コンパクト集合からなる de Bakker-Zucker プロセス領域の特徴付け

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概要. 完備距離空間の圏における２つの領域方程式

$$X \cong \wp_{\rm cl}(A \times id_{1/2}(X)) \quad および \quad X \cong \wp_{\rm co}(A \times id_{1/2}(X))$$

のユニークな解として与えられる２つのプロセス領域 $\mathbf{P}$ 及び $\widehat{\mathbf{P}}$ は, 各々プロセス代数の意味論で便利に使われる (ここで $\cong$ は左辺から右辺の上への等長写像の存在を表し, $\wp_{\rm cl}(X)$ と $\wp_{\rm co}(X)$ は各々 X の閉部分集合からなる空間と X のコンパクト部分集合からなる空間を表すものとする). ある種の目的に対しては, $\widehat{\mathbf{P}}$ の方が $\mathbf{P}$ より便利である. しかし, $\widehat{\mathbf{P}}$ の定義と計算上の意味は $\mathbf{P}$ のそれらより複雑であり, また $\widehat{\mathbf{P}}$ と $\mathbf{P}$ を異なる方程式の解として定義するのみでは, この２つの関係は明らかではない. ここでは, $\widehat{\mathbf{P}}$ の $\mathbf{P}$ の部分領域としての特徴付けを与え, それにより $\widehat{\mathbf{P}}$ と $\mathbf{P}$ の関係, 及び $\widehat{\mathbf{P}}$ の計算上の意味を明らかにする.

Characterizing the de Bakker-Zucker Process Domain of Compact Sets

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Abstract. In the denotational semantics of process algebras, we conveniently use the process domains $\mathbf{P}$ and $\widehat{\mathbf{P}}$ which are respectively obtained as the unique solutions of the following domain equations in the category complete metric spaces:

$$X \cong \wp_{\rm cl}(A \times id_{1/2}(X)) \quad \text{and} \quad X \cong \wp_{\rm co}(A \times id_{1/2}(X)),$$

where $\cong$ denotes the existence of an isometry from the left-hand side onto the right-hand side, and $\wp_{\rm cl}(X)$ (resp. $\wp_{\rm co}(X)$) denotes the space consisting of closed subsets of X (resp. compact subsets of X). For certain purposes, the domain $\widehat{\mathbf{P}}$ is more convenient than $\mathbf{P}$. On the other hand, the definition and computational meaning of $\widehat{\mathbf{P}}$ are more complicated than those of $\mathbf{P}$, and little is known about the relationship between $\mathbf{P}$ and $\widehat{\mathbf{P}}$ by defining these just as the solutions of different equations. In this paper, we give a characterization $\widehat{\mathbf{P}}$ as a subdomain of $\mathbf{P}$, thereby clarifying the relationship between $\widehat{\mathbf{P}}$ and $\mathbf{P}$, and the computational meaning of $\widehat{\mathbf{P}}$.

1 Introduction

In the denotational semantics of process algebras, we conveniently use the process domains $\mathbf{P}$ and $\widehat{\mathbf{P}}$ which are respectively obtained as the unique solutions of the following domain equations (1) and (2) in the category of *complete metric spaces* (*cms*'s):

$$X \cong \wp_{\rm cl}(A \times id_{1/2}(X)), \tag{1}$$

$$X \cong \wp_{\rm co}(A \times id_{1/2}(X)), \tag{2}$$

where A ($\neq \emptyset$) is an arbitrarily given set (of actions), $\cong$ denotes the existence of an isometry from the left-hand side onto the right-hand side, and $\wp_{\rm cl}(X)$ (resp. $\wp_{\rm co}(X)$) denotes the space consisting of closed subsets of X (resp. compact subsets of X).[1] For the definitions of the operators $id_{1/2}$, $\times$, $\wp_{\rm cl}$, $\wp_{\rm co}$ and for how

[1] In this paper, we only consider metric spaces $\langle X, d\rangle$ such that the metric function d is bounded by 1, (i.e., such that $d[X \times X] \subseteq [0,1]$). Equation (1) (resp. (2)) has a unique solution in the category of complete metric spaces whose metric functions are bounded by 1. The existence (resp. uniqueness) of a solution of (1) has been proved in [5] (resp. [1]). The existence and uniqueness of a solution of (2) can also be established along the lines of [1] (see [16] for a related topic).

metrics are defined on $\wp_{\rm cl}(A \times id_{1/2}(X))$ and $\wp_{\rm co}(A \times id_{1/2}(X))$, see Definition 1 in Sect. 2.

For certain purposes, the domain $\widehat{\mathbf{P}}$ is more convenient than $\mathbf{P}$ (we call $\widehat{\mathbf{P}}$ the *de Bakker-Zucker process domain of compact sets*). On the other hand, the definition and computational meaning of $\widehat{\mathbf{P}}$ are more complicated than those of $\mathbf{P}$, and little is known about the relationship between $\mathbf{P}$ and $\widehat{\mathbf{P}}$ by defining these just as the solutions of different equations. In this paper, we give a characterization $\widehat{\mathbf{P}}$ as a subdomain of $\mathbf{P}$, thus clarifying the relationship between $\mathbf{P}$ and $\widehat{\mathbf{P}}$, and the computational meaning of $\widehat{\mathbf{P}}$.

Although various process domains have been proposed for use in denotational semantics for concurrent languages (especially for process algebras), the importance of the two domains $\mathbf{P}$ and $\widehat{\mathbf{P}}$ lies in the fact that they give denotational semantics for a wide class of concurrent languages for which labeled transition systems are given by arbitrary transition rules of a certain format. The domain $\widehat{\mathbf{P}}$ is used for languages without value-passing (see [13] and [3, Chap. 2]), and $\mathbf{P}$ is suited for languages with value-passing (see [7] and [3, Chap. 5]).

Since $\mathbf{P}$ is the solution of (1), there exists an isometry ι from $\mathbf{P}$ onto $\wp_{\rm cl}(A \times id_{1/2}(\mathbf{P}))$. Thus, we have

$$\mathbf{P} \overset{\iota}{\cong} \wp_{\rm cl}(A \times id_{1/2}(\mathbf{P})), \tag{3}$$

where $\overset{\iota}{\cong}$ denotes that ι is an isometry from the left-hand side onto the right-hand side.

We will show that the set $\tilde{\mathbf{P}}$ $(\subseteq \mathbf{P})$ defined by the following equation (4) is the solution of domain equation (2), and therefore, it is isomorphic to $\widehat{\mathbf{P}}$ (in the category of complete metric spaces).

$$\tilde{\mathbf{P}} = \bigcap_{n \in \omega} [\tilde{\mathbf{P}}_n], \tag{4}$$

where $\tilde{\mathbf{P}}_n$ $(n \in \omega)$ is inductively defined by the following clauses (i) and (ii):

(i) $\tilde{\mathbf{P}}_0 = \mathbf{P}$.

(ii) For each $n \in \omega$,

$$\begin{aligned}&\tilde{\mathbf{P}}_{n+1} = \\ &\iota^{-1}[\{ X \in \wp_{\rm cl}(A \times id_{1/2}(\mathbf{P}))\mid \\ &\quad X \subseteq A \times \tilde{\mathbf{P}}_n \wedge \\ &\quad \forall n \in \omega[\ \tilde{\pi}_n[X] \text{ is finite }]\}],\end{aligned} \tag{5}$$

where $\tilde{\pi}_n$ is the nth projection on $A \times id_{1/2}(\mathbf{P})$ (for the definition of the projections on $A \times id_{1/2}(\mathbf{P})$, see Definition 4 in Sect. 2). We call elements of $\tilde{\mathbf{P}}$ *hereditarily precompact processes* (see Theorem 2 in Sect. 3, for this terminology).

For a similar characterization of the space of compact sets in the linear-time context, cf. [12, Theorem 3.18].

2 Preliminaries

To prove the characterization result described in Sect. 1, we need a few preliminaries in notation and in metric topology.

For a set X, the powerset of X is denoted by $\wp(X)$. For a function $f : X \to Y$ and $X' \in \wp(X)$, we denote by $f[X']$ the *image* of X under f. We use the standard λ-notation $(\lambda x \in X.\ E(x))$ to denote the mapping which maps $x \in X$ to $E(x)$. We sometimes write $(E_x)_{x \in X}$ or $(E_x \mid x \in X)$ for $(\lambda x \in X.\ E(x))$. For two sets X and Y, the function space from X to Y is denoted by $(X \to Y)$. The set of natural numbers $0, 1, \ldots$ is denoted by ω.

The notions of *isometry, closed set, compact set, complete metric space,* and *Cauchy sequence* are assumed to be known (the reader might consult [2, 11, 14] for these notions).

We use the following operations on complete metric spaces.

Definition 1 Let $\langle M, d\rangle$, $\langle M_1, d_1\rangle$, $\langle M_2, d_2\rangle$ be complete metric spaces.

(1) An arbitrary set A can be supplied with a metric d_A, called the *discrete metric*, defined by

$$d_A(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

(2) We define a metric $d_{\rm P}$ on the Cartesian product $M_1 \times M_2$ as follows: For $\langle x_1, x_2\rangle, \langle y_1, y_2\rangle \in M_1 \times M_2$,

$$\begin{aligned}&d_{\rm P}(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) \\ &= \max_{j \in \{1,2\}}[\, d_j(x_j, y_j)\,].\end{aligned}$$

(3) We define a metric $d_{\rm H}$ on $\wp_{\rm cl}(M)$, called the *Hausdorff distance*, as follows: For every $X, Y \in \wp_{\rm cl}(M)$,

$$d_H(X,Y) = \max\{\sup_{x\in X}[\underline{d}(x,Y)], \sup_{y\in Y}[\underline{d}(y,X)]\},$$

where $\underline{d}(x,Z) = \inf_{z\in Z}[d(x,z)]$ for every $x \in M$ and $Z \subseteq M$ (we use the convention that $\sup\emptyset = 0$ and $\inf\emptyset = 1$).[2]

The space $\wp_{co}(M)$ is supplied with a metric by taking the restriction of d_H to it.[3]

(4) For a real number $\varepsilon \in [0,1]$, we define

$$id_\varepsilon(\langle M,d\rangle) = \langle M,d'\rangle,$$

where $\forall x,y \in M[\ d'(x,y) = \varepsilon \cdot d(x,y)\ ]$. ▮

Definition 2 (Projections) Let $\langle X,d\rangle$ a be cms. A family $(\pi_n)_{n\in\omega} \in (\omega \to (X \to X))$ is said to be a *projection family* on X iff

$$\begin{cases} \text{(i) } \forall n,m \in \omega[\ n \le m \Rightarrow \pi_n \circ \pi_m = \pi_n\ ], \\ \text{(ii) } \forall x_1,x_2 \in X[\ \pi_0(x_1) = \pi_0(x_2)\ ], \end{cases} \quad (6)$$

and the following holds for every $x_1,x_2 \in X$:

$$d(x_1,x_2) = \inf\{(1/2)^n \mid n \in \omega \wedge \pi_n(x_1) = \pi_n(x_2)\}. \quad (7)$$

A cms $\langle X,d\rangle$ is said to *have a projection family* iff there exists a projection family on X (see [6] for a related concept of a *projection space*).[4] ▮

Definition 3 (Finite Characterization) Let X be a cms having a projection family $(\pi_n)_{n\in\omega}$, and $Y \subseteq X$. We say Y is a *finitely characterized subset* of X iff

$$\exists k \in \omega, \exists Y' \in \wp(X), \forall x \in X[\ x \in Y \Leftrightarrow \pi_k(x) \in Y'\ ]. \quad (8)$$ ▮

[2]The fact that $\langle\wp_{cl}(M), d_H\rangle$ is a cms was first proved by Hahn. An accessible proof of it has been given in [5, Appendix A] with its minor errata in [4, pages 79–80].

[3]The fact $\wp_{co}(M)$ is a closed subset of $\wp_{cl}(M)$ can be shown by using Proposition 2 below. Thus, the completeness of $\wp_{co}(M)$ follows from the completeness of $\wp_{cl}(M)$.

[4]Unlike in [6], we do not demand that

$$\forall n,m \in \omega[\ m \le n \Rightarrow \pi_n \circ \pi_m = \pi_m\ ].$$

Actually, the projection family $(\pi_n)_{n\in\omega}$ introduced in the proof of Lemma 2 does not satisfy this condition.

Lemma 1 *Let X be a cms having a projection family, and Y the intersection of finitely characterized subsets of X. Then, Y is closed.* ▮

Proof. See [10, Lemma 2]. ■

Lemma 2 *The cms* $\mathbf{P}$ *has a projection family.* ▮

Proof. Let us fix an arbitrary element $\hat{p}$ of $\mathbf{P}$. We inductively define

$$(\pi_n)_{n\in\omega} \in (\omega \to (\mathbf{P} \to \mathbf{P}))$$

as follows:

(i) $\forall p \in \mathbf{P}[\ \pi_0(p) = \hat{p}\ ]$.

(ii) $\forall n \in \omega, \forall p \in \mathbf{P}[\ \pi_{n+1}(p) = \iota^{-1}[\{(a, \pi_n(p')) \mid \langle a,p'\rangle \in p\}]\ ]$.

We can check, by induction, that (6)(i) holds. Condition (6)(ii) clearly holds. Condition (7) follows from the following two propositions (9) and (10).

$$\forall n \in \omega, \forall p_1,p_2 \in \mathbf{P}[\ \pi_n(p_1) = \pi_n(p_2) \Rightarrow d(p_1,d_2) \le (1/2)^n\ ]. \quad (9)$$

$$\forall n \in \omega, \forall p_1,p_2 \in \mathbf{P}[\ \pi_{n+1}(p_1) \ne \pi_{n+1}(p_2) \Rightarrow d(p_1,d_2) \ge (1/2)^n\ ]. \quad (10)$$

Propositions (9) and (10) can be proved by induction (see [8, Lemma 2.1] for their proofs). ■

From $(\pi_n)_{n\in\omega}$, we define projection families $(\tilde{\pi}_n)_{n\in\omega}$ on $A \times id_{1/2}(\mathbf{P})$ and $(\tilde{\tilde{\pi}}_n)_{n\in\omega}$ on $\wp_{cl}(A \times id_{1/2}(\mathbf{P}))$ by:

Definition 4 **(1)** We first fix an arbitrary element $\hat{a}$ of A.

(i) For each $\langle a,p\rangle \in A \times \mathbf{P}$,

$$\tilde{\pi}_0(\langle a,p\rangle) = \langle\hat{a}, \pi_0(p)\rangle.$$

(ii) For $n \in \omega$ and $\langle a,p\rangle \in A \times \mathbf{P}$,

$$\tilde{\pi}_{n+1}(\langle a,p\rangle) = \langle a, \pi_n(p)\rangle.$$

(2) For each $X \in \wp_{cl}(A \times id_{1/2}(\mathbf{P}))$ and $n \in \omega$,

$$\tilde{\tilde{\pi}}_n(X) = \tilde{\pi}_n[X].$$ ▮

It is easy to check that $(\tilde{\pi}_n)_{n\in\omega}$ (resp. $(\tilde{\tilde{\pi}}_n)_{n\in\omega}$) is a projection family on $A \times id_{1/2}(\mathbf{P})$ (resp. on $\wp_{\rm cl}(A \times id_{1/2}(\mathbf{P}))$).

A topological space X is said to be *sequentially compact* iff every sequence of elements of X has a converging subsequence (see [11, Chap. 5]). It is well known that compactness and sequential compactness coincide for metric spaces (see, e.g., [2, Sect. 11.3], for the proof):

Proposition 1 *A metric space* $\langle X, d\rangle$ *is* compact *iff it is* sequentially compact. ∎

The concept of *precompactness* defined next is used to formulate another characterization of compactness.

Definition 5 A metric space $\langle X, d\rangle$ is said to be *precompact* (or *totally bounded*) iff for every $\epsilon > 0$, there exists finite family $\mathcal{U} \subseteq \wp(X)$ such that $\bigcup \mathcal{U} = X$ and

$$\forall U \in \mathcal{U}[\ \sup\{d(x,y) \mid x, y \in U\} \leq \epsilon\ ]. ∎$$

The next proposition characterizes compactness in terms of precompactness (see, e.g., [11, Theorem 5.32] for the proof):

Proposition 2 *A metric space* $\langle X, d\rangle$ *is compact iff it is complete and precompact.* ∎

We remark that the characterization result described in Sect. 1 is analogous to Prop. 2 (it might be suggestive to say that the characterization result is a *recursive version* of Prop. 2, in the setting of branching-time process domains).

3 Characterization

In this section, we prove the characterization result described in Sect. 1.

The next lemma, which is a generalization of Theorem 3.18 of [12], gives a characterization of the space of compact subsets.

Lemma 3 *Let* $\langle X, d\rangle$ *be a cms having a projection family* $(\pi_n)_{n\in\omega}$, *and* $Y \in \wp(X)$. *Then*

$$\begin{aligned} &Y \textit{ is compact} \\ &\Leftrightarrow Y \textit{ is closed} \\ &\quad \wedge \forall k \in \omega[\ \pi_k[Y] \textit{ is finite}\ ]. ∎ \end{aligned} \tag{11}$$

Proof. Since X is a cms, it immediately follows that Y (equipped with the relative topology) is complete iff Y is closed. It is easy to check that Y is precompact iff

$$\forall k \in \omega[\ \pi_k[Y] \text{ is finite}\ ].$$

Thus, (11) follows from Prop. 2. ∎

The next proposition is standard in general topology (see, e.g., [2, Sect. 3.7], for the proof).

Proposition 3 *Let* X *be a topological space, and* Y *a closed subset of* X. *Then*

$$\wp_{\rm cl}(Y) = \wp_{\rm cl}(X) \cap \wp(Y), \tag{12}$$

where Y *in the left-hand side is taken to be equipped with the relative topology.* ∎

From Lemma 3, we obtain the next lemma.

Lemma 4 (1) *Let* $\langle X, d\rangle$ *be a cms having a projection family* $(\pi_n)_{n\in\omega}$. *Then the set*

$$\begin{aligned} &\{Y \in \wp_{\rm cl}(X) \mid \\ &\quad \forall n \in \omega[\ \pi_n[Y] \textit{ is finite}\ ]\} \end{aligned} \tag{13}$$

is a closed subset of the cms $\wp_{\rm cl}(X)$.

(2) *For any closed subset* $\mathbf{P}'$ *of* $\mathbf{P}$, *the set*

$$\begin{aligned} &\{X \in \wp_{\rm cl}(A \times id_{1/2}(\mathbf{P})) \mid \\ &\quad X \subseteq A \times \mathbf{P}' \wedge \\ &\quad \forall n \in \omega[\ \tilde{\pi}_n[X] \textit{ is finite}\ ]\} \end{aligned} \tag{14}$$

is a closed subset of $\wp_{\rm cl}(A \times id_{1/2}(\mathbf{P}))$. ∎

Proof. Part (1) follows from Lemma 1, and Part (2) follows from Part (1). ∎

From Lemmas 3 and 4, we obtain the next theorem, which gives a characterization of the de Bakker-Zucker process domain of compact sets.

Theorem 1 *Let* $\tilde{\mathbf{P}}$ *be defined as in Sect. 1. Then,* $\tilde{\mathbf{P}}$ *is the solution of domain equation (2). That is,*

$$\tilde{\mathbf{P}} \cong \wp_{\rm co}(A \times id_{1/2}(\tilde{\mathbf{P}})). ∎ \tag{15}$$

Proof. Remember that the solution $\mathbf{P}$ of domain equation (1) satisfies (3), with ι being an isometry (see Sect. 1).

For $X \in \wp(A \times \mathbf{P})$, let $\Phi(X)$ denote that

$$\forall n \in \omega[\ \tilde{\pi}_n[X] \text{ is finite}\ ], \tag{16}$$

where $(\tilde{\pi}_n)_{n\in\omega}$ is the projection family on $A \times id_{1/2}(\mathbf{P})$. For $\mathbf{P}' \subseteq \mathbf{P}$, we put

$$\wp^*_{\mathrm{cl}}(A \times id_{1/2}(\mathbf{P}')) \\ = \{X \in \wp^*_{\mathrm{cl}}(A \times id_{1/2}(\mathbf{P}'))\mid \Phi(X)\}. \tag{17}$$

For closed $\mathbf{P}' \subseteq \mathbf{P}$, we have

$$\wp^*_{\mathrm{cl}}(A \times id_{1/2}(\mathbf{P}')) = \wp_{\mathrm{co}}(A \times id_{1/2}(\mathbf{P}')) \tag{18}$$

by Lemma 3.

By Lemma 4(2), $\tilde{\mathbf{P}}_n$ is a closed subset of $\mathbf{P}$, for every $n \in \omega$. Therefore, $\tilde{\mathbf{P}} = \bigcap_n[\tilde{\mathbf{P}}_n]$ is a closed subset of $\mathbf{P}$. Thus, we have

$$\wp^*_{\mathrm{cl}}(A \times id_{1/2}(\tilde{\mathbf{P}})) = \wp_{\mathrm{co}}(A \times id_{1/2}(\tilde{\mathbf{P}})). \tag{19}$$

By Prop. 3, we have

$$\begin{aligned}
&\wp_{\mathrm{cl}}(A \times id_{1/2}(\tilde{\mathbf{P}})) \\
&= \wp_{\mathrm{cl}}(A \times id_{1/2}(\mathbf{P})) \cap \wp(A \times \tilde{\mathbf{P}}) \quad \text{(by (12))} \\
&= \wp_{\mathrm{cl}}(A \times id_{1/2}(\mathbf{P})) \cap \bigcap_{n\in\omega}[\,\wp(A \times \tilde{\mathbf{P}}_n)\,] \\
&\qquad \text{(by (4))} \\
&= \bigcap_{n\in\omega}[\,\wp_{\mathrm{cl}}(A \times id_{1/2}(\mathbf{P})) \cap \wp(A \times \tilde{\mathbf{P}}_n)\,] \\
&= \bigcap_{n\in\omega}[\,\wp_{\mathrm{cl}}(A \times id_{1/2}(\tilde{\mathbf{P}}_n))\,] \quad \text{(by (12))}.
\end{aligned}$$

Thus,

$$\wp_{\mathrm{cl}}(A \times id_{1/2}(\tilde{\mathbf{P}})) \\ = \bigcap_{n\in\omega}[\,\wp_{\mathrm{cl}}(A \times id_{1/2}(\tilde{\mathbf{P}}_n))\,]. \tag{20}$$

Furthermore,

$$\begin{aligned}
&\wp_{\mathrm{co}}(A \times id_{1/2}(\tilde{\mathbf{P}})) \\
&= \wp^*_{\mathrm{cl}}(A \times id_{1/2}(\tilde{\mathbf{P}})) \quad \text{(by (18))} \\
&= \{X \in \wp_{\mathrm{cl}}(A \times id_{1/2}(\tilde{\mathbf{P}}))\mid \Phi(X)\} \\
&= \{X \in \bigcap_{n\in\omega}[\,\wp_{\mathrm{cl}}(A \times id_{1/2}(\tilde{\mathbf{P}}_n))\,]\mid \Phi(X)\} \\
&\qquad \text{(by (20))} \\
&= \bigcap_{n\in\omega}[\,\{X \in \wp_{\mathrm{cl}}(A \times id_{1/2}(\tilde{\mathbf{P}}_n))\mid \Phi(X)\}\,] \\
&= \bigcap_{n\in\omega}[\,\{X \in \wp_{\mathrm{cl}}(A \times id_{1/2}(\mathbf{P}))\mid \\
&\qquad X \subseteq A \times \tilde{\mathbf{P}}_n \wedge \Phi(X)\}\,] \\
&\qquad \text{(by (12))} \\
&= \bigcap_{n\in\omega}[\,\iota[\tilde{\mathbf{P}}_{n+1}]\,] \quad \text{(by (5))} \\
&= \iota[\bigcap_{n\in\omega}[\,\tilde{\mathbf{P}}_{n+1}\,]] \quad \text{(since } \iota \text{ is 1-1)} \\
&= \iota[\bigcap_{n\geq 1}[\,\tilde{\mathbf{P}}_n\,]] = \iota[\tilde{\mathbf{P}}] \quad \text{(by (4))}.
\end{aligned}$$

Thus,

$$\wp_{\mathrm{co}}(A \times id_{1/2}(\tilde{\mathbf{P}})) = \iota[\tilde{\mathbf{P}}]. \tag{21}$$

Consequently, by putting $\tilde{\iota} = \iota\restriction\tilde{\mathbf{P}}$, we obtain the desired consequence that

$$\tilde{\mathbf{P}} \overset{\tilde{\iota}}{\cong} \wp_{\mathrm{co}}(A \times id_{1/2}(\tilde{\mathbf{P}})). \blacksquare \tag{22}$$

Remark 1 When A is infinite, $\tilde{P}_{n+1}$ is a proper subset of $\tilde{P}_n$, for each $n \in \omega$. For example, if $A = \{a_i\mid i \in \omega\}$ with $a_i \neq a_j$ for $i \neq j$, then we can construct a process p such that $p \in \tilde{\mathbf{P}}_1$ but $p \notin \tilde{\mathbf{P}}_2$ as follows: First, put

$$p' = \iota^{-1}(\{\langle a_i, \hat{p}\rangle\mid i \in \omega\}),$$

where $\hat{p}$ is an arbitrary element of $\mathbf{P}$. And put $p = \iota^{-1}(\{\langle a, p'\rangle\})$. Clearly, $p \in \tilde{\mathbf{P}}_1$. But $p \notin \tilde{\mathbf{P}}_2$, since $p' \notin \tilde{\mathbf{P}}_1$.

We remark that $\mathbf{P}$ and $\tilde{\mathbf{P}}$ coincide, when A is finite. ∎

Remark 2 For every $p \in \tilde{\mathbf{P}}$, $\iota(p)$ is compact by Theorem 1. However $\tilde{\mathbf{P}}$ itself is *not* compact, when A is infinite. ∎

Next, we give an alternative characterization of the BZ process domain of compact sets. This characterization is given in terms of *hereditary pointwise precompactness*, which we defined as follows:

Definition 6 A subset $\mathbf{P}'$ of $\mathbf{P}$ is said to be a *hereditarily pointwise precompact subset* of $\mathbf{P}$ iff the following three conditions (i)–(iii) hold:

(i) $\forall p \in \mathbf{P}', \forall k \in \omega[\ \tilde{\pi}_k[\iota(p)]$ is finite $]$.

(ii) $\forall p \in \mathbf{P}', \forall a \in A, \forall p' \in \mathbf{P}[$
$\iota(p) \ni \langle a, p'\rangle \Rightarrow p' \in \mathbf{P}'\ ]$.

(iii) $\mathbf{P}'$ is a closed subset of $\mathbf{P}$. ∎

The next theorem characterizes $\tilde{\mathbf{P}}$ as the largest hereditarily pointwise precompact subset.

Theorem 2 *$\tilde{\mathbf{P}}$ is the largest of all hereditarily pointwise precompact subsets of* $\mathbf{P}$. ∎

Proof. First, we show that $\tilde{\mathbf{P}}$ is a hereditarily pointwise precompact subset of $\mathbf{P}$. It suffices to show that $\tilde{\mathbf{P}}$ satisfies conditions (i)–(iii) of Definition 6. The fact that $\tilde{\mathbf{P}}$ satisfies condition (i) follows from the fact that $\tilde{\mathbf{P}} \subseteq \tilde{\mathbf{P}}_1$ and the definition of $\tilde{\mathbf{P}}_1$ (see (5)). By (15), $\tilde{\mathbf{P}}$ satisfies condition (ii). As stated in the proof of Theorem 1, $\tilde{\mathbf{P}}$ is a closed subset of $\mathbf{P}$, i.e., it satisfies condition (iii). Thus, $\tilde{\mathbf{P}}$ satisfies conditions (i)–(iii), and therefore, it is hereditarily pointwise precompact subset of $\mathbf{P}$.

Next, let $\mathbf{P}'$ be an arbitrary hereditarily pointwise precompact subset of $\mathbf{P}$. Then, we can show, by induction, that

$$\forall n \in \omega[\, \mathbf{P}' \subseteq \tilde{\mathbf{P}}_n \,].$$

Hence we have

$$\mathbf{P}' \subseteq \bigcap_{n\in\omega} [\, \tilde{\mathbf{P}}_n \,] = \tilde{\mathbf{P}}.$$

Thus, we obtain the claim of the theorem. ■

4 Concluding Remarks

We conclude this paper with a few remarks about future work.

In this paper, we give a characterization of the solution of a domain equation $X \cong \wp_{co}(A \times id_{1/2}(X))$ as a subdomain of the solution of $X \cong \wp_{cl}(A \times id_{1/2}(X))$. Here we note that the operator $\wp_{co}(A \times id_{1/2}(\cdot))$ is obtained from $\wp_{cl}(A \times id_{1/2}(\cdot))$ by replacing $\wp_{cl}$ by $\wp_{co}$. It might be possible to generalize this result so that for a class of operators F, the solution of $X = F'(X)$ is characterized as a subdomain of the the solution of $X = F(X)$, where F' is obtained from F by replacing $\wp_{cl}$ by $\wp_{co}$.

In the proof of the characterization result, Theorem 1, we conveniently use the property that the base space $\mathbf{P}$ has a projection family. It remains for future research to clarify the condition for a cms to have a projection family.

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