$ has a principal type-scheme $\alpha^*$. By Lemma 2, $\alpha^*$ is minimal formula of $\alpha$. This defines a function from the set of closed BCK-$\lambda$-terms in $\beta$-$\eta$-normal form which is stratifiable to $\alpha$ to the set of minimal formulas of $\alpha$. By Lemma 2, this function is surjective. By Lemma 1, this funcition is injective. Thus the theorem holds. ∎

**Corollary 1** *A BCK-formula has the unique proof in $\beta$-$\eta$-normal form iff it has the unique minimal formula.* ∎

## 参考文献


[1] Babaev, A.A., Solov'ev, S.V., A coherence theorem for canonical morphism in cartesian closed categories, *Zapiski nauchnykh Seminarov Lenigradskogo Otdeleniya matematichskogo Instituta im. V.A. Steklova An SSSR* **88** (1979) 3-29.

[2] Hindley, J.R. Seldin, J.P., Introduction to Combinators and Lambda-Calculus, (Cambridge University Press, London, 1986)

[3] Hindley, J.R., BCK-combinators and linear lambda-terms have types, *Theoret. Comp. Sci.* **64**(1989) 97-105.

[4] Hirokawa, S., Principal types of BCK-lambda terms, submitted

[5] Hirokawa, S., Linear lambda-terms and coherence theorem, submitted

[6] Hirokawa, S., BCK-formulas having unique normal form proofs, Manuscript 1990.

[7] Howard, W.A., The formulae-as-types notion of construction, in: Hindley and Seldin (eds.), *To H.B. Curry, Essays on Combinatory Logic, Lambda Calculus and Formalism* (Academic Press 1980) 479-490.

[8] Komori, Y., BCK algebras and lambda calculus, in: Proc. 10th Symp. on Semigroups, Sakado 1986, (Josai University, Sakado 1987) 5-11.

[9] Mints, G.E., A simple proof of the coherence theorem for cartesian closed categories, Manuscript 1982.