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分母分離形３次元状態空間ディジタルフィルタの直接的設計法

A DIRECT DESIGN METHOD OF SEPARABLE DENOMINATOR 3-D STATE-SPACE DIGITAL FILTERS

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I. INTRODUCTION

The problem of designing digital filters can be divided into two steps as approximation and synthesis. Since these two interrelated steps have been studied separately until now, many redundant computation steps are involved in the traditional design procedure. To design digital filters more efficiently, the authors have proposed a direct design method(1). The basic idea of the direct design method is based on a "balanced approximation method" proposed by Kung(2) and an equivalent relation between balanced realizations and optimal realizations (minimum roundoff noise realizations) of digital filters(1). This method can perform the approximation and synthesis simultaneously with much less computational complexity. Resulting filters of this direct design method can approximate given impulse responses closely. In addition, they are always guaranteed to be stable, nearly optimal and free of overflow oscillations. Moreover, in Ref. (3), the authors have shown that this direct design method can also be extended to design CRSD (causal, recursive and separable in denominator) 2-D digital filters.

Over the recent years, much research work has been carried out in the area of multi-dimensional digital filters. Although the main research effort has focused on 2-D digital filters, the study on 3-D digital filters is being expected in many areas such as moving picture or photo processing, seismic or geophysical data processing, and so on. In this paper, we will extend the direct design method to design CRSD 3-D digital filters.

This paper is arranged as follows. Sec. 2 gives some preliminaries about CRSD 3-D digital filters. Sec. 3 and Sec. 4 propose, respectively, a balanced approximation method and a synthesis method of optimal realizations of CRSD 3-D digital filters. Since the basic ideas are direct extensions of those used by Lashgari, et al.(4) and Kawamata and Higuchi(5), the main results will be given without detailed derivation. Sec. 4 is concluded by showing the absence of overflow oscillations in the optimal realizations of CRSD 3-D digital filters. In Sec. 5, we first reveal the equivalent relation between balanced realizations and optimal realizations of CRSD 3-D digital filters, and then propose the direct design method on the basis of this relation and the balanced approximation method given in Sec. 3. At the end of this section, a numerical example is given to show the validity of the direct design method. Finally, Sec. 6 gives some concluding remarks.

In this paper, the following partial ordering will be used for integer 3-tuples:

$$\begin{aligned} &(i,j,k) \le (p,q,r) && \text{iff } i\le q,\ j\le q \text{ and } k\le r \\ &(i,j,k) = (p,q,r) && \text{iff } i=p,\ j=q \text{ and } k=r \\ &(i,j,k) < (p,q,r) && \text{iff } (i,j,k)\le(p,q,r) \\ &&& \quad\text{and } (i,j,k)\ne(p,q,r). \end{aligned} \tag{1}$$

II. STATE-SPACE REPRESENTATION OF CRSD 3-D DIGITAL FILTERS

In Refs. (6) and (7), some basic concepts about 3-D digital filters have been studied. Due to limited space of this paper, these concepts will be adopted without explanation. The filter to be studied in this paper is of the following form:

$$x'(i,j,k) = A\,x(i,j,k) + B\,u(i,j,k) \tag{2.1}$$

$$y(i,j,k) = C\,x(i,j,k) + d\,u(i,j,k) \tag{2.2}$$

where

$$x(i,j,k) = \begin{pmatrix} x^h(i,j,k) \\ x^v(i,j,k) \\ x^d(i,j,k) \end{pmatrix}, \quad x'(i,j,k) = \begin{pmatrix} x^h(i+1,j,k) \\ x^v(i,j+1,k) \\ x^d(i,j,k+1) \end{pmatrix} \tag{2.3}$$

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad C = (c_1,c_2,c_3). \tag{2.4}$$

and where $x^h(i,j,k)$, $x^v(i,j,k)$ and $x^d(i,j,k)$ are the ℓ-th order horizontal state vector, the m-th order vertical state vector and the n-th order depth state vector, respectively, u(i,j,k) and y(i,j,k) are the scalar input and the scalar output, respectively, and A_{11}-A_{33}, b_1-b_3, and c_1-c_3 are coefficient matrices with appropriate dimensions. Further, we restrict ourselves to filters whose impulse responses satisfy the following causality condition:

$$h_{i,j,k} = 0, \quad \text{for } i < 0 \text{ or } j < 0 \text{ or } k < 0. \tag{3}$$

The transfer function of this kind of filter takes the following form:

$$H(z_1,z_2,z_3) = \frac{N(z_1,z_2,z_3)}{D_1(z_1)D_2(z_2)D_3(z_3)} \tag{4.1}$$

under the initial conditions

$$x^h(0,j,k) = 0,\ x^v(i,0,k) = 0 \text{ and } x^d(i,j,0) = 0 \quad \text{for } (i,j,k) \geq (0,0,0). \tag{4.2}$$

Thus, we call it a causal, recursive and separable denominator (CRSD) digital filter, and denote it as CRSD(A,B,C,d) for simplicity. The block diagram is shown in Fig. 1.

Calculating the impulse response of CRSD(A, B,C,d) explicitly under the initial condition (3), we have

$$h_{i,j,k} = \begin{cases} c_1A_{11}^{i-1}b_1 & \text{for } i>0,\ j=0 \text{ and } k=0 \\ c_2A_{22}^{j-1}b_2 & \text{for } i=0,\ j>0 \text{ and } k=0 \\ c_3A_{33}^{k-1}b_3 & \text{for } i=0,\ j=0 \text{ and } k>0 \\ c_3A_{33}^{k-1}A_{32}A_{22}^{j-1}b_2 & \text{for } i=0,\ j>0 \text{ and } k>0 \\ c_3A_{33}^{k-1}A_{31}A_{11}^{i-1}b_1 & \text{for } i>0,\ j=0 \text{ and } k>0 \\ c_2A_{22}^{j-1}A_{21}A_{11}^{i-1}b_1 & \text{for } i>0,\ j>0 \text{ and } k=0 \\ c_3A_{33}^{k-1}A_{32}A_{22}^{j-1}A_{21}A_{11}^{i-1}b_1 & \\ & \text{for } i>0,\ j>0 \text{ and } k>0. \end{cases} \tag{5}$$

There are an infinite number of realizations with the same transfer function or impulse response. Specifically, if CRSD(A,B,C,d) is a realization of $H(z_1,z_2,z_3)$, then, by any equivalent transformation $x(i,j,k)=T^{-1}x(i,j,k)$, we can get a new realization $\text{CRSD}(T^{-1}AT,T^{-1}B,CT,d)$, where

$$T = T_1 \oplus T_2 \oplus T_3 \tag{6}$$

and T_1, T_2 and T_3 are $\ell \times \ell$, m x m and n x n nonsingular matrices, respectively, and $\oplus$ expresses the direct sum of matrices.

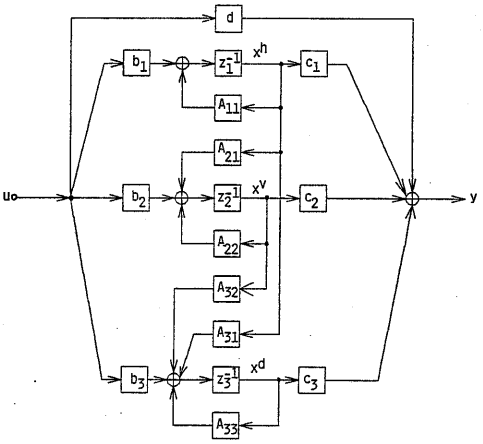


Fig. 1 Block diagram of separable denominator 3-D digital filter

III. APPROXIMATION OF CRSD 3-D DIGITAL FILTERS

Similar to 2-D digital filters, the controllability matrix G and the observability matrix O of a 3-D digital filter are defined as follows:

$$G = [g(1,0,0),g(2,0,0),\ldots,g(L,0,0),g(0,1,0), \ldots,g(L,1,0),\ldots,g(L,M,0),g(0,0,1), g(1,0,1),\ldots,g(L,M,N)], \tag{7}$$

$$O^t = [f(0,0,0),f(0,0,1),\ldots,f(0,0,N),f(0,1,0), f(0,1,1),\ldots,f(0,M,N),f(1,0,0),f(1,0,1), \ldots,f(L,M,N)] \tag{8}$$

where L, M and N are integers, and

$$g(i,j,k)= A^{i-1,j,k}B^{1,0,0} + A^{i,j-1,k}B^{0,1,0} + A^{i,j,k-1}B^{0,0,1} \tag{9.1}$$

$$f(i,j,k) = (CA^{i,j,k})^t \tag{9.2}$$

and where, $A^{i,j,k}$, $B^{1,0,0}$, etc. are defined in a similar way as $A^{i,j}$ and $B^{1,0}$, etc. of 2-D digital filters(6),(7).

For CRSD 3-D digital filters, computing these two matrices explicitly, we can get

$$G = \left(\begin{array}{ccc} G_1 & 0 & 0 \\ \hdashline 0 & G_2 & 0 \\ \hdashline 0 & 0 & G_3 \end{array}\right) \begin{array}{l} \}\ \ell \\ \}\ m \\ \}\ n \end{array} \tag{10}$$

where

$$G_1 = [\bar{b}_1,\ A_{11}\bar{b}_1,\ldots,\ A_{11}^{L-1}\bar{b}_1] \tag{11.1}$$

$$\bar{b}_1 = b_1 \tag{11.2}$$

$$G_2 = [\bar{b}_2,\ A_{22}\bar{b}_2,\ldots,\ A_{22}^{M-1}\bar{b}_2] \tag{11.3}$$

$$\bar{b}_2 = [b_2,\ A_{21}G_1] \tag{11.4}$$

$$G_3 = [\bar{b}_3, A_{33}\bar{b}_3, \ldots, A_{33}^{N-1}\bar{b}_3] \qquad (11.5)$$

$$\bar{b}_3 = [b_3, A_{31}G_1, A_{32}G_2], \qquad (11.6)$$

and

$$O = \left(\begin{array}{c:c:c} & & O_3(N+1) \\ & O_2(M+1) & 0 \\ O_1(L+1) & 0 & 0 \end{array}\right) \qquad (12)$$
$$\quad \underbrace{\qquad}_{\ell} \quad \underbrace{\qquad}_{m} \quad \underbrace{\qquad}_{n}$$

where

$$O_1 = \begin{pmatrix} \bar{c}_1 \\ \bar{c}_1A_{11} \\ \cdots \\ \bar{c}_1A_{11}^{L-1} \end{pmatrix}, \qquad \bar{c}_1 = \begin{pmatrix} c_1 \\ O_3A_{31} \\ O_2A_{21} \end{pmatrix} \qquad (13.1)$$

$$O_2 = \begin{pmatrix} \bar{c}_2 \\ \bar{c}_2A_{22} \\ \cdots \\ \bar{c}_2A_{22}^{M-1} \end{pmatrix}, \qquad \bar{c}_2 = \begin{pmatrix} c_2 \\ O_3A_{32} \end{pmatrix} \qquad (13.2)$$

$$O_3 = \begin{pmatrix} \bar{c}_3 \\ \bar{c}_3A_{33} \\ \cdots \\ \bar{c}_3A_{33}^{N-1} \end{pmatrix}, \qquad \bar{c}_3 = c_3 \qquad (13.3)$$

and where $O_1(L+1)$, $O_2(M+1)$ and $O_3(N+1)$ are obtained from O_1, O_2 and O_3 by substituting L, M and N with L+1, M+1 and N+1, respectively.

Now let us define three 1-D digital filters $DF(A_{11},\bar{b}_1,\bar{c}_1)$, $DF(A_{22},\bar{b}_2,\bar{c}_2)$ and $DF(A_{33},\bar{b}_3,\bar{c}_3)$, with controllability and observability matrices given by (O_1,G_1), (O_2,G_2) and (O_3,G_3), respectively. Then, the controllability and the observability of a CRSD 3-D digital filter are completely determined by these 1-D filters. In this paper, we will call these 1-D digital filters the characteristic filters of the corresponding CRSD 3-D digital filter.

From the definition of characteristic filters, we can see that the coefficient matrices A_{11}, A_{22} and A_{33} of a CRSD 3-D digital filter are the same as the transition matrices of the corresponding characteristic filters. Further, from Eqs.(11) and (13), we can easily get the following relations between other coefficient matrices:

$$b_i = \text{First column of } \bar{b}_i, \quad \text{for } i=1,2,3 \qquad (14.1)$$
$$c_i = \text{First row of } \bar{c}_i, \quad \text{for } i=1,2,3 \qquad (14.2)$$
$$A_{21} = \bar{b}_2^{\leftarrow 1} G_1^{+} = O_2^{+}\bar{c}_1^{\uparrow N+1} \qquad (14.3)$$
$$A_{31} = \bar{b}_3^{\leftarrow 1} G_1^{+} = O_3^{+}\bar{c}_1^{\uparrow 1} \qquad (14.4)$$
$$A_{32} = \bar{b}_3^{\leftarrow L+1} G_2^{+} = O_3^{+}\bar{c}_2^{\uparrow 1} \qquad (14.5)$$

where $(\bullet)^{+}$ expresses the pseudo-inverse of a matrix, and $(\bullet)^{\leftarrow i}$ ($(\bullet)^{\uparrow i}$) expresses the operation of shifting a matrix for i columns (rows) leftward (upward), and filling the right columns (bottom rows) with zeros.

Computing the impulse responses of the characteristic filters explicitly, we have

$$h_i^h = \bar{c}_1A_{11}^{i-1}\bar{b}_1 = [h_{i,0,0}, h_{i,0,1}, h_{i,0,2}, \ldots, h_{i,0,N}, h_{i,1,0}, h_{i,1,1}, \ldots, h_{i,1,N}, \ldots, h_{i,M,N}]^t \qquad (15.1)$$

$$h_j^v = \bar{c}_2A_{22}^{j-1}\bar{b}_2 = \begin{pmatrix} h_{0,j,0} & h_{1,j,0} & \cdots & h_{L,j,0} \\ h_{0,j,1} & h_{1,j,1} & \cdots & h_{L,j,1} \\ \cdots & \cdots & \cdots & \cdots \\ h_{0,j,1} & h_{1,j,1} & \cdots & h_{L,j,N} \end{pmatrix} \qquad (15.2)$$

and

$$h_k^d = \bar{c}_3A_{33}^{k-1}\bar{b}_3 = [h_{0,0,k}, h_{1,0,k}, \ldots, h_{L,0,k}, h_{0,1,k}, h_{1,1,k}, \ldots, h_{L,1,k}, h_{0,2,k}, \ldots, h_{N,2,k}, \ldots, h_{L,M,k}] \qquad (15.3)$$

where h_i^h, h_j^v and h_k^d are the impulse responses of $DF(A_{11},\bar{b}_1,\bar{c}_1)$, $DF(A_{22},\bar{b}_2,\bar{c}_2)$ and $DF(A_{33},\bar{b}_3,\bar{c}_3)$, respectively.

Thus, if the desired 3-D impulse response $h_{i,j,k}$ is given, then the desired impulse responses of the characteristic filters can be obtained using Eq. (15). From these impulse responses, we can find the characteristic filters by any time-domain approximation method of 1-D digital filters, and then the desired CRSD 3-D digital filter can be immediately obtained using (14). Adopting Kung's method for finding the characteristic filters, we have the following algorithm for approximating CRSD digital filters.

Algorithm 1:

Suppose that the specification is given by $h_{i,j,k}$, $(0,0,0)\le(i,j,k)\le(L,M,N)$, then the desired filter is obtained by the following steps.

Step 1: Form the impulse responses h_i^h (i = 0,1, of the characteristic filters using (15).

Step 2: Form the Hankel matrices from h_i^h, h_j^v and h_k^d as follows:

$$h = \begin{pmatrix} h_1^h & h_2^h & \cdots & h_L^h \\ h_2^h & h_3^h & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ h_L^h & 0 & \cdots & 0 \end{pmatrix} \qquad (16.1)$$

$$v = \begin{pmatrix} h_1^v & h_2^v & \cdots & h_M^v \\ h_2^v & h_3^v & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ h_M^v & 0 & \cdots & 0 \end{pmatrix} \qquad (16.2)$$

$$d = \begin{pmatrix} h_1^d & h_2^d & \cdots & h_N^d \\ h_2^d & h_3^d & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ h_N^d & 0 & \cdots & 0 \end{pmatrix} \qquad (16.3)$$

Step 3: Find the characteristic filters using Kung's method. For illustration, we will only show how DF(A_{11},$\overline{b}_1$,$\overline{c}_1$) is found.

The singular value decomposition (SVD) of Φ^h is as follows:

$$\Phi^h = U_{11}\ \Sigma_{11} V_{11}^t + U_{12}\ \Sigma_{12} V_{12}^t \tag{17}$$

where U_{11}, U_{12}, V_{11} and V_{12} are orthogonal matrices, and

$$\Sigma_{11} = (\ \alpha_1,\ \alpha_2, \ldots,\ \alpha_\ell) \tag{18.1}$$
$$\Sigma_{12} = (\ \alpha_{\ell+1},\ \alpha_{\ell+2}, \ldots) \tag{18.2}$$

and where α_i's are the singular values of the Φ^h, and satisfy

$$\alpha_1 \geq \alpha_2 \geq \ldots, \quad \text{and} \quad \alpha_\ell >> \alpha_{\ell+1} \quad \text{for some } \ell < L. \tag{19}$$

Then, the coefficient matrices of DF(A_{11},$\overline{b}_1$,$\overline{c}_1$) can be found by

$$A_{11} = (\ \Sigma_{11}^{-1/2} U_{11}^t)(U_{11}\ \Sigma_{11}^{1/2}) \tag{20.1}$$

$$\overline{b}_1 = \text{First column of } \Sigma_{11}^{1/2} V_{11}^t \tag{20.2}$$

$$\overline{c}_1 = \text{First (N+1) x (M+1) rows of } U_{11}\ \Sigma_{11}^{1/2} \tag{20.3}$$

Coefficient matrices of DF(A_{22},$\overline{b}_2$,$\overline{c}_2$) and DF(A_{33},$\overline{b}_3$,$\overline{c}_3$) can be found in a similar manner.

Step 4: Find the coefficient matrices of CRSD(A,B,C,d) by using Eq. (14), where

$$O_2^+ = \Sigma_{21}^{-1/2} U_{21}^t, \text{ and } \quad O_3^+ = \Sigma_{31}^{-1/2} U_{31}^t$$
$$G_1^+ = V_{11}\ \Sigma_{11}^{-1/2}, \quad \text{and} \quad G_2^+ = V_{21}\ \Sigma_{21}^{-1/2}. \tag{21}$$

and where U_{21}, V_{21} and Σ_{21}, and U_{31} and Σ_{31} are obtained from the SVD of Φ^v and Φ^d, respectively.

IV. SYNTHESIS OF CRSD 3-D DIGITAL FILTERS

In this section, we will analyze the roundoff noise in CRSD 3-D digital filters, and then give a method for synthesizing optimal realizations. Since the basic ideas are simple extensions of those used by Kawamata and Higuchi in Ref. (5), we will not discuss the problem in detail, but give the main results only.

4.1 Variance of Roundoff Noise in CRSD 3-D Digital Filters

Due to roundoff after multiplication, the actual CRSD 3-D digital filter implemented by a finite wordlength machine is described by

$$\tilde{x}'(i,j,k) = A\ \tilde{x}(i,j,k) + Bu(i,j,k) + \alpha(i,j,k)$$
$$\tilde{y}(i,j,k) = C\ \tilde{x}(i,j,k) + du(i,j,k) + \beta(i,j,k) \tag{22}$$

where $\tilde{x}(i,j,k)$, and $\tilde{y}(i,j,k)$ are the actual state vector and the actual output, respectively, $\alpha(i,j,k)$ and $\beta(i,j,k)$ are, respectively, error vectors generated due to roundoff after multiplication in (A,B) and (C,d).

Subtracting (2) from (22), we obtain the output error $\Delta y(i,j,k) = \tilde{y}(i,j,k) - y(i,j,k)$ as

$$e'(i,j,k) = A\ e(i,j,k) + \alpha(i,j,k) \tag{23.1}$$
$$\Delta y(i,j,k) = C\ e(i,j,k) + \beta(i,j,k) \tag{23.2}$$

where

$$e(i,j,k) = \tilde{x}(i,j,k) - x(i,j,k). \tag{23.3}$$

Assume that the product quantization errors are white noise and are statistically independent from source to source, and from point to point. Then we have

$$E\left(\begin{pmatrix}\alpha(i,j,k)\\ \beta(i,j,k)\end{pmatrix}\begin{pmatrix}\alpha(p,q,r)\\ \beta(p,q,r)\end{pmatrix}^t\right)$$
$$= \sigma^2 \delta_{i-p,j-q,l-r} \text{ block diag}(Q^h,Q^v,Q^d,q) \tag{24}$$

where E[•] is the expectation, $\sigma^2=2^{-2w}$ is the variance of each noise sources and w is the word-length, $\delta_{i,j,k}$ is the Kronecker delta, and Q^h is a diagonal matrix whose i-th diagonal element is the number of noninteger coefficients of the i-th rows of A_{11} and b_1, and in a similar way, the diagonal matrices Q^v and Q^d, and the scalar q are defined for (A_{21},A_{22},b_2), (A_{31},A_{32},A_{33},b_3) and (c_1,c_2,c_3,d), respectively. Under the above assumption, we can obtain the variance of roundoff noise by following a similar procedure used in Ref. (5). The result is given by

$$E[\Delta y^2] = \sigma^2 \text{tr}[Q^h W^h] + \sigma^2 \text{tr}[Q^v W^v] + \sigma^2 \text{tr}[Q^d W^d] + \sigma^2 q \tag{25}$$

where W^h, W^v and W^d are referred to as the horizontal, vertical and depth noise matrices (or observability gramians) of CRSD(A,B,C,d), respectively, and satisfy the following Lyapunov equations:

$$W^h = A_{11}^t W^h A_{11} + A_{21}^t W^v A_{21} + A_{31}^t W^d A_{31} + c_1^t c_1$$
$$W^v = A_{22}^t W^v A_{22} + A_{32}^t W^d A_{32} + c_2^t c_2$$
$$W^d = A_{33}^t W^d A_{33} + c_3^t c_3. \tag{26}$$

4.2 ℓ_2 Norm Scaling

It is desirable that, when a state-space digital filter is synthesized, no overflow occurs in this filter. To prevent the overflow problem, state variables must be scaled. In this paper, the l_2 norm scaling is adopted, and it is performed via equivalent transformation such that the following constraints are satisfied:

$$(K^h)_{ii} = 1, \quad (K^v)_{jj} = 1 \text{ and } (K^d)_{kk} = 1 \tag{27}$$

where $(\bullet)_{ii}$ is the i-th diagonal element of the matrix, and K^h, K^v and K^d are the horizontal, vertical and depth covariance matrices (or controllability gramians) of CRSD(A,B,C,d) which satisfy the following Lyapunov equations:

$$K^h = A_{11}K^hA_{11}^t + b_1b_1^t$$

$$K^v = A_{22}K^vA_{22}^t + A_{21}K^hA_{21}^t + b_2b_2^t$$

$$K^d = A_{33}K^dA_{33}^t + A_{31}K^hA_{31}^t + A_{32}K^vA_{32}^t + b_3b_3^t. \tag{28}$$

4.3 Synthesis of Optimal Realizations

From Eqs. (10)-(13), (26) and (28) we can easily get the following relations by using the properties of solutions of the Lyapunov equations:

$$K^h = G_1G_1^t, \quad K^v = G_2G_2^t, \quad K^d = G_3G_3^t \tag{29.1}$$

$$W^h = O_1^tO_1, \quad W^v = O_2^tO_2, \quad W^d = O_3^tO_3 \tag{29.2}$$

provided that L, M and N are sufficiently large. Thus, (K^h,W^h), (K^v,W^v), and (K^d,W^d) are the controllability gramians and observability gramians of the characteristic filters $DF(A_{11},\overline{b_1},\overline{c_1})$, $DF(A_{22},\overline{b_2},\overline{c_2})$ and $DF(A_{33},\overline{b_3},\overline{c_3})$, respectively.

From Eq. (25), we can easily see that the roundoff noise of CRSD(A,B,C,d) can be minimized by minimizing the first three terms independently. These three terms, in turn, can be minimized by optimizing the characteristic filters. Thus, applying Hwang's synthesis method(8) to the characteristic filters, we can get the optimal realization of CRSD(A,B,C,d). That is, the equivalent transformation $T = (T_1 \oplus T_2 \oplus T_3)$ required to minimize the roundoff noise of CRSD(A,B,C,d) can be obtained by finding T_1, T_2 and T_3 separately by applying Hwang's method to $DF(A_{11},\overline{b_1},\overline{c_1})$, $DF(A_{22},\overline{b_2},\overline{c_2})$ and $DF(A_{33},\overline{b_3},\overline{c_3})$, respectively. For instance, T_1 can be found as follows:

$$T_1 = L_1R_1 \Lambda_1U_1^t \tag{30}$$

where the nonsingular matrix L_1, the orthogonal matrices R_1 and U_1, and the diagonal matrix Λ_1 are, respectively, as follows:

$$L_1L_1^t = K^h \tag{31.1}$$

$$R_1^t(L_1^tW^hL_1)R_1 = \text{diag}(\alpha_1^2, \alpha_2^2,\dots,\alpha_\ell^2) \tag{31.2}$$

$$\Lambda_1 = \text{diag}(\lambda_{11},\lambda_{12},\dots,\lambda_{1\ell}),$$

$$\lambda_{1i} = \Big(\sum_{j=1}^{\ell}\alpha_j/\ell\alpha_i\Big)^{1/2} \tag{31.3}$$

$$(U_1(\Lambda_1)^{-2}U_1^t)_{ii} = 1 \quad \text{for } i=1,2,\dots,\ell \tag{31.4}$$

where α_i's are the second order modes of $DF(A_{11},\overline{b_1},\overline{c_1})$.

The normalized minimum variance of roundoff noise is then given by

$$\begin{aligned} E[\Delta y^2]_{min}/\sigma^2 \\ &= (\ell+1)\Big(\sum_{i=1}^{\ell} \alpha_i\Big)^2/\ell + (\ell+m+1)\Big(\sum_{j=1}^{m} \beta_j\Big)^2/m \\ &\quad + (\ell+m+n+1)\Big(\sum_{k=1}^{n} \gamma_k\Big)^2/n + (\ell+m+n+1). \end{aligned} \tag{32}$$

where β's and γ's are the second order modes of $DF(A_{22},\overline{b_2},\overline{c_2})$ and $DF(A_{33},\overline{b_3},\overline{c_3})$, respectively.

4.4 Absence of Overflow Oscillations in Optimal Realizations

In the previous subsection, we have proposed a synthesis method of optimal realizations of CRSD 3-D digital filters. In the following, we will show that these optimal realizations are also free of overflow oscillations, in addition to minimum roundoff noise. First, we have the following theorem:

Theorem 1: A realization CRSD(A,B,C,d) of a CRSD 3-D digital filter is free of overflow oscillations under zero input conditions if its coefficient matrices satisfy

$$\|A_{11}\|_2 < 1, \quad \|A_{22}\|_2 < 1 \quad \text{and} \quad \|A_{33}\|_2 < 1. \tag{33}$$

Proof: Consider the realization CRSD(A,B,C,d) which satisfies (33). For simplicity, let us denote the spectral norms of its coefficient matrices A_{11}-A_{33} as r_{11}-r_{33}. All these values are finite because the coefficient matrices are real constant matrices. Under the zero input condition, any realization of a CRSD 3-D digital filter with overflow nonlinearity can be described by

$$\tilde{x}^h(i,j,k) = f^h[A_{11}\tilde{x}^h(i-1,j,k)]$$

$$\tilde{x}^v(i,j,k) = f^v[A_{21}\tilde{x}^h(i,j-1,k) + A_{22}\tilde{x}^v(i,j-1,k)]$$

$$\begin{aligned} \tilde{x}^d(i,j,k) = f^d[&A_{31}\tilde{x}^h(i,j,k-1) + A_{32}\tilde{x}^v(i,j,k-1) \\ &+ A_{33}\tilde{x}^d(i,j,k-1)] \end{aligned} \tag{34}$$

where the nonlinear vector functions $f^h[\bullet]$, $f^v[\bullet]$ and $f^d[\bullet]$ representing overflow characteristics satisfy the following properties, respectively,

$$|f_p^h(\xi)| \le |\xi|, \qquad \text{for } p=1,2,\dots,1$$

$$|f_q^v(\xi)| \le |\xi|, \qquad \text{for } q=1,2,\dots,m$$

and

$$|f_r^d(\xi)| \le |\xi|, \qquad \text{for } r = 1,2,\dots,n$$

$$\text{for any scalar } \xi \tag{35}$$

and where the initial states are bounded as

$$\|\widetilde{x}^h(0,j,k)\|_2 \leq B_1 < \infty, \quad \text{for all } j>0,\ k>0$$

$$\|\widetilde{x}^v(i,0,k)\|_2 \leq B_2 < \infty, \quad \text{for all } i>0,\ k>0$$

$$\|\widetilde{x}^d(i,j,0)\|_2 \leq B_3 < \infty, \quad \text{for all } i>0,\ j>0. \tag{36}$$

From Eqs. (33)-(36), we have the following inequalities

$$\begin{aligned}\|\widetilde{x}^h(i,j,k)\|_2 &= \|f^h[A_{11}\widetilde{x}^h(i-1,j,k)]\|_2 \\ &\leq \|A_{11}\widetilde{x}^h(i-1,j,k)\|_2 \\ &= \|A_{11}^i\widetilde{x}^h(0,j,k)\|_2 \\ &\leq r_{11}^iB_1\end{aligned} \tag{37}$$

$$\begin{aligned}\|\widetilde{x}^v(i,j,k)\|_2 &= \|f^v[A_{21}\widetilde{x}^h(i,j-1,k) \\ &\qquad + A_{22}\widetilde{x}^v(i,j-1,k)]\|_2 \\ &\leq \|A_{21}\widetilde{x}^h(i,j-1,k)+A_{22}\widetilde{x}^v(i,j-1,k)\|_2 \\ &= \|A_{21}A_{11}^i\widetilde{x}^h(0,j-1,k) \\ &\quad + A_{22}A_{21}A_{11}^i\widetilde{x}^h(0,j-2,k) + \ldots \\ &\quad + A_{22}^{j-1}A_{21}A_{11}^i\widetilde{x}^h(0,0,k) \\ &\quad + A_{22}^j\widetilde{x}^v(i,0,k)\|_2 \\ &\leq R_2r_{21}r_{11}^iB_1 + r_{22}^jB_2\end{aligned} \tag{38.1}$$

where

$$\begin{aligned}R_2 &= \|I_m + A_{22} +\ldots+ A_{22}^{j-1}\|_2 \\ &\leq 1 + r_{22} + \ldots + r_{22}^{j-1} \\ &\leq 1/(1-r_{22}) \qquad \text{for any integer } j>0.\end{aligned} \tag{38.2}$$

Similarly

$$\begin{aligned}\|\widetilde{x}^d(i,j,k)\|_2 &= \|f^d[A_{31}\widetilde{x}^h(i,j,k-1) \\ &\qquad + A_{32}\widetilde{x}^v(i,j,k-1) \\ &\qquad + A_{33}\widetilde{x}^d(i,j,k-1)]\|_2 \\ &\leq \|A_{31}\widetilde{x}^h(i,j,k-1) \\ &\quad + A_{32}\widetilde{x}^v(i,j,k-1) \\ &\quad + A_{33}\widetilde{x}^d(i,j,k-1)\|_2 \\ &\leq R_3r_{31}r_{11}^iB_1 \\ &\quad + R_3r_{32}R_2r_{21}r_{11}^iB_1 \\ &\quad + R_3r_{32}r_{22}^jB_2+r_{33}^kB_3\end{aligned} \tag{39.1}$$

where

$$\begin{aligned}R_3 &= \|I_n + A_{33} + \ldots + A_{33}^{k-1}\|_2 \\ &\leq 1/(1-r_{33})\end{aligned}$$

Thus, from Eqs. (33), (36), (38.2) and (39.2), we have

$$\lim_{i\to\infty} \|\widetilde{x}^h(i,j,k)\|_2 = 0 \quad \text{for any integer } j>0 \text{ and } k>0$$

$$\lim_{i,j\to\infty} \|\widetilde{x}^v(i,j,k)\|_2 = 0 \quad \text{for any integer } k>0$$

and

$$\lim_{i,j,k\to\infty} \|\widetilde{x}^d(i,j,k)\|_2 = 0. \tag{40}$$

Therefore

$$\lim_{i\to\infty} \widetilde{x}^h(i,j,k) = 0 \quad \text{for any integer } j>0 \text{ and } k>0$$

$$\lim_{i,j\to\infty} \widetilde{x}^v(i,j,k) = 0 \quad \text{for any integer } k>0$$

and

$$\lim_{i,j,k\to\infty} \widetilde{x}^d(i,j,k) = 0. \tag{41}$$

Equation (41) shows that the state vector of the realization CRSD(A,B,C,d) with overflow nonlinearities converges to zero. Therefore, this realization is free of overflow oscillations under zero input condition. //

It has been proved that the coefficient matrix of the optimal realization of a 1-D digital filter DF(A,B,C,d) satisfies(9)

$$\|A\|_2 \leq 1 \tag{42}$$

and the strict inequality holds if all the second order modes of this filter are distinct. Using this fact and Theorem 1, and recalling that a realization of a CRSD 3-D digital filter is optimal iff its characteristic filters are optimal in the 1-D sense, we can conclude that optimal realizations of CRSD 3-D digital filters are also free of overflow oscillations.

V. SPATIAL-DOMAIN DIRECT DESIGN METHOD OF CRSD 3-D DIGITAL FILTERS

In the previous sections, we have proposed an approximation method and a synthesis method of CRSD 3-D digital filters. In order to design CRSD 3-D digital filters more efficiently, in this section, we will propose a direct design method which can perform the approximation and synthesis simultaneously with much less computational complexity.

5.1 Relation between Balanced Realizations and Optimal Realizations of CRSD 3-D Digital Filters

In 3-D case, a realization of a CRSD 3-D digital filter is called balanced iff its controllability and observability gramians satisfy

$$K^hW^h = \mathrm{diag}(\alpha_1,\alpha_2,\ldots,\alpha_\ell)$$
$$K^vW^v = \mathrm{diag}(\beta_1,\beta_2,\ldots,\beta_m)$$

$$K^dW^d = \mathrm{diag}(\gamma_1,\gamma_2,\ldots,\gamma_n) \tag{43}$$

where α_i, β_j and γ_k are the square roots of the eigenvalues of K^hW^h, K^vW^v and K^dW^d, respectively, and are called second order modes of the corresponding CRSD 3-D digital filter. Since (K^h,W^h), (K^v,W^v) and (K^d,W^d) are also controllability and observability gramians of the characteristic filters, we can also say that a realization of a CRSD 3-D digital filter is balanced iff its characteristic filters are balanced in the 1-D sense.

From the discussion of Sec. 3.2, we know that the optimal realization of CRSD 3-D digital filters can be synthesized by separately finding optimal realizations of their characteristic filters. Therefore, if the balanced realization of a CRSD 3-D digital filter is given, the optimal realization can be obtained by applying the equivalent transformation given in Ref. (1) to each characteristic filter. That is, the optimal realization of a CRSD 3-D digital filter can be obtained from its balanced realization by the following simple equivalent transformation:

$$T = \text{block diag}(\rho_1^{1/2}U_1^t,\rho_2^{1/2}U_2^t,\rho_3^{1/2}U_3^t) \tag{44}$$

where

$$\rho_1 = \sum_{i=1}^{\ell}\alpha_i/\ell,\quad \rho_2 = \sum_{j=1}^{m}\beta_j/m,\quad \rho_3 = \sum_{k=1}^{n}\gamma_k/n \tag{45}$$

and U_1, U_2 and U_3 are found from Eq. (31.4) or similar equations.

5.2 Direct Design Method of CRSD 3-D Digital Filters

From the definition of balanced realizations, it is clear that state-space digital filters resulted from the approximation method of Sec. 3 are nearly balanced realizations, because each of the characteristic filter obtained by Kung's method is nearly balanced(2). Using the equivalent transformation (44), and the approximation proposed in Sec. 3, we have the following direct design algorithm:

Algorithm 2:

Step 1 and Step 2 are the same as those in Algorithm 1,

Step 3: Instead of Eq. (20), the following equation is used:

$$A_{11} = U_1(\,\Sigma_{11}^{-1/2}U_{11}^t)(U_{11}\ \Sigma_{11}^{1/2})U_1^t$$

$$\overline{b}_1 = \rho_1^{-1/2}U_1(\text{First column of }\ \Sigma_{11}^{1/2}V_{11}^t)$$

$$\overline{c}_1 = (\text{First }(L+1)\text{ x }(M+1)\text{ rows of}$$

$$U_{11}\ \Sigma_{11}^{1/2})\rho_1^{1/2}U_1^t \tag{46}$$

Coefficient matrices of $DF(A_{22},\overline{b}_2,\overline{c}_2)$ and $DF(A_{33},\overline{b}_3,\overline{c}_3)$ can be found in a similar manner.

Step 4: Instead of Eq. (21), the following equation is used:

$$O_2^+ = \rho_2^{-1/2}U_2(\,\Sigma_{21}^{-1/2}U_{21}^t)$$

$$O_3^+ = \rho_3^{-1/2}U_3(\,\Sigma_{31}^{-1/2}U_{31}^t)$$

$$G_1^+ = (V_{11}\ \Sigma_{11}^{-1/2})\rho_1^{1/2}U_1^t$$

$$G_2^+ = (V_{21}\ \Sigma_{21}^{-1/2})\rho_2^{1/2}U_2^t. \tag{47}$$

5.3 A Numerical Example

In order to show the validity of the direct design method, in the following, we will give a numerical example. The specification to be approximated in this example is the impulse response of a Gaussian filter given by

$$h_{i,j,k} = \begin{cases} 0.256322\exp[-0.083203((i-5)^2 + (j-5)^2 \\ \qquad + (k-5)^2)] \\ \qquad\qquad \text{for } (0,0,0)\leq(i,j,k)\leq(10,10,10) \\ 0 \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{48}$$

This filter is designed as follows. First, form the impulse response and the Hankel matrices of the characteristic filters using Eqs. (15) and (16), and then find the SVD of the Hankel matrices. Since the impulse response is symmetric, the three groups of singular values are the same and given by

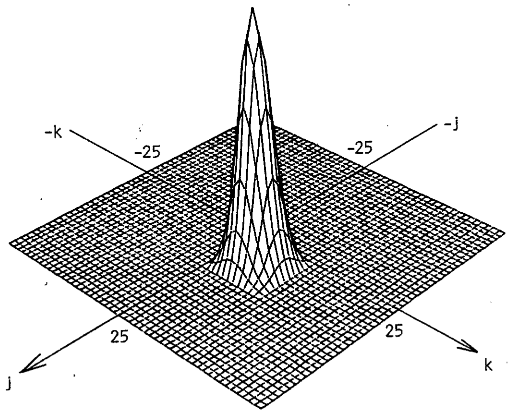
i=j=k	1	2	3	4	....
$\alpha_i = \beta_j = \gamma_k$	4.0339	1.7212	0.4907	0.1008	....

Since $\alpha_3 \gg \alpha_4$, $\beta_3 \gg \beta_4$ and $\gamma_3 \gg \gamma_4$, we set the filter order be (3 x 3 x 3). Then, from the SVD of the Hankel matrices, a nearly optimal realization of the desired CRSD 3-D digital filter is found using Eqs. (14), (46) and (47).

The coefficient matrices of the nearly optimal realization so designed is given in Table 1. For comparison, two other realizations with the same impulse response are also listed in this table. The impulse response and the frequency response of the ideal Gaussian filter are shown in Figs. 2-4, and those of the (3 x 3 x 3)-th order 3-D digital filter obtained by the direct design method are shown in Figs. 5-7. In addition, to gain some more insights about the characteristics of the 3-D digital filter in this example, the contour plots of the ideal Gaussian filter and those of the filter designed are also given in Figs. 8 and 9. The approximation error is given by

$$\max\left|h'_{i,j,k} - h_{i,j,k}\right| = 0.02047. \tag{49}$$

Further, the roundoff noise to output signal ratios of the three realizations in Table 1 are



Fig, 2 Impulse response of the ideal Gaussian filter $(i=5)$

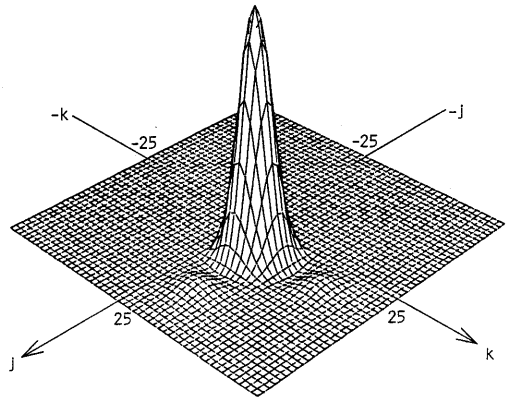


Fig. 5 Impulse response of the (3x3x3)-th order 3-D digital filter designed in the example $(i=5)$

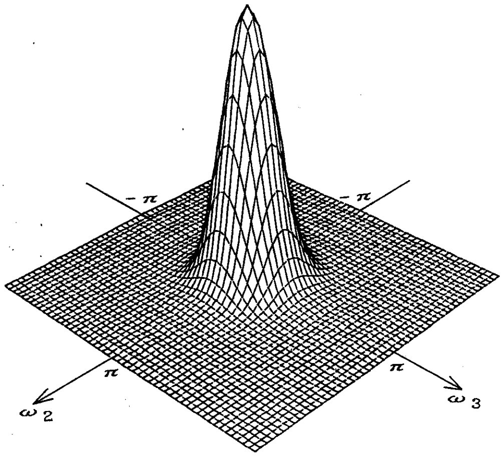


Fig. 3 Frequency response of the ideal Gaussian filter $(\omega_1=0)$

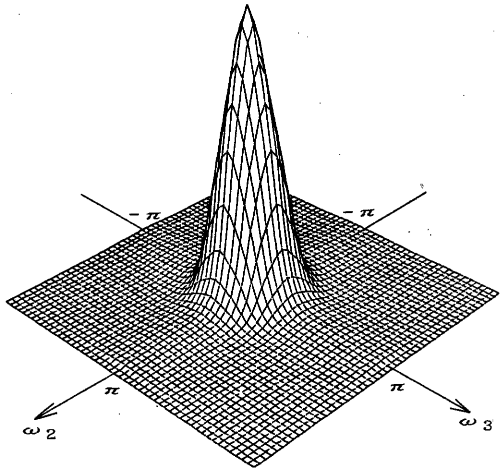


Fig. 6 Frequency response of the (3x3x3)-th order 3-D digital filter designed in the example $(\omega_1=0)$

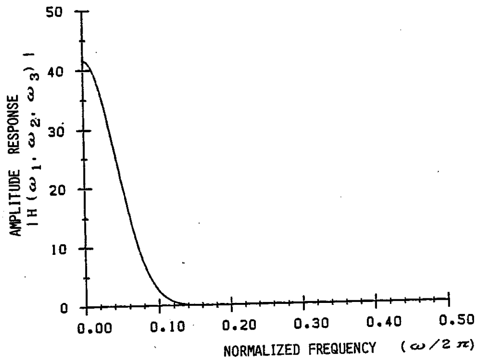


Fig. 4 Frequency response of the ideal Gaussian filter $(\omega_1=\omega_2=\omega_3)$

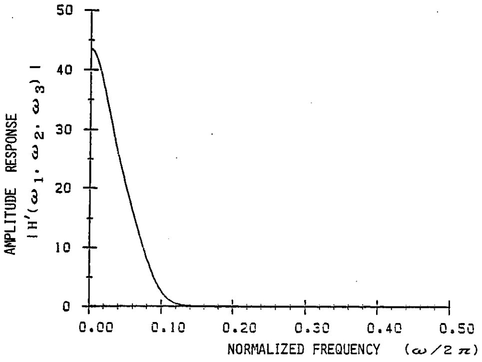


Fig. 7 Frequency response of the (3x3x3)-th order 3-D digital filter designed in the example $(\omega_1=\omega_2=\omega_3)$

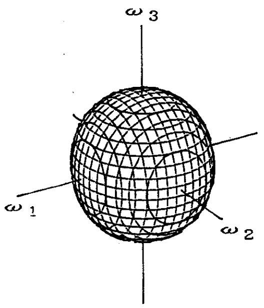


Fig. 8 Contour plots of the amplitude response of the ideal Gaussian filter (-12dB)

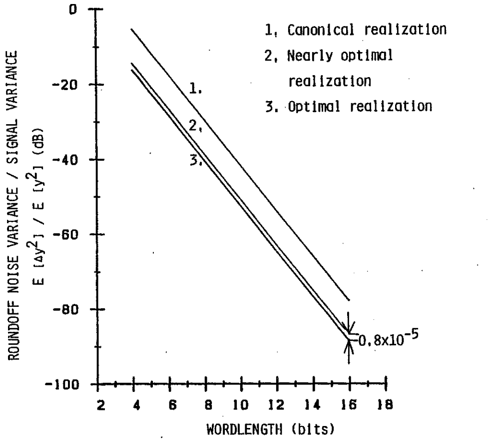


Fig. 10 Relation between roundoff noise to signal ratio and the wordlength of different structures of the (3x3x3)-th order 3-D digital filter designed in the example

Table 2 Roundoff noise to signal ratio

Realization	Noise to Signal Ratio $E[\Delta y^2]/E[y^2]$ ($\sigma^2=1$)
Canonical Realization	863.691604
Optimal Realization	72.195071
Nearly optimal Realization	72.195211

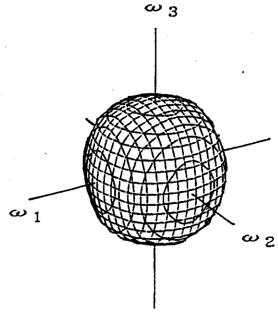


Fig. 9 Contour plots of the amplitude response of the (3x3x3)-th order 3-D digital filter designed in the example (-12dB)

given in Table 2. The relation between roundoff noise to output signal ratio and the wordlength is shown in Fig. 10. In Fig. 10 and Table 2, the output signal variance $E[y^2]$ is the value obtained by injecting a normal white signal with zero mean and unit variance into the filter, and can be evaluated by

$$E[y^2] = c_1K^hc_1^t + c_2K^vc_2^t + c_3K^dc_3^t + d^2. \qquad (50)$$

From Fig. 10 and Table 2 we can see that the roundoff noise of the nearly optimal realization is almost the same as that of the optimal realization. Moreover, the spectral norms of transition matrices A_{11}, A_{22} and A_{33} of the nearly optimal realization are

$$\|A_{11}\|_2 = \|A_{22}\|_2 = \|A_{33}\|_2 = 0.919394 < 1. \qquad (51)$$

Thus, from Theorem 1, the nearly optimal realization obtained can also suppress overflow oscillations.

VI. CONCLUDING REMARKS

In this paper, the design problem of causal, recursive and separable denominator (CRSD) 3-D digital filters has been studied. By introducing the 1-D characteristic filters, we have proposed a balanced approximation method and the synthesis method of optimal realizations of CRSD 3-D digital filters. Further, on the basis of the balanced approximation and the equivalent relation between balanced realizations and optimal realizations of CRSD 3-D digital filters, we have proposed a direct design method in the spatial domain. This direct design method can result in stable state-space digital filters which are nearly optimal with respect to roundoff noise, and free of overflow oscillations. Efficiency of direct design method has been shown by a numerical example. Further, this direct design method can also be extended to multi-dimensional case, and this will be discussed in other papers.

Table 1 Coefficient matrices of digital filters in the example (d = 1.1 x 10^{-4})

Realization	Matrices									
Canonical Realization	A =	0	1	0	0	0	0	0	0	0
		0	0	1	0	0	0	0	0	0
		0.3993	-1.4720	1.9940	0	0	0	0	0	0
		0.0000	0.0000	0.0000	0	1	0	0	0	0
		0.0000	0.0000	0.0000	0	0	1	0	0	0
		0.1436	-0.0275	0.1039	0.3993	-1.4720	1.9940	0	0	0
		0.0000	0.0000	0.0000	0.0010	-0.0002	0.0007	0	1	0
		0.0000	0.0000	0.0000	0.0005	-0.0001	0.0003	0	0	1
		0.0055	-0.0011	0.0040	0.1437	-0.0276	0.1040	0.3993	-1.4720	1.9940
	B^t =	0	0	0.2046	0	0	0.0079	0	0	0.0003
	C =	0.0020	-0.0004	0.0015	0.0531	-0.0102	0.0384	1.3740	-0.2496	0.9898
Optimal Realization	A =	0.6023	0.2280	0.0909	0	0	0	0	0	0
		0.0909	0.6957	0.4709	0	0	0	0	0	0
		0.2280	-0.4075	0.6957	0	0	0	0	0	0
		0.3550	0.3435	-0.1297	0.6023	0.2280	0.0909	0	0	0
		-0.1297	-0.1255	0.0474	0.0909	0.6957	0.4709	0	0	0
		0.3435	0.3324	-0.1255	0.2280	-0.4075	0.6957	0	0	0
		0.0136	0.0132	-0.0050	0.3550	0.3435	-0.1297	0.6023	0.0909	0.2280
		0.0132	0.0128	-0.0048	0.3435	0.3324	-0.1255	0.2280	0.6957	-0.4075
		-0.0050	-0.0048	0.0018	-0.1297	-0.1255	0.0474	0.0909	0.4709	0.6957
	B^t =	-0.5811	0.2124	-0.5624	-0.0223	0.0082	-0.0216	-0.0009	-0.0008	0.0003
	C =	-0.0018	-0.0017	0.0006	-0.0462	-0.0447	0.0169	-1.2020	0.4392	-1.1630
Nearly Optimal Realization	A =	0.7001	-0.4921	-0.0054	0	0	0	0	0	0
		0.3525	0.6831	-0.3574	0	0	0	0	0	0
		0.1022	-0.0679	0.6104	0	0	0	0	0	0
		-0.2123	0.0093	-0.1182	0.7001	-0.4921	-0.0054	0	0	0
		-0.2597	0.0114	-0.1446	0.3525	0.6831	-0.3574	0	0	0
		0.5475	-0.0241	0.3048	0.1022	-0.0679	0.6104	0	0	0
		0.0215	-0.0009	0.0120	0.5597	-0.0246	0.3116	0.7001	0.3515	0.1019
		-0.0009	0.0000	-0.0005	-0.0247	0.0011	-0.0137	-0.4936	0.6831	-0.0679
		0.0120	-0.0005	0.0067	0.3125	-0.0137	0.1740	-0.0054	-0.3574	0.6104
	B^t =	0.2766	0.3382	-0.7132	0.0106	0.0130	-0.0274	-0.0011	0.0000	-0.0006
	C =	-0.0022	0.0000	-0.0012	-0.0580	0.0025	-0.0323	0.5730	0.6987	-1.4730

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