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Finding shortest non-separating and non-disconnecting paths

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Abstract: For a connected graph $G = (V, E)$ and $s, t \in V$, a non-separating s-t path is a path P between s and t such that the set of vertices of P does not separate G, that is, $G - V(P)$ is connected. An s-t path is non-disconnected if $G - E(P)$ is connected. The problems of finding shortest non-separating and non-disconnecting paths are both known to be NP-hard. In this paper, we consider the problems from the viewpoint of parameterized complexity. We show that the problem of finding a non-separating s-t path of length at most k is W[1]-hard parameterized by k, while the non-disconnecting counterpart is fixed-parameter tractable parameterized by k. We also consider the shortest non-separating path problem on several classes of graphs and show that this problem is NP-hard even on bipartite graphs, chordal graphs, and planar graphs. As for positive results, the shortest non-separating path problem is fixed-parameter tractable parameterized by k on planar graphs and polynomial-time solvable on chordal graphs if k is the shortest path distance between s and t.

1. Introduction

Lovász' path removal conjecture states the following claim: There is a function $f: \mathbb{N} \to \mathbb{N}$ such that for every $f(k)$-connected graph G and every pair of vertices u and v, G has a path P between u and v such that $G - V(P)$ is k-connected. This claim remains still open and some spacial cases have been resolved [4, 14, 15, 20]. Tutte [20] proved that $f(1) = 3$, that is, every triconnected graph satisfies that for every pair of vertices, there is a path between them whose removal results a connected graph. Kawarabayashi et al. [14] proved a weaker version of this conjecture: There is a function $f: \mathbb{N} \to \mathbb{N}$ such that for every $f(k)$-connected graph G and every pair of vertices u and v, G has an induced path P between u and v such that $G - E(P)$ is k-connected.

As a practical application, let us consider a network represented by an undirected graph G, and we would like to build a private channel between a specific pair of nodes s and t. For some security reasons, the path used in this channel should be dedicated to the pair s and t, and hence all other connections do not use all nodes and/or edges on this path while keeping their connections. In graph-theoretic terms, the vertices (resp. edges) of the path between s and t does not form a separator (resp. a cut) of G. Tutte's result [20] indicates that such a path always exists in triconnected graphs, but may not exist in biconnected graphs. In addition to this connectivity constraint, the path between s and t is preferred to be short due to the cost of building a private channel. Motivated by such a natural application, the following two problems are studied.

Definition 1. *Given a connected graph G, $s, t \in V(G)$, and an integer k,* Shortest Non-Separating Path *asks whether there is a path P between s and t in G such that the length of P is at most k and $G - V(P)$ is connected.*

Definition 2. *Given a connected graph G, $s, t \in V(G)$, and an integer k,* Shortest Non-Disconnecting Path *asks whether there is a path P between s and t in G such that the length of P is at most k and $G - E(P)$ is connected.*

We say that a path P is *non-separating* (in G) if $G - V(P)$ is connected and is *non-disconnecting* (in G) if $G - E(P)$ is connected.

Related work. The shortest path problem in graphs is one of the most fundamental combinatorial optimization

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problems, which is studied under various settings. Indeed, our problems Shortest Non-Separating Path and Shortest Non-Disconnecting Path can be seen as variants of this problem. From the computational complexity viewpoint, Shortest Non-Separating Path is known to be NP-hard and its optimization version cannot be approximated with factor $|V|^{1-\varepsilon}$ in polynomial time for $\varepsilon > 0$ unless P = NP [21]. Shortest Non-Disconnecting Path is shown to be NP-hard on general graphs and polynomial-time solvable on chordal graphs [16].

Our results. We investigate the parameterized complexity of both problems. We show that Shortest Non-Separating Path is W[1]-hard and Shortest Non-Disconnecting Path is fixed-parameter tractable parameterized by k. A crucial observation for the fixed-parameter tractability of Shortest Non-Disconnecting Path is that the set of edges in a non-disconnecting path can be seen as an independent set of a cographic matroid. By applying the representative family of matroids [10], we show that Shortest Non-Disconnecting Path can be solved in $2^{\omega k}|V|^{O(1)}$ time, where ω is the exponent of the matrix multiplication. We also show that Shortest Non-Disconnecting Path is OR-compositional, that is, there is no polynomial kernelization unless coNP $\subseteq$ NP/poly. To cope with the intractability of Shortest Non-Separating Path, we consider the problem on planar graphs and show that it is fixed-parameter tractable parameterized by k. This result can be generalized to wider classes of graphs which have the *diameter-treewidth property* [8]. We also consider Shortest Non-Separating Path on several classes of graphs. We can observe that the complexity of Shortest Non-Separating Path is closely related to that of Hamiltonian Cycle (or Hamiltonian Path with specified end vertices). This observation readily proves the NP-completeness of Shortest Non-Separating Path on bipartite graphs, chordal graphs, and planar graphs. For chordal graphs, we devise a polynomial-time algorithm for Shortest Non-Separating Path for the case where k is the shortest path distance between s and t.

Due to the space limitation, we just provide an outline of the proof for each result.

2. Shortest Non-Separating Path

We discuss our complexity and algorithmic results for Shortest Non-Separating Path.

2.1 Hardness on graph classes

We obverse that, in most cases, Shortest Non-Separating Path is NP-hard on classes of graphs for which Hamiltonian Path (with distinguished end vertices) is NP-hard. Let $G = (V, E)$ be a graph and $s, t \in V$ be distinct vertices of G. We add a pendant vertex p adjacent to s and denote the resulting graph by G'. Then, we have the following observation.

Observation 1. *For every non-separating path P between s and t in G', $V(G) \setminus V(P) = \{p\}$.*

Suppose that for a class $\mathcal{C}$ of graphs,

- the problem of deciding whether given graph $G \in \mathcal{C}$ has a Hamiltonian path between specified vertices s and t in G is NP-hard and
- $G \in \mathcal{C}$ implies $G' \in \mathcal{C}$.

By Observation 1, G' has a non-separating s-t path if and only if G has a Hamiltonian path between s and t. This implies that the problem of finding a non-separating path between specified vertices is NP-hard on class $\mathcal{C}$.

Theorem 1. *The problem of deciding if an input graph has a non-separating s-t path is NP-complete even on planar graphs, bipartite graphs, and chordal graphs.*

The proof of the theorem is done by performing a polynomial-reduction from Hamiltonian Cycle to Hamiltonian Path (with specified end vertices) for planar graphs, bipartite graphs, and chordal graphs. Since Hamiltonian Cycle is known to be NP-complete on these classes of graphs [13, 17].

2.2 W[1]-hardness

Next, we show that Shortest Non-Separating Path is W[1]-hard parameterized by k. The proof is done by giving a reduction from Multicolored Clique, which is known to be W[1]-complete [9]. In Multicolored Clique, we are given a graph G with a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ and asked to determine whether G has a clique C such that $|V_i \cap C| = 1$ for each $1 \leq i \leq k$.

From an instance $(G, \{V_1, \ldots, V_k\})$ of Multicolored Clique, we construct an instance of Shortest Non-Separating Path as follows. Without loss of generality, we assume that G contains more than k vertices. We add two vertices s and t and edges between s and all $v \in V_1$ and between t and all $v \in V_k$. For any pair of $u \in V_i$ and $v \in V_j$ with $|i-j| \geq 2$, we do the following. If $\{u, v\} \in E$, then we remove it. Otherwise, we add a path $P_{u,v}$ of length 2 and a pendant vertex that is adjacent to the internal vertex w of $P_{u,v}$. Finally, we add a vertex v^*, add an edge between v^* and each original vertex

$v \in V(G)$, and add a pendant vertex p adjacent to v^*. The constructed graph is denoted by H.

Lemma 1. *There is a clique C in G such that $|C \cap V_i| = 1$ for $1 \le i \le k$ if and only if there is a non-separating s-t path of length at most $k+1$ in H.*

Thus, we have the following theorem.

Theorem 2. Shortest Non-Separating Path *is W[1]-hard parameterized by k.*

2.3 Graphs with the diameter-treewidth property

By Theorem 2, Shortest Non-Separating Path is unlikely to be fixed-parameter tractable on general graphs. To overcome this intractability, we focus on sparse graph classes. We first note that algorithmic meta-theorems for FO Model Checking [11, 12] does not seem to be applied to Shortest Non-Separating Path as we need to care about the connectivity of graphs, while it can be expressed by a formula in MSO logic, which is as follows. The property that vertex set X forms a non-separating s-t path can be expressed as:

$$\texttt{conn}(V \setminus X) \wedge \texttt{hampath}(X, s, t),$$

where $\texttt{conn}(Y)$ and $\texttt{hampath}(Y, s, t)$ are formulas in MSO_2 that are true if and only if the subgraph induced by Y is connected and has a Hamiltonian path between s and t, respectively. We omit the details of these formulas, which can be found in [6] for example[*1]. By Courcelle's theorem [5] and its extension due to Arnborg et al. [1], we can compute a shortest non-separating s-t path in $O(f(\mathrm{tw}(G))n)$ time, where n is the number of vertices and $\mathrm{tw}(G)$ is the treewidth[*2] of G. As there is an $O(\mathrm{tw(G)}^{\mathrm{tw(G)}^3} n)$-time algorithm for computing the treewidth of an input graph G [2], we have the following theorem.

Theorem 3. Shortest Non-Separating Path *is fixed-parameter tractable parameterized by the treewidth of input graphs.*

A class $\mathcal{C}$ of graphs is *minor-closed* if every minor of a graph $G \in \mathcal{C}$ also belongs to $\mathcal{C}$. We say that $\mathcal{C}$ has the *diameter-treewidth property* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{C}$, the treewidth of G is upper bounded by $f(\mathrm{diam}(G))$, where $\mathrm{diam}(G)$ is the diameter of G. It is well known that every planar graph G has treewidth at most $3 \cdot \mathrm{diam}(G) + 1$ [19][*3], which implies that the class of planar graphs has the diameter-treewidth property. This can be generalized to more wider classes of graphs. A graph is called an *apex graph* if it has a vertex such that removing it makes the graph planar.

Theorem 4 ([7, 8])**.** *Let $\mathcal{C}$ be a minor-closed class of graphs. Then, $\mathcal{C}$ has the diameter-treewidth property if and only if it excludes some apex graph.*

Theorem 5. *Suppose that a minor-closed class $\mathcal{C}$ of graphs has the diameter-treewidth property. Then,* Shortest Non-Separating Path *is fixed-parameter tractable parameterized by k on $\mathcal{C}$.*

The proof goes as follows. If the distance between s and t is more than k, the instance is trivially infeasible. Suppose otherwise. Then, every non-separating path between s and t contains only vertices of distance at most k from s. This implies that the vertices to which the distance from s more than k is easily handled. From this observation, we construct an equivalent instance of diameter $O(k)$ and by the diameter-treewidth property and Theorem 3, the theorem follows.

2.4 Chordal graphs with $k = \mathrm{dist}(s, t)$

In Section 2.1, we have seen that Shortest Non-Separating Path is NP-complete even on chordal graphs. To overcome this intractability, we restrict ourselves to finding a non-separating s-t path of length $\mathrm{dist}(s, t)$ on chordal graphs.

Theorem 6. *There is a polynomial-time algorithm for* Shortest Non-Separating Path *on chordal graphs, provided that k is equal to the shortest path distance between s and t.*

The idea of proving this theorem is as follows. In chordal graphs, every shortest path between s and t that is non-separating does not contain some minimal separators, and we can show that this condition is also a sufficient condition for such a path. Using nontrivial observations on chordal graphs, we can find a shortest path that satisfies this condition in polynomial time.

3. Shortest Non-Disconnecting Path

The goal of this section is to establish the fixed-parameter tractability and a conditional lower bound on polynomial kernelizations for Shortest Non-Disconnecting Path.

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*1 In [6], they give an MSO_2 sentence `hamiltonicity` expressing the property of having a Hamiltonian cycle, which can be easily transformed into a formula expressing $\texttt{hampath}(X, s, t)$.

*2 We do not give the definition of treewidth and (the optimization version of) Courcelle's theorem. We refer to [6] for details.

*3 More precisely, the treewidth of a planar graph is upper bounded by $3r + 1$, where r is the radius of the graph.

3.1 Fixed-parameter tractability

Theorem 7. Shortest Non-Disconnecting Path *can be solved in time* $2^{\omega k} n^{O(1)}$*, where* ω *is the matrix multiplication exponent and* n *is the number of vertices of the input graph* G*.*

The algorithm is based on representative families of matroids due to [10]. It is well known that the set of edges of a non-disconnected path forms an independent set in the cographic matroid of G [18]. We give a dynamic programming algorithm with the aid of representative families of linear matroids.

3.2 Kernel lower bound

It is well known that a parameterized problem is fixed-parameter tractable if and only if it admits a kernelization (see [6], for example). By Theorem 7, Shortest Non-Disconnecting Path admits a kernelization. A natural step next to this is to explore the existence of polynomial kernelizations for Shortest Non-Disconnecting Path. However, the following theorem conditionally rules out the possibility of polynomial kernelization.

Theorem 8. *Unless* coNP $\subseteq$ NP/poly*,* Shortest Non-Disconnecting Path *does not admit a polynomial kernelization (with respect to parameter* k*).*

The proof of the theorem is done by showing that Shortest Non-Disconnecting Path is OR-compositional, and by [3], the problem does not admit a polynomial kernelization unless coNP $\subseteq$ NP/poly.

References

[1] Stefan Arnborg, Jens Lagergren, and Detlef Seese. "Easy Problems for Tree-Decomposable Graphs". In: *J. Algorithms* 12.2 (1991), pp. 308–340.

[2] Hans L. Bodlaender. "A Linear-Time Algorithm for Finding Tree-Decompositions of Small Treewidth". In: *SIAM J. Comput.* 25.6 (1996), pp. 1305–1317.

[3] Hans L. Bodlaender, Rodney G. Downey, Michael R. Fellows, and Danny Hermelin. "On problems without polynomial kernels". In: *J. Comput. Syst. Sci.* 75.8 (2009), pp. 423–434.

[4] Guantao Chen, Ronald J. Gould, and Xingxing Yu. "Graph Connectivity After Path Removal". In: *Comb.* 23.2 (2003), pp. 185–203.

[5] Bruno Courcelle. "The Monadic Second-Order Logic of Graphs. I. Recognizable Sets of Finite Graphs". In: *Inf. Comput.* 85.1 (1990), pp. 12–75.

[6] Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms.* 1st. Springer Publishing Company, Incorporated, 2015.

[7] Erik D. Demaine and Mohammad Taghi Hajiaghayi. "Diameter and Treewidth in Minor-Closed Graph Families, Revisited". In: *Algorithmica* 40.3 (2004), pp. 211–215.

[8] David Eppstein. "Diameter and Treewidth in Minor-Closed Graph Families". In: *Algorithmica* 27.3 (2000), pp. 275–291.

[9] Michael R. Fellows, Danny Hermelin, Frances A. Rosamond, and Stéphane Vialette. "On the parameterized complexity of multiple-interval graph problems". In: *Theor. Comput. Sci.* 410.1 (2009), pp. 53–61.

[10] Fedor V. Fomin, Daniel Lokshtanov, Fahad Panolan, and Saket Saurabh. "Efficient Computation of Representative Families with Applications in Parameterized and Exact Algorithms". In: *J. ACM* 63.4 (2016), 29:1–29:60.

[11] Martin Grohe and Stephan Kreutzer. "Methods for Algorithmic Meta Theorems". In: *AMS-ASL Joint Special Session.* Vol. 558. Contemporary Mathematics. American Mathematical Society, 2009, pp. 181–206.

[12] Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. "Deciding First-Order Properties of Nowhere Dense Graphs". In: *J. ACM* 64.3 (2017), 17:1–17:32.

[13] Alon Itai, Christos H. Papadimitriou, and Jayme Luiz Szwarcfiter. "Hamilton Paths in Grid Graphs". In: *SIAM J. Comput.* 11.4 (1982), pp. 676–686.

[14] Ken-ichi Kawarabayashi, Orlando Lee, Bruce A. Reed, and Paul Wollan. "A weaker version of Lovász' path removal conjecture". In: *J. Comb. Theory, Ser. B* 98.5 (2008), pp. 972–979.

[15] Matthias Kriesell. "Induced paths in 5-connected graphs". In: *J. Graph Theory* 36.1 (2001), pp. 52–58.

[16] Xiao Mao. "Shortest non-separating st-path on chordal graphs". In: *CoRR* abs/2101.03519 (2021). arXiv: 2101.03519.

[17] Haiko Müller. "Hamiltonian circuits in chordal bipartite graphs". In: *Discret. Math.* 156.1-3 (1996), pp. 291–298.

[18] James G. Oxley. *Matroid Theory (Oxford Graduate Texts in Mathematics)*. USA: Oxford University Press, Inc., 2006.

[19] Neil Robertson and Paul D. Seymour. "Graph minors. III. Planar tree-width". In: *J. Comb. Theory, Ser. B* 36.1 (1984), pp. 49–64.

[20] William T. Tutte. "How to Draw a Graph". In: *Proceedings of the London Mathematical Society* s3-13.1 (1963), pp. 743–767.

[21] Bang Ye Wu and Hung-Chou Chen. "The approximability of the minimum border problem". In: *The 26th Workshop on Combinatorial Mathematics and Computation Theory*. 2009.