Chewの定理の新しい証明

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Chew の定理 [Che81] は，互換な (compatible) 項書き換え系の正規形の一意性を主張する．しかし，その原論文の証明はギャップを含み理解し難いものであった．

我々は，条件付き項書き換え系の証明の頂点除去という手法を用いて同定理に新たな証明を与える．書き換えの独立性の概念を導入することによって，頂点除去過程の停止性が示される．

A new proof of Chew's theorem

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Chew's theorem [Che81] states that the unique normal form property (UN) holds in a compatible term rewriting system (TRS), i.e., normal forms are unique up to conversion. The original proof is "rather intricate"[Klo92]. There is a general feeling of doubt about the original proof. In fact, there is a gap in the proof.

We present a new proof of the theorem using a peak elimination process of a proof in a conditional TRS. The notion of independence of reductions is introduced in order to show termination of the process.

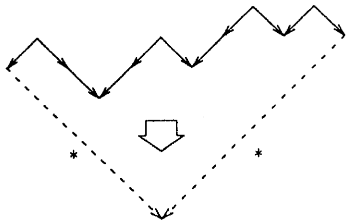

Figure 1: Peak elimination

1 Introduction

Chew's theorem [Che81] states that the unique normal form property (UN) holds in a compatible term rewriting system (TRS), i.e., normal forms are unique up to conversion. The original proof in [Che81] is "rather intricate" [Klo92]. There is a general feeling of doubt about the original proof. In fact, there is a gap in the proof[1].

Several people have made partial attempts at new proofs [dV90, Oga92, OO93, TO94].

Let us briefly outline the methodology of our proof. Given a compatible term rewriting system R, we transform it into the conditional linearization $\widehat{R}$ due to Toyama and Oyamaguchi [TO94], which is a variant of de Vrijer's [dV90]. As de Vrijer observed, it is suffice to prove that the Church-Rosser property (CR) holds for $\widehat{R}$ in order to conclude that R is **UN**. We will prove **CR** of $\widehat{R}$ by a peak elimination process.

Given a conversion, or proof, $t_1 \leftrightarrow \cdots \leftrightarrow t_n$ in $\widehat{R}$, the peak elimination replaces a peak $t_{i-1} \leftarrow t_i \rightarrow t_{i+1}$ in this proof with a conversion $t_{i-1} \leftrightarrow^* t_{i+1}$ in $\widehat{R}$ according to the peak elimination rules. If all peaks are eliminated by applying peak eliminations to the given proof repeatedly, then we find a term s such that $t_1 \rightarrow^* s \leftarrow^* t_n$ as shown in figure 1.

The main difficulty of this proof is to show termination of the peak elimination process. Thereto we introduce the notion of *independence* of two reductions in a proof in $\widehat{R}$. It is shown that independence is preserved during the peak elimination process.

Descendant trees are then defined for the reductions in the starting proof of the process. Using properties of independence, it is shown that these descendant trees are finite, which indicates that one particular peak elimination rule is applied only finitely many times. Since peak elimination with only the other rules is terminating, the result follows.

2 Preliminaries

The definitions and terminologies of abstract reduction systems, terms or term rewriting systems are taken from [Klo92].

[1] See section 2.3 for the detail.

2.1 Abstract reduction systems

Let $\rightarrow$ be an (abstract) reduction system, that is a binary relation on some underlying domain. The symmetric closure, the reflexive transitive closure and the reflexive transitive symmetric closure of $\rightarrow$ are written as $\leftrightarrow$, $\rightarrow^*$ and $\leftrightarrow^*$, respectively. If there is no a' such that $a \rightarrow a'$, then a is a *normal form* of the reduction system. A sequence $a_1 \leftrightarrow \cdots \leftrightarrow a_n$ is called a *proof*. A (sub)proof of the form $a' \leftarrow a \rightarrow a''$ is called a *peak*.

A reduction system $\rightarrow$ is *strongly normalizing* (SN) if there is no infinite sequence such that $a_1 \rightarrow a_2 \rightarrow \cdots$.

A reduction system $\rightarrow$ has the *unique normal form property* (UN) if $a \leftrightarrow^* a' \Rightarrow a \equiv a'$ for each pair of normal forms a, a'. We say $\rightarrow$ has the *Church-Rosser property* (CR) if, for any $a \leftrightarrow^* a'$, there exists b such that $a \rightarrow^* b$ and $a' \rightarrow^* b$. A reduction system $\rightarrow$ has the *weak Church-Rosser property* (**WCR**) if, for $a \rightarrow b$ and $a \rightarrow c$, there exists d such that $b \rightarrow^* d$ and $c \rightarrow^* d$. It is well-known that **SN** & **WCR** implies **CR**.

2.2 Terms

Let F be a set of *function symbols* and V a countably infinite set of *variables* satisfying that F and V are disjoint from each other. For every $f \in F$, a natural number *arity* is associated with f. Function symbols with arity 0 are called *constants*. The set of all *terms* built from F and V is defined as usual. The set of variables occurring in a term t is denoted by $V(t)$. A term t is called a *ground term* if $V(t) = \emptyset$. A term t with no repeated occurrence of a variable is said to be *linear*.

A *substitution* is a map from variables to terms and the domain is naturally expanded to whole terms. The application of a substitution σ to a term t is written as $t\sigma$. A substitution σ is also written as $\{x_1 := t_1, \cdots, x_n := t_n\}$, where x_i are variables such that $x_i\sigma \not\equiv x_i$.

Let $\Box$ be a fresh special constant symbol. A *context* $C[\,]$ is a term in $T(F \cup \Box, V)$. When $C[\,]$ is a context with n $\Box$'s and $t_1, \cdots, t_n$ are terms, $C[t_1, \cdots, t_n]$ denotes the term obtained by replacing all $\Box$'s in $C[\,]$ with t_i's in a left-to-right manner. A term t is called a *subterm* of a term s if there is a context $C[\,]$ such that $C[t] \equiv s$. When $C[\,] \not\equiv \Box$, then t is said to be a *proper* subterm of $C[t]$.

The set of *positions* $P(t)$ of a term t is defined as below:

1. $P(t) = \Lambda$ if t is either a constant or a variable.
2. $P(t) = \{\Lambda\} \cup \{i \cdot u \mid 1 \leq i \leq n \text{ and } u \in P(t_i)\}$ if $t \equiv f(t_1, \cdots, t_n)$.

For a position $p \in P(t)$, t/p is the subterm occurring at p. For terms t, s and a position $p \in P(t)$, $t[p \leftarrow s]$ is the term obtained by replacing the subterm at p in t with s.

For positions p_1 and p_2, $p_1 \leq p_2$ if p_1 is a prefix of p_2. We write $p_1 < p_2$ if $p_1 \leq p_2$ and $p_1 \neq p_2$. When neither $p_1 \leq p_2$ nor $p_2 \leq p_1$, p_1 and p_2 are said to be *parallel*, notation $p_1 \perp p_2$. The longest common prefix of p_1 and p_2 is denoted by $\wedge(p_1, p_2)$.

2.3 Term rewriting systems

A *term rewriting system* (TRS) is a finite set R of *rewrite rules*. A rewrite rule is a pair of terms denoted by $l \rightarrow r$

satisfying following properties:

1. l is not a variable.
2. $V(l) \supseteq V(r)$.

The term l (r) is called the *left-hand side* (*right-hand side*) of $l \to r$.

The reduction system $\to_R$ on the set of terms is defined from a TRS R as follows:

$$\to_R = \{C[l\theta] \to_R C[r\theta] \mid C[\,] \text{ is a context, } \theta \text{ is a substitution and } l \to r \in R\}.$$

A term $l\theta$ is called a *redex* of R if $l \to r \in R$. For a reduction $\alpha : C[l\theta] \to_R C[r\theta]$, the position of the redex $l\theta$ in $C[l\theta]$ is denoted by $p(\alpha)$.

A rewrite rule $l \to r$ is *left-linear* if l is linear. A TRS with only left-linear rewrite rules is also said to be *left-linear*.

Let t and t' be terms satisfying that $t\sigma \equiv t'\sigma$ with a substitution σ. Then t and t' are *unifiable* and σ is called a *unifier* of t and t'. If t and t' are unifiable, then there exists a substitution θ called a *most general unifier* of t and t' such that there is σ' satisfying $\sigma = \sigma' \circ \theta$ for all unifiers σ of t and t'.

Let t and t' be terms such that $t \equiv C[s]$ with a context $C[]$ and a non-variable term s. Suppose that s and t' are unifiable with a most general unifier θ. Then, $C[s\theta]$ is called a *superposition* of t and t'.

Let $C[]$ be a context with n $\Box$'s and let $t_i \leftrightarrow_R^* t_i'$ be proofs in R for $1 \le i \le n$. The *embedding* of the proofs into $C[]$ is the following:

$$\begin{array}{ll} & C[t_1, t_2, \cdots, t_n] \\ \leftrightarrow_R^* & C[t_1', t_2, \cdots, t_n] \\ \leftrightarrow_R^* & C[t_1', t_2', \cdots, t_n] \\ \leftrightarrow_R^* & \cdots \\ \leftrightarrow_R^* & C[t_1', t_2', \cdots, t_n'], \end{array}$$

which is denoted by $C[t_1, \cdots, t_n] \Leftrightarrow_R^* C[t_1', \cdots, t_n']$.

When we think of a pair of rules S and S', it is supposed that S and S' are *standardized apart*, i.e., the variables in S and S' are renamed appropriately so that S and S' do not share variables.

Definition 2.1 Let $S : l \to r$ and $S' : l' \to r'$ be rewrite rules. Rules S and S' are *nonoverlapping* if there is no superposition of l and l'. Rules S and S' are said to be *overlay* if the superposition of l and l' exists only where the context $C[] = \Box$. If S and S' are overlay and $r\sigma \equiv r'\sigma$ for all unifiers σ of l and l', then S and S' are *almost nonoverlapping*. A TRS R is nonoverlapping (overlay, almost nonoverlapping) if each pair of rules in R is nonoverlapping (overlay, almost nonoverlapping).

Definition 2.2 A term $\bar{t}$ is a *linearization* of a term t if

- $\bar{t}$ is linear, and
- there is a substitution σ s.t. $\bar{t}\sigma = t$ and $x\sigma \in V$ for all $x \in V$.

For a rewrite rule $l \to r$, $\bar{l} \to \bar{r}$ as following is called a linearization of $l \to r$:

- $\bar{l}$ is a linearization of l s.t. $\bar{l}\sigma = l$, and
- $\bar{r}\sigma = r$.

Definition 2.3 ([Che81, dV90]) Rewrite rules S and S' are said to be *compatible*[2] if there exist linearizations $\bar{S}$, $\bar{S}'$ of S, S' such that $\bar{S}$ and $\bar{S}'$ are almost nonoverlapping. A TRS R is compatible if each pair of rules is compatible.

Example 2.1 Combinatory logic CL can be regarded as a TRS [Klo92]. The TRS CL-pc is the union of CL and the following *parallel-conditional* rules. CL-pc is a compatible TRS.

$$\overset{SKI}{\left\{\begin{array}{lll} Sxyz & \to & xz(yz) \\ Kxy & \to & x \\ Ix & \to & x \end{array}\right\}} + \overset{\textit{parallel-conditional}}{\left\{\begin{array}{lll} CTxy & \to & x \\ CFxy & \to & y \\ Czxx & \to & x \end{array}\right\}}$$

The aim of this paper is the proof of Chew's theorem [Che81].

Theorem 2.1 A compatible TRS is UN.

The proof of Lemma 6.1 in [Che81], which is essential for the proof of the theorem, is incomplete [vO]. It states a correspondence between a system with markers α, β, and a system without markers. More precisely, removing the markers in a step in the former system (by means of the choice function ch), should give a step in the latter system.

The induction on the length of $B \to_R^* A$ in the proof does not work for the distributivity rule αd. Let us consider the following example:

$$\underset{B}{g(\alpha(h(t_1,t_2),h(t_3,t_4)))} \underset{\alpha d}{\to} \underset{A}{g(h(\alpha(t_1,t_3),\alpha(t_2,t_4)))},$$

where t_i are arbitrary terms. Any way of removing the markers (by selecting an argument via the choice function ch), we obtain $C_B = \{g(h(t_1,t_2)), g(h(t_3,t_4))\}$ from B and $C_A = \{g(h(t_1,t_2)), g(h(t_1,t_4)), g(h(t_3,t_2)), g(h(t_3,t_4))\}$ from A. In the induction step, it must be shown that for each $s_A \in C_A$, there exists $s_B \in C_B$ such that $s_B \to_G^* s_A$, which is impossible in the case $s_A = g(h(t_1,t_4))$ or $g(h(t_3,t_2))$.

The problem would be avoided by assuming some synchronization mechanism of ch. But it apparently gets problem with $\alpha-$ rule.

3 Conditional linearization and Peak elimination

3.1 A property of compatible rewrite rules

Definition 3.1 The set of *non-common positions* $NC_{t,t'}$ of terms t and t' is the set of all minimal elements in $\{p \mid TopSym(t/p) \ne TopSym(t'/p)\}$, where $TopSym(t)$ is the top symbol of the term t. The *common context* $C_{t,t'}[]$ of t and t' is $t[p \leftarrow \Box \mid p \in NC_{t,t'}]$ ($\equiv t'[p \leftarrow \Box \mid p \in NC_{t,t'}]$).

[2] De Vrijer's terminology [dV90] is used here. The corresponding notion in Chew's original paper is "strongly nonoverlapping and compatible".

Definition 3.2 For terms t, t', a relation $\sim_{t,t'}$ is defined as follows:

$$s \sim_{t,t'} s' \text{ iff } s \equiv t/p \text{ and } s' \equiv t'/p \text{ for some } p \in NC_{t,t'}.$$

Note that if $s \sim_{t,t'} s'$, then s and s' are subterms of t and t', respectively.

Lemma 3.1 Let t, t' be terms without shared variables. Assume $s \sim_{t,t'} C[u]$ and $u \sim_{t,t'} u'$. Then u is a ground term. ∎

Lemma 3.2 Let t, t' be unifiable terms without shared variables. Suppose t and t' are linear. Then, the substitution defined as below is a unifier of t and t':

$$\theta_{t,t'} = \{x := s' \mid x \sim_{t,t'} s'\} \cup \{x' := s \mid s \sim_{t,t'} x' \text{ and } s \notin V\}.$$
∎

Lemma 3.3 Let $S : l \to r$, $S' : l' \to r'$ be compatible rewrite rules with unifiable linearizations of left-hand sides, i.e., there exist linearizations $\bar{S} : \bar{l} \to \bar{r}$, $\bar{S}' : \bar{l}' \to \bar{r}'$ of S, S' respectively such that $\bar{l}$ and $\bar{l}'$ are unifiable and $\bar{r}\sigma \equiv \bar{r}'\sigma$ for each unifier σ of $\bar{l}$ and $\bar{l}'$. Then, for all $q \in NC_{\bar{r},\bar{r}'}$, either of the following holds:

1. $\bar{r}/q \in V$, and there exist a context $C'_q[]$ with m' □'s ($m' \geq 0$), ground terms $g_1, \cdots, g_{m'}$ and variables $x'_1, \cdots, x'_{m'}$ in S' s.t.
   - $\bar{r}/q \sim_{\bar{l},\bar{l}'} C'_q[g_1, \cdots, g_{m'}]$,
   - $g_k \sim_{\bar{l},\bar{l}'} x'_k$ for all k, and
   - $\bar{r}'/q \equiv C'_q[x'_1, \cdots, x'_{m'}]$.
2. $\bar{r}'/q \in V$, and there exist a context $C[]$ with n' □'s ($n' \geq 0$), ground terms $g'_1, \cdots, g'_{n'}$ and variables $x_1, \cdots, x_{n'}$ in S s.t.
   - $C_q[g'_1, \cdots, g'_{n'}] \sim_{\bar{l},\bar{l}'} \bar{r}'/q$,
   - $x_k \sim_{\bar{l},\bar{l}'} g'_k$ for all k, and
   - $\bar{r}/q \equiv C_q[x_1, \cdots, x_{n'}]$.

Proof Since $\bar{r}$ and $\bar{r}'$ are unifiable, $\bar{r}/q \in V$ or $\bar{r}'/q \in V$. We only check the former case. Let $C'_q[] = \bar{r}/q\{x := \Box \mid x\theta_{l,l'} \neq x\}$, where $\theta_{l,l'}$ is the unifier defined in lemma 3.2. Since $\theta_{l,l'}$ is a unifier of $\bar{r}$ and $\bar{r}'$, there are terms $g_1, \cdots, g_{m'}$ and variables $x'_1, \cdots, x'_{m'}$ satisfying the three conditions. From lemma 3.1, g_k are ground terms. ∎

3.2 Left-right separated CTRS and conditional linearization

Definition 3.3 A *left-right separated conditional term rewriting system* is a finite set of conditional rewrite rules with extra variables of the form $l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n$ satisfying following conditions:

1. l is left-linear, $V(l) = \{x_1, \cdots, x_n\}$,
2. $V(r) \subseteq \{y_1, \cdots, y_n\}$,
3. $\{x_1, \cdots, x_n\} \cap \{y_1, \cdots, y_n\} = \emptyset$, and
4. $x_i \not\equiv x_j$ if $i \neq j$.[3]

[3] $y_i \equiv y_j$ may hold for $i \neq j$.

$l \to r$ is called the *unconditional part* and $x_1 = y_1, \cdots, x_n = y_n$ is called the *condition part* of $l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n$. In the rest of this paper, for convenience:

1. A condition part is often abbreviated by Q, Q', etc.
2. Variables $x_1, \cdots, x_n$ are assumed to appear in the left-to-right order in l.

Definition 3.4 Let $\widehat{R}$ be a left-right separated CTRS. The reduction $\overset{\nabla}{\to}_{\widehat{R}_i}$ is inductively defined as follows:

$$\begin{cases} \overset{\nabla}{\to}_{\widehat{R}_0} &= \emptyset, \\ \overset{\nabla}{\to}_{\widehat{R}_{i+1}} &= \{C[l\theta] \overset{\nabla}{\to}_{\widehat{R}_{i+1}} C[r\theta] \mid \widehat{l} \to \widehat{r} \Leftarrow x_1 = y_1, \cdots, \\ & \quad x_n = y_n \in \widehat{R} \text{ and } x_j\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}_i} y_j\theta\}. \end{cases}$$

Then, $\overset{\nabla}{\to}_{\widehat{R}} = \cup_i \overset{\nabla}{\to}_{\widehat{R}_i}$.

Proofs $x_j\theta \overset{\nabla *}{\leftrightarrow}_{R_i} y_j\theta$ are called *subproofs* associated with $C[\widehat{l}\theta] \overset{\nabla}{\to}_{\widehat{R}_{i+1}} C[\widehat{r}\theta]$. Subproofs of an R_1 reduction are called trivial subproofs and we eventually denote $\overset{\nabla}{\to}_{\widehat{R}_1}$ as $\to_{\widehat{R}_1}$.

When a reduction $t \overset{\nabla}{\to}_{\widehat{R}} t'$ is done by a rewrite rule $\widehat{S} \in \widehat{R}$, it is also denoted by $t \overset{\nabla}{\to}_{\widehat{S}} t'$. For a reduction $C[\widehat{l}\theta] \overset{\nabla}{\to}_{\widehat{R}} C[\widehat{r}\theta]$, $\widehat{l}\theta$ is called a *redex* of $\widehat{R}$.

Reductions are often treated as more than a relation; we assume a reduction in $\widehat{R}$ is associated with the following "information" implicitly:

- the rule used,
- the position, and
- the subproofs.

Similarly, a rewrite proof $A : t_1 \overset{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_n$ is regarded as a hierarchical construct. Reductions $t_i \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_{i+1}$ itself are top-level components, reductions in subproofs of them are second-level components, etc. A reduction α is *in* A when α is a component of the hierarchical construct. Also, a reduction α is called a *top-level reduction* if α is a top-level component.

Definition 3.5 For a rewrite rule $S : l \to r$, a *conditional linearization* $\widehat{S} : \widehat{l} \to \widehat{r} \Leftarrow Q$ is a left-right separated conditional rewrite rule constructed as follows:

1. $\widehat{l}$ is a linearization of l s.t. $\widehat{l}\sigma = l$ and $V(\widehat{l}) \cap V(l) = \emptyset$,
2. $\widehat{r} \equiv r$, and
3. add $x\sigma = x$ to the condition part Q for all $x \in V(l)$.

A conditional linearization $\widehat{R}$ of a TRS R is a collection of conditional linearizations of all rules in R.

Note that conditional linearizations of S are unique up to renaming of variables in $\widehat{l}$. In the rest of this paper, R denotes the TRS and $\widehat{R}$ denotes the conditional linearization of R.

Example 3.1 $\widehat{R}$ is the conditional linearization of R.

$$R = \left\{ \begin{array}{lll} d(x,x) & \to & 0 \\ f(y) & \to & d(y, f(y)) \\ 1 & \to & f(1) \end{array} \right\}$$

$$\widehat{R} = \left\{ \begin{array}{lllll} d(x_1,x_2) & \to & 0 & \Leftarrow & x_1 = x, x_2 = x \\ f(y_1) & \to & d(y, f(y)) & \Leftarrow & y_1 = y \\ 1 & \to & f(1) & & \end{array} \right\}$$

The following theorem appeared in [TO94] with the condition of non-duplicating. The expansion to the general case is straightforward.

Theorem 3.1 ([TO94]) If $\widehat{R}$ is CR, then R is UN. ∎

For left-right separated conditional rewrite rules $\widehat{S}$, $\widehat{S}'$, $\widehat{S}$ and $\widehat{S}'$ are said to be nonoverlapping (almost nonoverlapping, overlay) if their unconditional parts are overlapping (almost nonoverlapping, overlay). A left-right separated CTRS $\widehat{R}$ is nonoverlapping (almost nonoverlapping, overlay) when every pair of rules in $\widehat{R}$ is nonoverlapping (almost nonoverlapping, overlay). A left-right separated CTRS $\widehat{R}$ is compatible if there exists a compatible TRS R such that $\widehat{R}$ is a conditional linearization of R.

Definition 3.6 A term t is a *head normal form* of $\widehat{R}$ if s is not a redex of $\widehat{R}$ for all s such that $t \overset{\nabla *}{\to}_{\widehat{R}} s$. A term t is a *quasi-ground normal form* of $\widehat{R}$ *wrt* $q \in P(t)$ if

1. for each $q' \leq q$, t/q' is a head normal form of $\widehat{R}$, and
2. t/q is a ground normal form of $\widehat{R}$.

Lemma 3.4 Let $\widehat{l} \to \widehat{r} \Leftarrow Q \in \widehat{R}$. Suppose $\widehat{R}$ be compatible. Then, for each non-variable proper subterm t of l and substitution θ, $t\theta$ is a head normal form of $\widehat{R}$. ∎

3.3 Conditional peak elimination

Lemma 3.5 If R is a compatible TRS, then $\widehat{R}$ is an overlay system. ∎

In the rest of this section, the following notations will be established:

1. R is a compatible TRS.
2. $S : l \to r$, $S' : l' \to r' \in R$.
3. $\bar{S} : \bar{l} \to \bar{r}$, $\bar{S}' : \bar{l}' \to \bar{r}'$ are linearizations of S and S' s.t. $\bar{r}\sigma \equiv \bar{r}'\sigma$ for all unifiers σ of $\bar{l}$ and $\bar{l}'$.
4. $\widehat{S} : \widehat{l} \to \widehat{r} \Leftarrow x_1 = y_1, \cdots, x_n = y_n$, $\widehat{S}' : \widehat{l}' \to \widehat{r}' \Leftarrow x'_1 = y'_1, \cdots, x'_m = y'_m$ are conditional linearizations of S and S' s.t. $\widehat{l} \equiv \bar{l}$ and $\widehat{l}' \equiv \bar{l}'$.

Definition 3.7 Suppose there is a peak of the form $C[\widehat{r}\theta] \overset{\nabla}{\leftarrow}_{\widehat{S}} C[\widehat{l}\theta] \equiv C[\widehat{l}'\theta] \overset{\nabla}{\to}_{\widehat{S}'} C[\widehat{r}'\theta]$. For $p \in NC_{\widehat{l},\widehat{l}'}$, the *left connecting proof* A_p of the peak is defined as follows:

$$A_p = \left\{ \begin{array}{r} y_i\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} x_i\theta \equiv C'_p[x'_j\theta, \cdots, x'_{j+j'}\theta] \overset{\nabla *}{\Leftrightarrow}_{\widehat{R}} C'_p[y'_j\theta, \cdots, y'_{j+j'}\theta] \\ \text{if } \widehat{l}/p \equiv x_i \text{ and } \widehat{l}'/p = C'_p[x'_j, \cdots, x'_{j+j'}], \\ C_p[y_i\theta, \cdots, y_{i+i'}\theta] \overset{\nabla *}{\Leftrightarrow}_{\widehat{R}} C_p[x_i\theta, \cdots, x_{i+i'}\theta] \equiv x'_j\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} y'_j\theta \\ \text{if } l/p = C_p[x_i, \cdots, x_{i+i'}] \notin V \text{ and } l'/p \equiv x'_j. \end{array} \right.$$

where $x_k\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} y_k\theta$ ($x'_k\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} y'_k\theta$) are the subproofs of $C[\widehat{l}\theta] \overset{\nabla}{\to}_{\widehat{S}} C[\widehat{r}\theta]$ ($C[\widehat{l}'\theta] \overset{\nabla}{\to}_{\widehat{S}'} C[\widehat{r}'\theta]$), $V(\widehat{l}/p) = \{x_i, \cdots, x_{i+i'}\}$ and $V(\widehat{l}'/p) = \{x'_j, \cdots, x'_{j+j'}\}$.

Definition 3.8 For a rewrite rule $\widehat{S} : \widehat{l} \to \widehat{r} \Leftarrow x_1 = y_1, \cdots, x_n = y_n$, $\mathcal{T}_{\widehat{S}}$ is a substitution defined as follows:

$$\mathcal{T}_{\widehat{S}} = \{x_1 := y_1, \cdots, x_n := y_n\}.$$

Lemma 3.6 Suppose there is a peak of the form $C[\widehat{r}\theta] \overset{\nabla}{\leftarrow}_{\widehat{S}} C[\widehat{l}\theta] \equiv C[\widehat{l}'\theta] \overset{\nabla}{\to}_{\widehat{S}'} C[\widehat{r}'\theta]$. Assume $t \rightsquigarrow_{\widehat{l},\widehat{l}'} t'$. Then, there exits $p \in NC_{\widehat{l},\widehat{l}'}$ such that $t\mathcal{T}_{\widehat{S}}\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} t'\mathcal{T}_{\widehat{S}'}\theta$ is the left connecting proof A_p.

Proof Definition 3.8 gives the proof. ∎

Lemma 3.7 Suppose there is a peak of the form $C[\widehat{r}\theta] \overset{\nabla}{\leftarrow}_{\widehat{S}} C[\widehat{l}\theta] \equiv C[\widehat{l}'\theta] \overset{\nabla}{\to}_{\widehat{S}'} C[\widehat{r}'\theta]$. For $q \in NC_{\widehat{r},\widehat{r}'}$, either of the following holds:

1. $\widehat{r}/q \in V$, and there exist a context $C'_q[]$ with m' □'s ($m' \geq 0$), ground terms $g_1, \cdots, g_{m'}$ and variables $y'_{j_1}, \cdots, y'_{j_{m'}}$ s.t.
   - $\widehat{r}/q\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} C'_q[g_1, \cdots, g_{m'}]\mathcal{T}_{\widehat{S}'}\theta$ is a left connecting proof of the peak,
   - $g_k \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} y'_{j_k}\theta$ are left connecting proofs of the peak for all k,
   - $\widehat{r}'/q\theta \equiv C'_q[y'_{j_1}, \cdots, y'_{j_{m'}}]\mathcal{T}_{\widehat{S}'}\theta$, and
   - $C'_q[g_1, \cdots, g_{m'}]\mathcal{T}_{\widehat{S}'}\theta$ is a quasi-ground normal form of $\widehat{R}$ wrt q_k for each position q_k of □ in $C'_q[]$.
2. $\widehat{r}'/q \in V$, and there exist a context $C_q[]$ with n' □'s ($n' \geq 0$), ground terms $g'_1, \cdots, g'_{n'}$ and variables $y_{i_1}, \cdots, y_{i_{n'}}$ s.t.
   - $C_q[g'_1, \cdots, g'_{n'}]\mathcal{T}_{\widehat{S}}\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} \widehat{r}/q\theta$ is a left connecting proof of the peak,
   - $y_{i_k}\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} g'_k$ are left connecting proofs of the peak for all k,
   - $\widehat{r}/q\theta \equiv C_q[y_{i_1}, \cdots, y_{i_{n'}}]\mathcal{T}_{\widehat{S}}\theta$, and
   - $C_q[g'_1, \cdots, g'_{n'}]\mathcal{T}_{\widehat{S}}\theta$ is a quasi-ground normal form of $\widehat{R}$ wrt q_k for each position q_k of □ in $C_q[]$.

Proof We only check the former case. The first three conditions are satisfied by lemma 3.3, 3.6, $\widehat{r} \equiv \bar{r}\mathcal{T}_{\widehat{S}}$ and $\widehat{r}' \equiv \bar{r}'\mathcal{T}_{\widehat{S}'}$. The last condition follows from lemma 3.4 and the fact that $C'_q[g_1, \cdots, g_{m'}]$ is a proper subterm of $\widehat{l}'$. ∎

Definition 3.9 Suppose there is a peak of the form $C[\widehat{r}\theta] \overset{\nabla}{\leftarrow}_{\widehat{S}} C[\widehat{l}\theta] \equiv C[\widehat{l}'\theta] \overset{\nabla}{\to}_{\widehat{S}'} C[\widehat{r}'\theta]$. For $q \in NC_{\widehat{r},\widehat{r}'}$, the *right connecting proof* B_q of the peak is a proof connecting $\widehat{r}/q\theta$ and $\widehat{r}'/q\theta$ described in the previous lemma, i.e.,

$$B_q = \left\{ \begin{array}{r} \widehat{r}/q\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} C'_q[g_1, \cdots, g_{m'}]\mathcal{T}_{\widehat{S}'}\theta \overset{\nabla *}{\Leftrightarrow}_{\widehat{R}} C'_q[y'_{j_1}, \cdots, y'_{j_{m'}}]\mathcal{T}_{\widehat{S}'}\theta \\ \text{if } \widehat{r}/q \in V, \\ C_q[y_{i_1}, \cdots, y_{i_{n'}}]\mathcal{T}_{\widehat{S}}\theta \overset{\nabla *}{\Leftrightarrow}_{\widehat{R}} C_q[g'_1, \cdots, g'_n]\mathcal{T}_{\widehat{S}}\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} \widehat{r}'/q\theta \\ \text{if } \widehat{r}/q \notin V \text{ and } \widehat{r}'/q \in V. \end{array} \right.$$

Definition 3.10 For a proof $A : t_1 \overset{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_n$ in $\widehat{R}$, a *peak elimination* is a transformation of A where a peak in A, e.g., $t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$, is replaced with the sequence defined below. If A' is obtained from A by a conditional peak elimination of A, we write $A \mapsto A'$.

There are three *peak elimination rules* corresponding to the relative positions of the reductions of the peak.

($P_\perp$) Two reductions of the peak occur at parallel positions. That is,

1. $t_{i-1} \equiv C[s'_1, s_2]$, $t_i \equiv C[s_1, s_2]$, $t_{i+1} \equiv C[s_1, s'_2]$, and
2. $s_j \overset{\nabla}{\rightarrow}_{\widehat{S}_j} s'_j$ for $j = 1, 2$.

Then, the replacement sequence is

$$t_{i-1} \equiv C[s'_1, s_2] \overset{\nabla}{\rightarrow}_{\widehat{S}_2} C[s'_1, s'_2] \overset{\nabla}{\leftarrow}_{\widehat{S}_1} C[s_1, s'_2] \equiv t_{i+1},$$

where the subproofs are not modified.

($P_<$) Two reductions of the peak are nesting. Suppose $t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i$ occurs below $t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$. That is,

1. $t_{i-1} \equiv C_2[s_2[C_1[s'_1]]]$, $t_i \equiv C_2[s_2[C_1[s_1]]]$, $t_{i+1} \equiv C_2[s'_2]$,
2. $s_2[C_1[s_1]] \overset{\nabla}{\rightarrow}_{\widehat{S}_2} s'_2$ with a subproof $C_1[s_1] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} u$, and
3. $s'_1 \overset{\nabla}{\leftarrow}_{\widehat{S}_1} s_1$.

Then the replacement sequence is

$$t_{i-1} \equiv C_2[s_2[C_1[s'_1]]] \overset{\nabla}{\rightarrow}_{\widehat{S}_2} C_2[s'_2] \equiv t_{i+1},$$

which has a modified subproof $C_1[s'_1] \overset{\nabla}{\leftarrow}_{\widehat{S}_1} C_1[s_1] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} u$. If $t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i$ occurs above $t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$, then the replacement sequence is $t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_{i+1}$, which is defined similarly.

(P_C) Two reductions of the peak overlap. In this case, the peak is called a *critical peak*. Since $\widehat{R}$ is overlay by lemma 3.5, the reductions occur at the same position in t_i. That is,

1. $S : \widehat{l} \rightarrow \widehat{r} \Leftarrow Q$ is the rule used in $t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i$,
2. $S' : \widehat{l'} \rightarrow \widehat{r'} \Leftarrow Q'$ is the rule used in $t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$,
3. $t_i \equiv C[\widehat{l}\sigma] \equiv C[\widehat{l'}\sigma]$, $t_{i-1} \equiv C[\widehat{r}\sigma]$ and $t_{i+1} \equiv C[\widehat{r'}\sigma]$.

Then the replacement sequence is

$$\underbrace{C[C_{\widehat{r},\widehat{r'}}[s_1, \cdots, s_k]]}_{t_{i-1}} \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} \underbrace{C[C_{\widehat{r},\widehat{r'}}[s'_1, \cdots, s'_k]]}_{t_{i+1}},$$

where $C_{\widehat{r},\widehat{r'}}$ is the common context of $\widehat{r}$ and $\widehat{r'}$, and $s_j \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} s'_j$ are right connecting proofs of the peak.

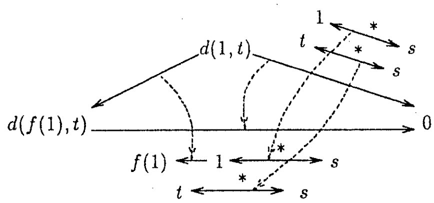

Figure 2: Rule $P_<$

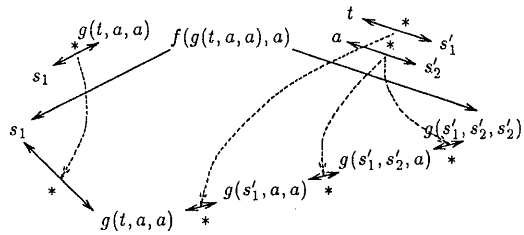

Figure 3: Rule P_C

Example 3.2 Let $\widehat{R}$ be that in example 3.1. Suppose $1 \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} s$ and $t \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} s$. Then, there is a peak of the form $d(f(1), t) \overset{\nabla}{\leftarrow}_{\widehat{R}} d(1, t) \overset{\nabla}{\rightarrow}_{\widehat{R}} 0$, where the left-oriented reduction is by the third rule, and the right-oriented reduction by the first rule. By $P_<$, it is replaced with $d(f(1), t) \overset{\nabla}{\rightarrow}_{\widehat{R}} 0$ as shown in Figure 2.

Example 3.3 Let $\widehat{R}$ be the following:

$$\widehat{R} = \left\{ \begin{array}{l} \widehat{S} : f(x_1, a) \rightarrow y_1 \Leftarrow x_1 = y_1, \\ \widehat{S'} : f(g(x'_1, a, a), x'_2) \rightarrow g(y'_1, y'_2, y'_2) \\ \qquad \Leftarrow x'_1 = y'_1, x'_2 = y'_2 \end{array} \right\}.$$

Suppose $t \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} s'_1$, $a \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} s'_2$ and $g(t, a, a) \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} s_1$. Then, there is a peak of the form $s_1 \overset{\nabla}{\leftarrow}_{\widehat{S}} f(g(t, a, a), a) \overset{\nabla}{\rightarrow}_{\widehat{S'}} g(s'_1, s'_2, s'_2)$. By P_C, it is replaced with $s_1 \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} g(s'_1, s'_2, s'_2)$ (which itself is the right connecting proof of the peak) as shown in figure 3. Here, $s_1 \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} g(t, a, a) \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} g(s'_1, a, a)$ and $a \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} s'_2$ are left connecting proofs. Note that $g(s'_1, a, a)$ is a quasi-ground normal form wrt p_1 and wrt p_2, where p_i are the positions of a^i in $g(t, a^1, a^2)$.

Definition 3.11 A rewrite proof in $\widehat{R}$ is a proof of the form $t_1 \overset{\nabla}{\rightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\rightarrow}_{\widehat{R}} t_k \overset{\nabla}{\leftarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftarrow}_{\widehat{R}} t_n$.

Lemma 3.8 Let $\widehat{R}$ be compatible.

1. Let A_1 be a proof in $\widehat{R}$. If a peak elimination process $A_1 \mapsto A_2 \mapsto \cdots$ terminates at A_n, then A_n is a rewrite proof in $\widehat{R}$.
2. If a peak elimination process $A_1 \mapsto A_2 \mapsto \cdots$ terminates for every proof A_1 in $\widehat{R}$, then $\widehat{R}$ is **CR**. ∎

4 Independence of reductions

4.1 Flattening and independence

In this section, the notion of independence is introduced. Independence is first defined for reductions in a proof in $\widehat{R}_1$. Then, it is lifted up to any proof in $\widehat{R}$ by flattening.

Lemma 4.1 For each non-$\widehat{R}_1$ reduction $t \stackrel{\nabla}{\to}_{\widehat{R}} t'$, there is a proof $t \equiv C[s_1, \cdots, s_m] \stackrel{\nabla *}{\leftrightarrow}_{\widehat{R}} C[s'_1, \cdots, s'_m] \to_{\widehat{R}_1} t'$ satisfying

1. $s_i \stackrel{\nabla *}{\leftrightarrow}_{\widehat{R}} s'_i$ are the subproofs of $t \stackrel{\nabla}{\to}_{\widehat{R}} t'$, and
2. in the reduction $C[s'_1, \cdots, s'_m] \to_{\widehat{R}_1} t'$, the same rule is used at the same position as in $t \stackrel{\nabla}{\to}_{\widehat{R}} t'$.

Proof Let $l \to r \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ be the rewrite rule for the reduction $t \stackrel{\nabla}{\to}_{\widehat{R}} t'$, $t \equiv C'[l\theta]$ and $t' \equiv C'[r\theta]$. Let $C''[\ ]$ be a context such that $C''[x_1, \cdots, x_m] \equiv l$. Then, the result follows by setting $C[\ ] = C'[C''[\ ]]$. ∎

Definition 4.1 For a non-$\widehat{R}_1$ reduction $t \stackrel{\nabla}{\to}_{\widehat{R}} t'$, the proof $t \equiv C[s_1, \cdots, s_m] \stackrel{\nabla *}{\leftrightarrow}_{\widehat{R}} C[s'_1, \cdots, s'_m] \to_{\widehat{R}_1} t'$ described in lemma 4.1 is called the *flattening* of $t \stackrel{\nabla}{\to}_{\widehat{R}} t'$. The flattening of a proof $A : t_1 \stackrel{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \stackrel{\nabla}{\leftrightarrow}_{\widehat{R}} t_n$ at the i-th non-R_1 reduction is obtained by replacing $\alpha : t_i \stackrel{\nabla}{\leftrightarrow}_{\widehat{R}} t_{i+1}$ with its flattening.

Lemma 4.2 When a flattening operation is regarded as a reduction on the set of proofs, there exists a unique normal forms for each proof A. The normal form is called the *flat proof of* A and is denoted by $A^\flat$.

Proof Since flattening operation is **WCR** and **SN**, it is **CR**. ∎

Note that $A^\flat$ contains only $\widehat{R}_1$ reductions.

Definition 4.2 The notations are the same as those used in lemma 4.1. The mapping $flat$ is a bijection from reductions in A to ones in its flattening as follows:

1. If α is the top-level reduction $t \stackrel{\nabla}{\to}_{\widehat{R}} t'$, then $flat(\alpha)$ is $C[s'_1, \cdots, s'_m] \to_{\widehat{R}_1} t'$.
2. If α is in the i-th subproof $s_i \stackrel{\nabla *}{\leftrightarrow}_{\widehat{R}} s'_i$, $flat(\alpha)$ is the corresponding reduction in $C[\cdots, s_i, \cdots] \stackrel{\nabla *}{\leftrightarrow}_{\widehat{R}} C[\cdots, s'_i, \cdots]$.

For a reduction α in A, $\alpha^\flat$ in $A^\flat$ is obtained by repeated applications of $flat$.

Example 4.1 Let $\widehat{R}$ be that in example 3.1. Let A be a one-step proof of the form $A : d(d(1, f(1)), 1) \stackrel{\nabla}{\to}_{\widehat{R}} 0$ with subproofs $d(1, f(1)) \leftarrow_{\widehat{R}_1} f(1)$ and $1 \to_{\widehat{R}_1} f(1)$. In this case, applying a flattening operation to A, we obtain $A^\flat : d(d(1, f(1)), 1) \leftarrow_{\widehat{R}_1} d(f(1), 1) \to_{\widehat{R}_1} d(f(1), f(1)) \to_{\widehat{R}_1} 0$ as in figure 4.

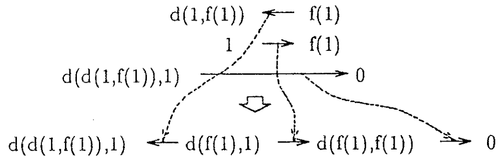

Figure 4: Flattening

Lemma 4.3 Let $A : t \stackrel{\nabla}{\to}_{\widehat{R}} t'$ be a one-step proof where the position of the reduction is p, and let $A^\flat : t \equiv t_1 \leftrightarrow_{\widehat{R}_1} \cdots \leftrightarrow_{\widehat{R}_1} t_n \to_{\widehat{R}_1} t'$. For all $i \leq n$, there is a reduction $\alpha : t_i \stackrel{\nabla *}{\to}_{\widehat{R}} t'$ satisfying $p(\alpha) = p$. ∎

Definition 4.3 Let $A_1 : t_1 \leftrightarrow_{\widehat{R}_1} \cdots \leftrightarrow_{\widehat{R}_1} t_n$ be a proof in $\widehat{R}_1$, and let $\alpha_i : t_i \leftrightarrow_{\widehat{R}_1} t_{i+1}$. Suppose that $i \leq j$. We say t_k is *between*

$$\begin{cases} t_i \text{ and } t_j \text{ (or } t_j \text{ and } t_i\text{)} & \text{when} \quad i \leq k \leq j, \\ \alpha_i \text{ and } t_j \text{ (or } t_j \text{ and } \alpha_i\text{)} & \text{when} \quad i+1 \leq k \leq j, \\ t_i \text{ and } \alpha_j \text{ (or } \alpha_j \text{ and } t_i\text{)} & \text{when} \quad i \leq k \leq j, \\ \alpha_i \text{ and } \alpha_j \text{ (or } \alpha_j \text{ and } \alpha_i\text{)} & \text{when} \quad i+1 \leq k \leq j. \end{cases}$$

A reduction $\alpha_k : t_k \leftrightarrow_{\widehat{R}_1} t_{k+1}$ is between terms (or a reduction and a term, or reductions) if both t_k and t_{k+1} are between terms (or a reduction and a term, or reductions).

Definition 4.4 Let A_1 be a proof in $\widehat{R}_1$. Relations $\perp\!\!\!\perp$, $\perp\!\!\!\perp_1$, $\perp\!\!\!\perp_2$ and $/\!\!/_2$ on reductions α, β in A_1 are defined as follows:

- $\alpha \perp\!\!\!\perp \beta$ if $\alpha \perp\!\!\!\perp_1 \beta$ or $\alpha \perp\!\!\!\perp_2 \beta$.
- $\alpha \perp\!\!\!\perp_1 \beta$ if
  1. $p(\alpha) \perp p(\beta)$, and
  2. $p(\gamma) \not\leq \wedge(p(\alpha), p(\beta))$ for all reductions γ between α and β.
- $\alpha \perp\!\!\!\perp_2 \beta$ if either $\alpha /\!\!/_2 \beta$ or $\beta /\!\!/_2 \alpha$.
- $\alpha /\!\!/_2 \beta$ if there are a term t between α and β, positions $p \in P(t)$ and $q \in P(t/p)$ s.t.
  1. t/p is a quasi-ground normal form of $\widehat{R}$ wrt q,
  2. $p(\alpha) \geq p \cdot q$,
  3. $p(\beta) \geq p$,
  4. $p(\gamma_1) \not< p \cdot q$ for all reductions γ_1 between α and t, and
  5. $p(\gamma_2) \not< p$ for all reductions γ_2 between t and β.

For a proof A in $\widehat{R}$ and reductions α, β in A, we also write $\alpha \perp\!\!\!\perp \beta$, $\alpha \perp\!\!\!\perp_1 \beta$, $\alpha \perp\!\!\!\perp_2 \beta$ and $\alpha /\!\!/_2 \beta$ if $\alpha^\flat \perp\!\!\!\perp \beta^\flat$, $\alpha^\flat \perp\!\!\!\perp_1 \beta^\flat$, $\alpha^\flat \perp\!\!\!\perp_2 \beta^\flat$ and $\alpha^\flat /\!\!/_2 \beta^\flat$, respectively.

Example 4.2 Let $\widehat{R}$ be that in example 3.3. Consider the following proof in $\widehat{R}_1$:

$$\begin{array}{ll} & \underline{f(g(f(a,a),a,a),a)} \\ \stackrel{\alpha_1}{\to}_{\widehat{S}'} & g(\underline{f(a,a)},a,a) \\ \stackrel{\alpha_2}{\to}_{\widehat{S}} & g(a,a,a) \\ \stackrel{\alpha_3}{\leftarrow}_{\widehat{S}} & g(a,\underline{f(a,a)},a) \\ \stackrel{\alpha_4}{\leftarrow}_{\widehat{S}} & g(a,f(a,a),\underline{f(a,a)}), \end{array}$$

where underlines indicate the redexes contracted. Then, $\alpha_2 \perp\!\!\!\perp_1 \alpha_3$, $\alpha_3 \perp\!\!\!\perp_1 \alpha_4$ and $\alpha_4 \perp\!\!\!\perp_1 \alpha_2$. Also, $\alpha_3 \not\perp\!\!\!\perp_2 \alpha_1$ and $\alpha_4 \not\perp\!\!\!\perp_2 \alpha_1$ since $g(f(a,a),a,a)$ is a quasi-ground normal form wrt p_1 and p_2, where p_i are the positions of a^i in $g(f(a,a),a^1,a^2)$.

Reductions α and β are *independent* if $\alpha \perp\!\!\!\perp \beta$. If $\alpha \perp\!\!\!\perp_2 \beta$, the term t in the definition of $\not\perp\!\!\!\perp_2$ is called a *split of* α *and* β. If t is a split and t/p is the quasi-ground normal form, then t/p is called the *body* of the split.

Definition 4.5 The notations are the same as those used in definition 3.10. Suppose $A \mapsto A'$. *Descendants* in A' of reductions in A is defined as follows:

- A reduction not in the replacement sequence of the peak is the descendant of the same reduction in A.
- For reductions in the replacement sequence, there are three cases according to the peak elimination rules:

($P_\perp$) Let $\gamma_1' : C[s_1', s_2'] \overset{\nabla}{\leftarrow}_{\widehat{S}_1} C[s_1, s_2']$, $\gamma_2' : C[s_1', s_2] \overset{\nabla}{\rightarrow}_{\widehat{S}_2} C[s_1', s_2']$ be reductions of the replacement sequence. γ_1' is the descendant of $\gamma_1 : t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{S}_1} t_i$. A reduction in a subproof B of γ_1' is the descendant of the same reduction in the subproof B of γ_1. It is similar for the case of γ_2'.

($P_<$) Let $\gamma_2' : t_{i-1} \overset{\nabla}{\rightarrow}_{\widehat{S}_2} t_{i+1}$ be the reduction of the replacement sequence, let $B : C_1[s_1'] \overset{\nabla}{\leftarrow}_{\widehat{S}_1} C_1[s_1] \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} u$ be the modified subproof and let γ_1' be the left-most top-level reduction in B. γ_2' is the descendant of $\gamma_2 : t_i \overset{\nabla}{\rightarrow}_{\widehat{S}_2} t_{i+1}$. A reduction in a subproof B' of γ_2' other than B is the descendant of the same reduction in the subproof B' of γ_2. A reduction in B other than γ_1'-part is the descendant of the same reduction in the subproof $C_1[s_1] \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} u$ of γ_2. γ_1' is the descendant of $\gamma_1 : t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{S}_1} t_i$. A reduction in a subproof of γ_1' is the same reduction in the subproof of γ_1. The dash lines in figure 2 indicates the descendants.

(P_C) In this case, the replacement sequence is a collection of subproofs of either of the reductions of the peak embedded into appropriate contexts. Suppose the replacement sequence is as follows:

$$t_{i-1} \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} C[s] \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} C[s'] \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} t_{i+1},$$

where $s \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} s'$ is a subproof of either of the reductions of the peak. Then, a reduction in $C[s] \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} C[s']$ is a descendant of corresponding reduction in $s \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} s'$. The dash lines in figure 3 indicates the descendants. Note that $t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{S}} t_i$ and $t_i \overset{\nabla}{\rightarrow}_{\widehat{S}'} t_{i+1}$ themselves have no descendant.

Moreover, for a peak elimination process $A_1 \mapsto \cdots \mapsto A_n$ and reductions α_i in A_i, we say α_n is a descendant of α_1 if α_{i+1} is a descendant of α_i for each $1 \leq i < n$.

Theorem 4.1 Let $\widehat{R}$ be compatible and let A, A' be proofs in $\widehat{R}$. Suppose that $A \mapsto A'$ and that reductions α', β' in A' are descendants of α, β in A, respectively. Then, $\alpha \perp\!\!\!\perp \beta$ implies $\alpha' \perp\!\!\!\perp \beta'$.

Proof From lemma 4.9, 4.10 and 4.12 below. ∎

4.2 Innocent swap

In this section, it is assumed that $\widehat{R}$ is a compatible left-right separated CTRS.

Lemma 4.4 Let t be a term such that t/p is a quasi-ground normal form wrt q. If $p' \not< p$ and t/p' is a redex, then $p' \perp p \cdot q$. ∎

Lemma 4.5 Let t be a term such that t/p is a quasi-ground normal form wrt q. If there is a reduction $\alpha : t \overset{\nabla}{\rightarrow}_{\widehat{R}} t'$ such that $p(\alpha) \not< p$, then t'/p is also a quasi-ground normal form wrt q. ∎

Lemma 4.6 Let $A : t_1 \overset{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_n$ be a proof with a reduction $\gamma : t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$, and let $t_i \equiv t_i^1 \leftrightarrow_{\widehat{R}_1} \cdots \leftrightarrow_{\widehat{R}_1} t_i^m \rightarrow_{\widehat{R}_1} t_{i+1}$ be the flat proof of $t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$. Suppose that there exist reductions α, β in A satisfying

1. $\alpha^\flat \perp\!\!\!\perp_2 \beta^\flat$,
2. both t_i and t_{i+1} in $A^\flat$ are between $\alpha^\flat$ and $\beta^\flat$, and
3. there exists j s.t. t_i^j is a split of $\alpha^\flat$ and $\beta^\flat$.

Then, t_{i+1} is also a split of $\alpha^\flat$ and $\beta^\flat$.

Proof Let t_i^j/p be the body of t_i^j. Since $\gamma^\flat$ is between $\alpha^\flat$ and $\beta^\flat$, $p(\gamma^\flat) \not< p$. Thus, from lemma 4.3, 4.4, 4.5, the result follows. ∎

Lemma 4.7 Let A be a proof in $\widehat{R}$ and let α, β be reductions in A. Suppose that $\alpha \perp\!\!\!\perp_2 \beta$ and that t is a split of $\alpha^\flat$ and $\beta^\flat$, where the body t/p is a quasi-ground normal form wrt q. Assume there is a position $p' \geq p$ satisfying

1. t/p' is a redex,
2. For each reduction γ between t and $\beta^\flat$, $p(\gamma) \not< p'$, and
3. $p(\beta^\flat) \geq p'$.

Then, $\alpha \perp\!\!\!\perp_1 \beta$.

Proof Since t/p' is a redex, $p' \perp p \cdot q$ from lemma 4.4, so $p(\beta) \perp p \cdot q$. Thus, $\alpha \not\perp\!\!\!\perp_2 \beta$. Hence, $p(\alpha^\flat) \geq p \cdot q$, and $p(\gamma') \not< p \cdot q$ for each reduction γ' between $\alpha^\flat$ and t. Moreover, $p(\beta^\flat) \geq p'$, and $p(\gamma) \not< p'$ for each reduction γ between t and $\beta^\flat$. Therefore, $\alpha \perp\!\!\!\perp_1 \beta$. ∎

Definition 4.6 Let $A : t_1 \overset{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_n$ be a proof, and let $\gamma_1 : t_{i-1} \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_i$ and $\gamma_2 : t_i \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_{i+1}$ be reductions such that $p(\gamma_1) \perp p(\gamma_2)$. Suppose that either $\gamma_2 : t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$ or $\gamma_1 : t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i$ holds.

When $\gamma_2 : t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$, the *innocent swap* of γ_1 and γ_2 is a transformation that changes the order of γ_1 and γ_2, i.e., A is transformed to

$$A' : t_1 \overset{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_{i-1} \overset{\nabla}{\rightarrow}_{\widehat{R}} t_i' \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_{i+1} \overset{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_n,$$

where $\gamma_1' : t_i' \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_{i+1}$ ($\gamma_2' : t_{i-1} \overset{\nabla}{\to}_{\widehat{R}} t_i'$) is a reduction with the same rule, position and subproofs as γ_1 (γ_2). In the case $\gamma_1 : t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i$, an innocent swap is similarly defined. For a reduction α in A, the *descendant* α' in A' is defined in the same way as that of peak eliminations by $P_\perp$.

Lemma 4.8 Let α, β be reductions in a proof A. Suppose that A' is obtained by an innocent swap on A and that α' and β' are descendants of α and β, respectively. Then $\alpha \perp\!\!\!\perp \beta \Rightarrow \alpha' \perp\!\!\!\perp \beta'$.

Proof Let $A : t_1 \overset{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_n$. Assume that the innocent swap is applied to $\gamma_1 : t_{i-1} \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_i$, $\gamma_2 : t_i \overset{\nabla}{\to}_{\widehat{R}} t_{i+1}$. Let $p_1 = p(\gamma_1)$ and $p_2 = p(\gamma_2)$. Let $C[] \equiv t_{i-1}[p_1 \leftarrow \square, p_2 \leftarrow \square]$, $t_{i-1} \equiv C[s_1, s_2]$, $t_i \equiv C[s_1', s_2]$, and $t_{i+1} \equiv C[s_1, s_2']$. We divide A and A' into the following proofs:

- $A_1 : t_1 \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} t_{i-1}$,
- $A_2 : t_{i+1} \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} t_n$,
- $B_1 : (t_{i-1} \equiv) C[s_1, s_2] \overset{\nabla}{\leftrightarrow}_{\widehat{R}} C[s_1', s_2] (\equiv t_i)$,
- $B_2 : (t_i \equiv) C[s_1', s_2] \overset{\nabla}{\to}_{\widehat{R}} C[s_1', s_2'] (\equiv t_{i+1})$,
- $B_2' : (t_{i-1} \equiv) C[s_1, s_2] \overset{\nabla}{\to}_{\widehat{R}} C[s_1, s_2'] (\equiv t_i')$,
- $B_1' : (t_i' \equiv) C[s_1, s_2'] \overset{\nabla}{\leftrightarrow}_{\widehat{R}} C[s_1', s_2'] (\equiv t_{i+1})$.

Since an innocent swap preserves the positions of reductions, it follows that $\alpha \perp\!\!\!\perp_1 \beta \Rightarrow \alpha' \perp\!\!\!\perp_1 \beta'$.

We will now prove that $\alpha \perp\!\!\!\perp_2 \beta \Rightarrow \alpha' \perp\!\!\!\perp \beta'$.

Without loss of generality, it can be assumed that $\alpha^\flat$ is on the "left-hand side" of $\beta^\flat$ in $A^\flat$. Let t be a split of $\alpha^\flat$ and $\beta^\flat$ in $A^\flat$, where the body t/p is a quasi-ground normal form wrt q. Then, then following cases exist:

1. Both α and β are in either A_1, A_2, B_1 or B_2.
2. α is in A_1, β is in B_1.
3. α is in A_1, β is in B_2.
4. α is in A_1, β is in A_2.
5. α is in B_1, β is in B_2.
6. α is in B_1, β is in A_2.
7. α is in B_2, β is in A_2.

Case 1. This is obvious.
Case 2. If the split t is in $A_1^\flat$, then it is obvious. If the split t is in $B_1^\flat$, then $t' \equiv t[p_2 \leftarrow s_2']$ in $B_1'^\flat$ is a split of $\alpha'^\flat$ and $\beta'^\flat$ from lemma 4.5. Thus, $\alpha' \perp\!\!\!\perp_2 \beta'$.
Case 3. If the split t is in $A_1^\flat$, then it is obvious. If the split t is in either $B_1^\flat$ or $B_2^\flat$, t/p_2 is a redex by lemma 4.3. For all reductions γ between t and $\beta^\flat$, $p(\gamma) \not< p_2$ since γ is in either $B_1^\flat$ or $B_2^\flat$. Suppose $p_2 \geq p$. Then, $\alpha \perp\!\!\!\perp_1 \beta$ from lemma 4.7 so $\alpha' \perp\!\!\!\perp_1 \beta'$. Next, suppose $p_2 \not\geq p$. Since $p_2 \leq p(\beta^\flat)$ and $p \leq p(\beta^\flat)$, $p_2 < p$. Hence, $t[p_1 \leftarrow s_1]$ in $B_2'^\flat$ is a split of $\alpha'^\flat$ and $\beta'^\flat$. Therefore, $\alpha' \perp\!\!\!\perp_2 \beta'$.
Case 4. If the split t is in either $A_1^\flat$ or $A_2^\flat$, then it is obvious. If the split t is in $B_1^\flat$, then $t[p_2 \leftarrow s_2']$ in $B_1'^\flat$ is a split of $\alpha'^\flat$ and $\beta'^\flat$ from lemma 4.5. If the split t is in $B_2^\flat$, then t_{i+1} is also a split of $\alpha^\flat$ and $\beta^\flat$ from lemma 4.6. Thus, t_{i+1} is a split of $\alpha'^\flat$ and $\beta'^\flat$. Therefore, $\alpha' \perp\!\!\!\perp_2 \beta'$.
Case 5. Since $p(\alpha^\flat) \geq p_1$, $p(\beta^\flat) \geq p_2$ and $p_1 \perp p_2$, $\alpha' \perp\!\!\!\perp_1 \beta'$.
Case 6. If the split t is in $A_2^\flat$, then it is obvious. Assume that the split t is in $B_1^\flat$. Since $p \leq p(\alpha^\flat)$ and $p_1 \leq p(\alpha^\flat)$, $p \not\perp p_1$. Thus, $p_2 \not< p$ from the assumption $p_1 \perp p_2$. Hence, $t[p_2 \leftarrow s_2']$ in $B_1'^\flat$ is a split of $\alpha'^\flat$ and $\beta'^\flat$ from lemma 4.5. Next, assume the split t is in $B_2^\flat$. Then t_{i+1} is also a split of $\alpha^\flat$ and $\beta^\flat$ from lemma 4.6. Therefore, $\alpha' \perp\!\!\!\perp_2 \beta'$.
Case 7. If the split t is in $A_2^\flat$, then it is obvious. If the split t is in $B_2^\flat$, then t_{i+1} in $A_2^\flat$ is also a split of $\alpha^\flat$ and $\beta^\flat$ from lemma 4.6. Thus, t_{i+1} is a split of $\alpha'^\flat$ and $\beta'^\flat$. Therefore, $\alpha' \perp\!\!\!\perp_2 \beta'$. ∎

Example 4.3 Let $\widehat{R} = \{g(x', y') \to f(x, y) \Leftarrow x' = x, y' = y,\ a \to b\}$. Consider the following proofs:

$$A : g(a,b) \overset{\nabla}{\to}_{\widehat{R}} f(a,b) \overset{\nabla}{\leftarrow}_{\widehat{R}} f(a,a) \overset{\nabla}{\to}_{\widehat{R}} f(b,a) \overset{\nabla}{\leftarrow}_{\widehat{R}} g(b,a),$$

$$A' : g(a,b) \overset{\nabla}{\to}_{\widehat{R}} f(a,b) \overset{\nabla}{\to}_{\widehat{R}} f(b,b) \overset{\nabla}{\leftarrow}_{\widehat{R}} f(b,a) \overset{\nabla}{\leftarrow}_{\widehat{R}} g(b,a).$$

Let $\gamma_1 : f(a,b) \overset{\nabla}{\leftarrow}_{\widehat{R}} f(a,a)$ and $\gamma_2 : f(a,a) \overset{\nabla}{\to}_{\widehat{R}} f(b,a)$. Then, an innocent swap of γ_1 for γ_2 in A produces A', whose descendants are $\gamma_1' : f(b,b) \overset{\nabla}{\leftarrow}_{\widehat{R}} f(b,a)$ and $\gamma_2' : f(a,b) \overset{\nabla}{\to}_{\widehat{R}} f(b,b)$.

Note that a swap of γ_2' for γ_1' in A' produces A, but that this does not preserve independence though $p(\gamma_2') \perp p(\gamma_1')$. In fact, it is not an innocent swap. Let us consider $\alpha : g(a,b) \overset{\nabla}{\to}_{\widehat{R}} f(a,b)$, $\beta : f(b,a) \overset{\nabla}{\leftarrow}_{\widehat{R}} g(b,a)$ in A, and the corresponding reductions α', β' in A'. Then, $\alpha' \perp\!\!\!\perp_2 \beta'$ since $f(b,b)$ is a split, while $\alpha \not\perp\!\!\!\perp_2 \beta$.

4.3 Proof of theorem 4.1

In this section, it is assumed that $\widehat{R}$ is a compatible left-right separated CTRS.

Lemma 4.9 Let A, A' be proofs in $\widehat{R}$ such that $A \overset{P_\perp}{\mapsto} A'$. Let reductions α', β' in A' be descendants of α, β in A. Then $\alpha \perp\!\!\!\perp \beta \Rightarrow \alpha' \perp\!\!\!\perp \beta'$.

Proof From lemma 4.8. ∎

Lemma 4.10 Let A, A' be proofs in $\widehat{R}$ such that $A \overset{P_<}{\mapsto} A'$. Let reductions α', β' in A' be descendants of α, β in A. Then $\alpha \perp\!\!\!\perp \beta \Rightarrow \alpha' \perp\!\!\!\perp \beta'$.

Proof Let $t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i \overset{\nabla}{\to}_{\widehat{R}} t_{i+1}$ be the peak in $A : t_1 \overset{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_n$ which $P_<$ is applied to. Let $l \to r \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ be the rule for the reduction $\gamma_2 : t_i \overset{\nabla}{\to}_{\widehat{R}} t_{i+1}$, where $t_i \equiv C[l\theta]$ and $t_{i+1} \equiv C[r\theta]$. Suppose $\gamma_1 : t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i$ occurs below the j-th substitution part of γ_2 and that $\gamma_2' : t_{i-1} \overset{\nabla}{\to}_{\widehat{R}} t_{i+1}$ is the replace sequence for the peak. Then, the flattening of A at γ_2 is

$$fA : \cdots \underline{t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i} \equiv t_i^0 \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} t_i^{j-1} \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} t_i^j \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} t_i^m \overset{\nabla}{\to}_{\widehat{R}} t_{i+1} \cdots,$$

where $t_i^{k-1} \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} t_i^k$ corresponds to the subproof $x_k\theta \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} y_k\theta$ of γ_2, and the flattening of A' at γ_2' is

$$fA' : \cdots t_{i-1} \equiv t_{i-1}^0 \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} \underline{t_{i-1}^{j-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i^{j-1}} \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} t_i^j \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} t_i^m \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1} \cdots,$$

where $t_{i-1}^k \equiv t_i^k[p(\gamma_1) \leftarrow t_{i-1}/p(\gamma_1)]$.

Thus, fA' is obtained from fA by repeated applications of innocent swaps to $flat(\gamma_1)$, γ_1^1 (a descendant of $flat(\gamma_1)$), γ_1^2 (a descendant of γ_1^1), $\cdots$ with their right adjacent reductions since $p(\gamma) \perp p(flat(\gamma_1))$ for each reduction γ in $t_i^0 \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} t_i^{j-1}$. From lemma 4.2 and lemma 4.8, independence is preserved. The proof is similar when γ_1 occurs above γ_2. ∎

Lemma 4.11 Let $t_1 \overset{\nabla}{\leftarrow}_{\widehat{R}} t \overset{\nabla}{\rightarrow}_{\widehat{R}} t_2$ be a critical peak and let A_p be a left connecting proof of the peak. Suppose α, β are reductions in subproofs of either of reductions of the peak such that the corresponding reductions, denoted by α_p, β_p, are in A_p. Then $\alpha \perp\!\!\!\perp \beta \Rightarrow \alpha_p \perp\!\!\!\perp \beta_p$.

Proof Suppose A_p is of the form $s' \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} s \equiv C_p[u_1, \cdots, u_n] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C_p[u_1', \cdots, u_n']$, where $s \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} s$ and $u_i \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} u_i'$ are subproofs. There are following cases:

1. Both α_p and β_p are in either $s' \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} s$ or $C_p[\cdots, u_i, \cdots] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C_p[\cdots, u_i', \cdots]$.
2. α_p is in $C_p[\cdots, u_i, \cdots] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C_p[\cdots, u_i', \cdots]$, β_p is in $C_p[\cdots, u_j, \cdots] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C_p[\cdots, u_j', \cdots]$ and $i \neq j$.
3. α_p is in $s' \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} s$ and β_p is in $C_p[\cdots, u_i, \cdots] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C_p[\cdots, u_i', \cdots]$ (or vice versa).

In case 1, it is obvious. In case 2, $\alpha_p \perp\!\!\!\perp_1 \beta_p$. The flat proofs of reductions of the peak can be written as following:

$$t_1 \leftarrow_{\widehat{R}_1} \cdots \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C_1[s'] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C_1[s] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} t$$

$$t \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C_2[C_p[u_1, \cdots, u_n]] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C_2[C_p[u_1', \cdots, u_n']] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} \cdots \rightarrow_{\widehat{R}_1} t_2$$

Note that the positions of $\Box$ in both $C_1[]$ and $C_2[]$ are p. Let p' be the position of reductions of the peak. Then, for all reductions γ in $C_1[s] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} t$ or $t \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C_2[C_p[u_1, \cdots, u_n]]$, $p(\gamma) \perp p' \cdot p$ from the definition of flattening. Therefore, the result follows in case 3. The proof is similar when A_p is of the form $C_p'[s_1', \cdots, s_n'] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C_p'[s_1, \cdots, s_n] \equiv u \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} u'$. ∎

Lemma 4.12 Let A, A' be a proof in $\widehat{R}$ such that $A \overset{P_C}{\mapsto} A'$. If reductions α, β in A have descendants α', β' in A', then $\alpha \perp\!\!\!\perp \beta \Rightarrow \alpha' \perp\!\!\!\perp \beta'$.

Proof Let $A : t_1 \overset{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_n$ and $t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$ be the critical peak eliminated in $A \mapsto A'$. Let $\gamma_1 : t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i$, $\gamma_2 : t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$ and $p' = p(\gamma_1) = p(\gamma_2)$.

Without loss of generality, it can be assumed that $\alpha^\flat$ is on the "left-hand side" of $\beta^\flat$ in $A^\flat$. We divide A into following proofs:

- $A_1 : t_1 \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} t_{i-1}$,
- $A_2 : t_{i+1} \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} t_n$,
- $B_1 : t_{i-1} \overset{\nabla}{\leftarrow}_{\widehat{R}} t_i$,
- $B_2 : t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$.

Let $B : t_{i-1} \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} t_{i+1}$ be the replacement sequence for the critical peak. Note that for each reduction γ in B, $p(\gamma) \geq p'$. Then, the following cases exist:

1. Both α and β are either A_1 or A_2.
2. α is in A_1 and β is in A_2.
3. α and β are in either B_1 or B_2.
4. α is in A_1 and β is in either B_1 or B_2 (or, α is in either B_1 or B_2 and β is in A_2).

Case 1. This is obvious.

Case 2. Since $p(\gamma) \geq p'$ for each reduction γ in B, it follows that $\alpha \perp\!\!\!\perp_1 \beta \Rightarrow \alpha' \perp\!\!\!\perp_1 \beta'$. Assume that $\alpha \perp\!\!\!\perp_2 \beta$ and t is a split of $\alpha^\flat$ and $\beta^\flat$. If t is in $B_1^\flat$, then t_{i-1} is also a split from lemma 4.6. If t is in $B_2^\flat$, t_{i+1} is also a split from lemma 4.6. Thus, we can assume that t is in either $A_1^\flat$ or $A_2^\flat$. Since $p(\gamma) \geq p'$ for each reduction γ in $B^\flat$, it follows that $\alpha' \perp\!\!\!\perp_2 \beta'$.

Case 3. Recall that B is a collection of the right connecting proofs B_q of the peak. Suppose α', β' are in B_q-part, $B_{q'}$-part of B, respectively. If $q \neq q'$, then $\alpha' \perp\!\!\!\perp_1 \beta'$. Hence, suppose $q = q'$. Again, recall that B_q is a collection of left connecting proofs. Suppose B_q is as follows:

$$s \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C[g_1, \cdots, g_m] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C[u_1, \cdots, u_m],$$

where $A_s : s \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C[g_1, \cdots, g_m]$ and $A_{u_i} : C[\cdots, g_i, \cdots] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C[\cdots, u_i, \cdots]$ are left connecting proofs. If α' (or β') is in A_s-part and β' (or α') is in A_{u_i}-part, then $\alpha' \perp\!\!\!\perp_2 \beta'$ by lemma 3.7. If α', β' are in A_{u_i}-part, A_{u_j}-part respectively such that $i \neq j$, then $\alpha' \perp\!\!\!\perp_1 \beta'$. The remaining case is both α' and β' are in either A_s-part or A_{u_i}-part, and the result follows from lemma 4.11. The proof is similar when B_q is of the form $C[s_1, \cdots, s_n] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} C[g_1, \cdots, g_n] \overset{\nabla^*}{\leftrightarrow}_{\widehat{R}} u$.

Case 4. From symmetry, we can assume that α is in A_1 and that β is in either B_1 or B_2. If β is either γ_1 or γ_2, β' does not exist. Thus, β is in a subproof of γ_1 or γ_2. Suppose $\alpha \perp\!\!\!\perp_1 \beta$. Then, $p(\gamma) \not\leq \wedge(p(\alpha^\flat), p')$ for each reduction γ between $\alpha^\flat$ and t_{i-1} since $\gamma_1^\flat$ is between $\alpha^\flat$ and $\beta^\flat$, and $p' \leq p(\beta^\flat)$. Since $p(\gamma') \geq p'$ for each reduction γ' in $B'^\flat$, it follows that $\alpha' \perp\!\!\!\perp_1 \beta'$.

Assume that $\alpha \perp\!\!\!\perp_2 \beta$ and that t is a split of $\alpha^\flat$ and $\beta^\flat$, where the body t/p is a quasi-ground normal form wrt q. If t is in $A_1^\flat$, then $\alpha' \perp\!\!\!\perp_2 \beta'$ since $p(\gamma') \geq p'$ for each reduction γ' in $B_1^\flat$. Suppose t is in either $B_1^\flat$ or $B_2^\flat$. Since $\gamma_1^\flat$ is between $\alpha^\flat$ and t, $p' \not< p$. Also, $p' \not\perp p$ since $p' \leq p(\beta^\flat)$ and $p \leq p(\beta^\flat)$. Hence, $p' \geq p$, so $\alpha \perp\!\!\!\perp_1 \beta$ from lemma 4.7. Thus, $\alpha' \perp\!\!\!\perp_1 \beta'$. ∎

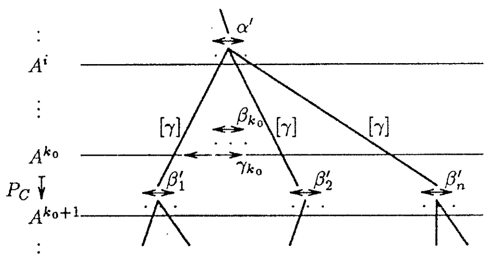

Figure 5: Descendant tree

5 Church-Rosser property of $\widehat{R}$

Let R be a compatible TRS, and let $\widehat{R}$ be the conditional linearization. Assume that $A^1 : t_1 \overset{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_n$ is an arbitrary proof in $\widehat{R}$ and that $A^1 \mapsto A^2 \mapsto \cdots$ is an arbitrary peak elimination process. The following section will show that the process $A^1 \mapsto A^2 \mapsto \cdots$ terminates. This implies that **CR** holds for $\widehat{R}$ by lemma 3.8.

Definition 5.1 The *initial labeling* on each reduction in A^i for $i = 1, 2, \cdots$ is defined as follows:

1. The set of *initial labels* is $\{[\alpha] \mid \alpha \text{ is in } A^1\}$.
2. Each reduction α in A^1 is labeled $[\alpha]$.
3. For each reduction β in A^i for $i \geq 2$, β is labeled $[\alpha]$ if β is a descendant of α in A^1.

Definition 5.2 Let α be a reduction in A^1. The *descendant tree* $T_{[\alpha]}$ is an edge-labeled tree defined as follows:

1. The root vertex is the reduction α in A^1.
2. Let α' in A^i be a vertex of $T_{[\alpha]}$. Suppose that there are $k > i$, β_k, γ_k satisfying following conditions:
   (a) In $A^k \mapsto A^{k+1}$, P_C is applied.
   (b) β_k is in a subproof of a reduction γ_k of the peak eliminated in $A^k \mapsto A^{k+1}$.
   (c) β_k is a descendant of α'.

   Let k_0 be the lowest value satisfying such conditions. Then, all the descendants $\beta'_1, \cdots, \beta'_n$ in A^{k_0+1} of β_{k_0} are the child vertices of α'. The label of the edges (α', β'_j) is the initial label of γ_{k_0}, e.g. $[\gamma]$ (figure 5).

Note that all vertices in $T_{[\alpha]}$ are labeled with $[\alpha]$.
We classify P_C into the following:

(P^1_C) The replacement sequence is empty.

(P^2_C) The replacement sequence is not empty.

Lemma 5.1 Suppose P^2_C is applied in $A^i \mapsto A^{i+1}$. Then, there are a reduction β in A^{i+1} and a descendant tree $T_{[\alpha]}$ such that β is a vertex of $T_{[\alpha]}$. ∎

Lemma 5.2 Suppose α is in a subproof of β. Then, $\alpha \not\perp\!\!\!\perp \beta$.

Proof Since $p(\alpha^\flat) > p(\beta^\flat)$, $\alpha \not\perp\!\!\!\perp_1 \beta$. Suppose there is a split t of $\alpha^\flat$ and $\beta^\flat$, where the body t/p is a quasi-ground normal form wrt q. Then, $p(\beta^\flat) \perp p \cdot q$ from lemma 4.3 and 4.4. Also, $p(\alpha^\flat) \perp p \cdot q$ since $p(\alpha^\flat) \geq p(\beta^\flat)$. This contradicts to the definition of $\perp\!\!\!\perp_2$. ∎

Lemma 5.3 Let A be a proof in $\widehat{R}$ with reductions α and β. If $\alpha \perp\!\!\!\perp \beta$ and β' is in a subproof of β, then $\alpha \perp\!\!\!\perp \beta'$.

Proof Suppose that $\beta^\flat$ is between $\alpha^\flat$ and $\beta'^\flat$. It is obvious that $\alpha \perp\!\!\!\perp_1 \beta \Rightarrow \alpha \perp\!\!\!\perp_1 \beta'$. If $\alpha \perp\!\!\!\perp_2 \beta$, then any split t of $\alpha^\flat$ and $\beta^\flat$ is also a split of $\alpha^\flat$ and $\beta'^\flat$.

Next, suppose that $\beta'^\flat$ is between $\alpha^\flat$ and $\beta^\flat$. It is obvious that $\alpha \perp\!\!\!\perp_1 \beta \Rightarrow \alpha \perp\!\!\!\perp_1 \beta'$. Suppose $\alpha \perp\!\!\!\perp_2 \beta$ and t is a split of α and β, where the body t/p is a quasi-ground normal form wrt q. The result is obvious when t is between $\alpha^\flat$ and $\beta'^\flat$. If t is between $\beta^\flat$ and β, $t/p(\beta^\flat)$ is a redex from lemma 4.3. Thus, $p(\beta^\flat) \perp p \cdot q$ from lemma 4.4. Hence, $p(\alpha^\flat) \geq p \cdot q$ so $\alpha \perp\!\!\!\perp_1 \beta$. Therefore, $\alpha \perp\!\!\!\perp_1 \beta'$. ∎

Lemma 5.4 Let A, A' be proofs such that $A \mapsto A'$. Suppose $\alpha'_1, \cdots, \alpha'_m$ in A' are descendants of α in A. Then, $m_1 \neq m_2 \Rightarrow \alpha'_{m_1} \perp\!\!\!\perp \alpha'_{m_2}$.

Proof Notations are the same as those used in definition 3.7 or lemma 3.7. It is clear that α has multiple descendants only when P_C is applied to a peak $C[\widehat{r}\theta] \overset{\nabla}{\leftarrow}_{\widehat{S}} C[\widehat{l}\theta] \equiv C[\widehat{l'}\theta] \overset{\nabla}{\rightarrow}_{\widehat{S'}} C[\widehat{r'}\theta]$ in A and when α is in a subproof of either of reductions of the peak.

The replacement sequence for the peak is a collection of right connecting proofs B_q. If α'_{m_1} is in B_q-part and α'_{m_2} is in B'_q-part such that $q \neq q'$, then $\alpha'_{m_1} \perp\!\!\!\perp_1 \alpha'_{m_2}$. Hence, suppose $q = q'$. Assume $\widehat{r}/q \equiv y_i \in V$. Then, B_q is as follows:

$$y_i\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} C'_q[g_1, \cdots, g_{m'}]T_{\widehat{S}'}\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} C'_q[y'_{j_1}, \cdots, y'_{j_{m'}}]T_{\widehat{S}'}\theta,$$

where $A_{p_k} : g_k \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} y'_{j_k}\theta$ are left connecting proofs of the peak. Note that p_k is the position of x'_{j_k} in $\widehat{l'}$. Since $\widehat{l'}/p_k = g_k$ are ground terms, A_{p_k} themselves are also subproofs of $C[\widehat{l'}\theta] \overset{\nabla}{\rightarrow}_{\widehat{S'}} C[\widehat{r'}\theta]$.

The other left connecting proof $A_p : y_i\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} C'_q[g_1, \cdots, g_{m'}]$ is rewritten as follows:

$$y_i\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} x_i\theta \equiv C'_p[x'_j\theta, \cdots, x'_{j+j'}\theta] \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} C'_p[y'_j\theta, \cdots, y'_{j+j'}\theta].$$

Then, A_{p_k} and $x'_{j+k'}\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} y'_{j+k'}\theta$ can not originated from the same subproof for any k, k'. For $p_{x'_{j+k'}} \geq p$ but $p_k \perp p$, where $p_{x'_{j+k'}}$ is the position of $x'_{j+k'}$ in $\widehat{l'}$.

It is clear that subproofs $x'_{j+k'}\theta \overset{\nabla *}{\leftrightarrow}_{\widehat{R}} y'_{j+k'}\theta$ are originated from different subproofs from each other. Hence, only the following case is possible: α'_{m_1} is in A_{p_k}-part and α'_{m_2} is in $A_{p_{k'}}$-part such that $k \neq k'$. Then, $\alpha'_{m_1} \perp\!\!\!\perp_1 \alpha'_{m_2}$.
The proof is similar in the case $\widehat{r}/q \notin V$. ∎

A path of $T_{[\alpha]}$ is a sequence of edges starting from the root. A *label path* is the sequence of labels of edges in a path. The set of all label paths of $T_{[\alpha]}$ is denoted by $Lpath_{T_{[\alpha]}}$.

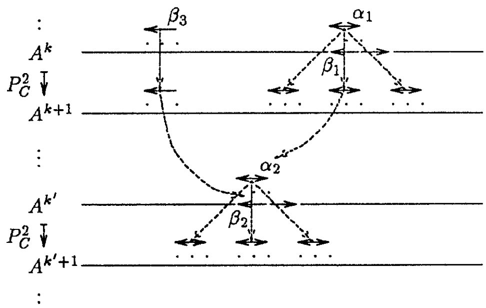

Figure 6: Proof of lemma 5.5

Lemma 5.5 Let $[\gamma_1], [\gamma_2], \cdots \in Lpath_{T_{[\alpha]}}$. Then, $[\gamma_i] \neq [\gamma_j]$ for all $i \neq j$.

Proof Suppose $[\gamma_i] = [\gamma_j] = [\beta]$ for some $i \neq j$. Then, there are descendants α_1, α_2 of α and descendants β_1, β_2, β_3 of β as shown in figure 6, where α_2 (β_2) is a descendant of α_1 (β_3).

Since β_1 and β_3 are descendants of the same reduction, $\beta_1 \perp\!\!\!\perp \beta_3$ from lemma 5.4 and theorem 4.1. Since α_1 is in a subproof of β_1, $\alpha_1 \perp\!\!\!\perp \beta_3$ from lemma 5.3. Hence, $\alpha_2 \perp\!\!\!\perp \beta_2$ from theorem 4.1. However, $\alpha_2 \not\perp\!\!\!\perp \beta_2$ by lemma 5.2. This leads to a contradiction. ∎

Lemma 5.6 For each initial label $[\alpha]$, the descendant tree $T_{[\alpha]}$ is finite.

Proof From lemma 5.5, each path of $T_{[\alpha]}$ has finite length (bounded by the number of reductions in A). Since $T_{[\alpha]}$ is obviously finitely branching, *König*'s lemma shows that $T_{[\alpha]}$ is finite. ∎

Lemma 5.7 In the peak elimination process $A^1 \mapsto A^2 \mapsto \cdots$, only finitely many peak eliminations occur with P_C^2.

Proof From lemma 5.7 and 5.1. ∎

Definition 5.3 Let $B : t_0 \overset{\nabla}{\leftrightarrow}_{\widehat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_n$ be a proof and let $\gamma_i : t_i \overset{\nabla}{\leftrightarrow}_{\widehat{R}} t_{i+1}$. A reduction γ_i is right-oriented (left-oriented) if $\gamma_i : t_i \overset{\nabla}{\rightarrow}_{\widehat{R}} t_{i+1}$ ($\gamma_i : t_i \overset{\nabla}{\leftarrow}_{\widehat{R}} t_{i+1}$). The *height* of γ_i is defined as follows:

$$height(\gamma_i) = \#\{\gamma_j \mid \gamma_j \text{ is left-oriented and } j < i\}.$$

The *mass* of B is defined as

$$mass(B) = \sum_{\text{right-oriented } \gamma_i} height(\gamma_i).$$

That is, the mass is the number of the tiles as shown in figure 7.

Lemma 5.8 Let B, B' be proofs such that $B \mapsto B'$ with either $P_\perp$, $P_<$ or P_C^1. Then, $mass(B) > mass(B')$. ∎

Corollary 5.1 Let $B_1 \overset{P_1}{\mapsto} B_2 \overset{P_2}{\mapsto} B_3 \overset{P_3}{\mapsto} \cdots$ be a peak elimination process starting from B_1. If each P_i is either $P_\perp$, $P_<$ or P_C^1, then the length of the process is finite. ∎

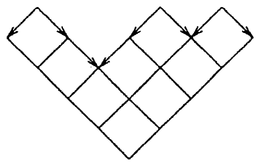

Figure 7: Mass

Theorem 5.1 Any peak elimination process $A^1 \mapsto A^2 \mapsto \cdots$ terminates.

Proof From corollary 5.7 and corollary 5.1. ∎

Corollary 5.2 Let R be a compatible TRS and let $\widehat{R}$ be the conditional linearization of R. Then, $\widehat{R}$ is CR. Therefore, R is UN. ∎

Acknowledgements

We would like to thank Vincent van Oostrom for pointing out the gap in the Chew's original proof, Fer-Jan de Vries, Yoshihito Toyama and Aart Middeldorp for helpful discussions.

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