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The Strong 3-rainbow Index of Comb Product of a Tree and a Connected Graph

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Abstract: Let G be a nontrivial connected graph of order n. Let k be an integer with $2 \leq k \leq n$. A strong k-rainbow coloring of G is an edge-coloring of G having property that for every set S of k vertices of G, there exists a tree with minimum size containing S whose all edges have distinct colors. The minimum number of colors required such that G admits a strong k-rainbow coloring is called the strong k-rainbow index $srx_k(G)$ of G. In this paper, we study the strong 3-rainbow index of comb product between a tree and a connected graph, denoted by $T_n \rhd_o H$. Notice that the size of $T_n \rhd_o H$ is the trivial upper bound for $srx_3(T_n \rhd_o H)$, which means we can assign distinct colors to all edges of $T_n \rhd_o H$. However, there are some connected graphs H such that some edges of $T_n \rhd_o H$ may be colored the same. Therefore, in this paper, we characterize connected graphs H with $srx_3(T_n \rhd_o H) = |E(T_n \rhd_o H)|$. We also provide a sharp upper bound for $srx_3(T_n \rhd_o H)$ where $srx_3(T_n \rhd_o H) \neq |E(T_n \rhd_o H)|$. In addition, we determine the $srx_3(T_n \rhd_o H)$ for some connected graphs H.

Keywords: comb product, rainbow coloring, strong k-rainbow index

1. Introduction

All graphs in this paper are simple, finite, and connected. We follow the terminology and notation of Diestel [12]. For two integers a and b, we define $[a, b]$ as a set of all integers x with $a \leq x \leq b$. Given an edge-colored graph G of order $n \geq 3$, where adjacent edges may be colored the same. A tree T in G is called a *rainbow tree*, if every edge of T has distinct colors. For further discussion, we always let k be an integer with $k \in [2, n]$ and $S \subseteq V(G)$ with $|S| = k$. A rainbow tree containing the vertices of S is called a *rainbow S-tree*. If $S = \{u, v\}$, then the rainbow S-tree is called the *rainbow $u-v$ path* [8]. The minimum number of colors needed in an edge-coloring of G such that there exists a rainbow S-tree for every set S in G is called the *k-rainbow index $rx_k(G)$* of G. These concepts were introduced by Chartrand et al. [10]. If $S = \{u, v\}$, then the 2-rainbow index of G is called the *rainbow connection number $rc(G)$* of G [8]. Such a graph G is called a *rainbow-connected* graph, i.e., G contains a rainbow $u - v$ path for every two vertices u and v of G [8]. It follows, for every nontrivial connected graph G of order n, that $rc(G) = rx_2(G) \leq rx_3(G) \leq ... \leq rx_n(G)$. Chartrand et al. [9] also introduced the generalization of rainbow connection number called the *rainbow l-connection number $rc_l(G)$* of G, that is the minimum number of colors needed in an edge-coloring of G such that there exist $l \geq 1$ internally disjoint rainbow $u - v$ paths for every two vertices u and v of G.

Caro et al. [6] conjectured that deciding whether a graph G has $rc(G) = 2$ is NP-Complete, in particular, computing $rc(G)$ is NP-Hard. In Ref. [7], Chakraborty et al. confirmed this conjecture. They also proved that it is NP-Complete to decide whether a given edge-colored graph is rainbow-connected. However, Li et al. [19] showed that deciding whether $rc(G) = 2$ becomes easy when G is a bipartite graph, whereas deciding whether $rc(G) = 3$ is still NP-Complete, even when G is a bipartite graph. Many authors also investigated bounds, algorithms, and computational complexity of the rainbow connection number of graphs (see Refs. [21], [22]). Other known results about rainbow connection number of graphs can be found in Refs. [4], [8], [9], [14], [18], [25], [26], [28], [29], [30].

The k-rainbow index has an interesting application for the secure transfer of information between some people in a communication network, which can be modeled by a graph. Ericksen [13] stated that the attacks on September 11, 2001, happened because some agencies cannot access the information and communicate with each other safely. In order to solve this problem, we can assign a large enough number of passwords to the line which connects these agencies so that no password is repeated. The mininum number of passwords which allows one secure line between every k agencies in a communication network (which may have other agencies as intermediaries) so that the passwords along the line are distinct is represented by the k-rainbow index of a graph.

The minimum size of a tree containing S is called the *Steiner distance $d(S)$* of S. The *k-Steiner diameter $sdiam_k(G)$* of G is the maximum Steiner distance of S among all sets S in G. If $S = \{u, v\}$, then $d(S) = d(u, v)$ ($d(u, v)$ is the *distance* between u and v, i.e., the length of a shortest $u - v$ path in G) and $sdiam_2(G) = diam(G)$ ($diam(G)$ is the *diameter* of G, i.e., the

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largest distance between two vertices of G). Hence, $diam(G) = sdiam_2(G) \leq sdiam_3(G) \leq \ldots \leq sdiam_n(G)$. In Ref. [10], Chartrand et al. gave simple lower and upper bounds for $rx_k(G)$, that is for every connected graph G of order $n \geq 3$ and each integer k with $k \in [3, n]$, $k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1$. They obtained the k-rainbow index of a cycle and a tree, where $rx_k(T_n) = n - 1$ which attains the upper bound for $rx_k(G)$. They also showed that the k-rainbow index of a unicyclic graph is $n - 1$ or $n - 2$. Therefore, Li et al. [20] characterized the graphs whose 3-rainbow index is $n - 1$ and $n - 2$. Liu and Hu [23] studied the 3-rainbow index with respect to three important graph product operations and also other graph operations. Graph operations are an interesting subject, which can be used to understand structures of graphs. Some other results about k-rainbow index can be found in Refs. [3], [5], [11], [17], [21], [22], [24].

In real life, one of the things that is being considered to make a secure communication network is the time needed so that every k people can access the information and communicate with each other as quickly as possible. To model this problem, the first and second authors generalized the concept of k-rainbow index [2]. A *Steiner S-tree* is a tree of size $d(S)$ which contains the vertices of S. If $S = \{u, v\}$, then the Steiner S-tree is called the *$u - v$ geodesic* [8]. An edge-coloring of G is called a *strong k-rainbow coloring*, if there exists a rainbow Steiner S-tree for every set S in G. The *strong k-rainbow index $srx_k(G)$* of G is the minimum number of colors needed in a strong k-rainbow coloring of G. Hence, $rx_k(G) \leq srx_k(G)$ for every connected graph G. If $S = \{u, v\}$, then the strong 2-rainbow index is called the *strong rainbow connection number $src(G)$* of G [8]. Therefore, $src(G) = srx_2(G) \leq srx_3(G) \leq \ldots \leq srx_n(G)$ for every connected graph G of order n. Chartrand et al. [8] gave lower and upper bounds for the strong rainbow connection number, that is $diam(G) \leq rc(G) \leq src(G) \leq |E(G)|$, where $|E(G)|$ is the size of G.

Note that the strong k-rainbow index is defined for every connected graph, since every coloring that assigns distinct colors to all edges of a connected graph is a strong k-rainbow coloring. Thus, it is easy to see that

$$sdiam_k(G) \leq srx_k(G) \leq |E(G)|. \qquad (1)$$

There is a connected graph of order $n \geq 3$ whose strong k-rainbow index attains the upper bound in Eq. (1) for every $k \in [3, n]$. To see this, let G be a connected graph which contains bridges and admits a strong k-rainbow coloring. Let $e = uv$ and $f = xy$ be two bridges of G. Then $G - e - f$ contains three components G_1, G_2, and G_3. Without loss of generality, let $u \in V(G_1)$, $y \in V(G_2)$, and $v, x \in V(G_3)$. If S is a set of k vertices containing u and y, then bridges e and f should be contained in every rainbow Steiner S-tree. This gives us the following fact.

Fact 1.1. *Let G be a connected graph of order n which contains bridges. Let $e, f \in E(G)$ be the bridges of G. For each integer $k \in [2, n]$, if c is a strong k-rainbow coloring of G, then $c(e) \neq c(f)$.*

The fact above implies the following theorem.

Theorem 1.1. *[2] Let T_n be a tree of order $n \geq 3$. For each integer $k \in [3, n]$, $srx_k(T_n) = |E(T_n)| = n - 1$.*

Note that a larger and complex communication network can be obtained by extending the previous networks, which can be done by doing some operation on the graphs. Therefore, we studied the srx_3 of vertex-amalgamation and edge-amalgamation of some graphs (see Refs. [1], [2]). In this paper, we study the srx_3 of comb product of a tree and a connected graph. Let G be a graph and $\mathcal{H}$ be a sequence of $|V(G)|$ rooted graphs $H_1, H_2, \ldots, H_{|V(G)|}$, where each H_i has a root vertex o_i. According to [15], the *rooted product* of G by $\mathcal{H}$, denoted by $G(\mathcal{H})$, is a graph obtained by identifying the root vertex o_i of H_i with the i-th vertex of G for all $i \in [1, |V(G)|]$. If $H_i \cong H$ and $o_i = o$ for each $i \in [1, |V(G)|]$, then Saputro et al. called this notion by *comb product* of G and H, denoted by $G \rhd_o H$ [27]. The study of comb product is needed when we have a communication network that contains some divisions (the division is modeled by a connected graph H) and some people in different divisions must pass through the head of their division, which is represented by vertex o, in order to transfer information to each other.

This paper is organized as follows. In Section 2, we first provide a connected graph H such that $srx_3(T_n \rhd_o H) = |E(T_n \rhd_o H)|$ and characterize connected graphs H with $srx_3(T_n \rhd_o H) = |E(T_n \rhd_o H)|$. We also provide a sharp upper bound for $srx_3(T_n \rhd_o H)$ where $srx_3(T_n \rhd_o H) \neq |E(T_n \rhd_o H)|$. In Section 3, we determine the exact values of $srx_3(T_n \rhd_o H)$ for some connected graphs H.

2. Sharp Upper Bound for $srx_3(T_n \rhd_o H)$

Let n and m be two integers at least 3. Let G and H be connected graphs of order n and m, respectively, with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{w_1, w_2, \ldots, w_m\}$. By the definition of comb product, we can say that $V(G \rhd_o H) = \{(u_i, w_p) : u_i \in V(G), w_p \in V(H)\}$ and $(u_i, w_p)(u_j, w_q) \in E(G \rhd_o H)$ whenever $u_i = u_j$ and $w_p w_q \in E(H)$, or $u_i u_j \in E(G)$ and $w_p = w_q = o$ [27]. For simplifying, we define $v_i^p = (u_i, w_p)$ for $i \in [1, n]$ and $p \in [1, m]$.

In this paper, we consider graphs $T_n \rhd_o H$. For further discussion, let H^i denote the i-th copy of H for each $i \in [1, n]$. Given c as a strong 3-rainbow coloring of $T_n \rhd_o H$. For $X \subseteq E(T_n \rhd_o H)$, let $c(X)$ denote the set of colors assigned to the edges in X. By using Theorem 1.1,

$$|c(E(T_n))| = n - 1. \qquad (2)$$

Following Eq. (1), $|E(T_n \rhd_o H)|$ is the natural upper bound for $srx_3(T_n \rhd_o H)$. In the next theorem, we determine the strong 3-rainbow index of $T_n \rhd_o T_m$ which is equal to its size.

Theorem 2.1. *Let n and m be two integers at least 3. Let T_n and T_m be trees of order n and m, respectively, and o be an arbitrary vertex of T_m. Then $srx_3(T_n \rhd_o T_m) = nm - 1$.*

Proof. Note that $T_n \rhd_o T_m$ is a tree, where $|E(T_n \rhd_o T_m)| = |E(T_n)| + n(|E(T_m)|)$. It follows by Theorem 1.1 that $srx_3(T_n \rhd_o T_m) = |E(T_n \rhd_o T_m)| = |E(T_n)| + n(|E(T_m)|) = nm - 1$. □

A natural thought is like this: Which connected graph H of order m except a tree that has the strong 3-rainbow index $|E(T_n \rhd_o H)|$? Since H is not a tree, H must contains cycles. Let $h \geq 3$ be the girth of H. Let C_h be a cycle of order h in H. We relabel vertices of H such that $V(C_h) = \{w_1, w_2, \ldots, w_h\}$, $E(C_h) =$

$\{w_iw_{i+1} : i \in [1,h]$ and $w_{h+1} = w_1\}$, and $d(o, w_1) \leq d(o, w_i)$ for all $i \in [2, h]$. We first provide the following observation.

Observation 2.1. *There exists an edge of C_h that is not contained in a shortest $o - w_i$ path for any $i \in [1, h]$.*

Proof. For $i \in [1, h]$, let $d(o, w_i) = l_i$. Recall that $l_1 \leq l_i$ for all $i \in [2, h]$. Thus, $l_i \in [l_1, l_1 + i - 1]$ for $i \in [1, \lfloor\frac{h}{2}\rfloor + 1]$ and $l_i \in [l_1, l_1 + h - i + 1]$ for $i \in [\lfloor\frac{h}{2}\rfloor + 2, h]$. Now, we consider two cases.

Case 1. $o \in V(C_h)$

It means $o = w_1$. Thus, $l_i = i - 1$ for $i \in [1, \lfloor\frac{h}{2}\rfloor + 1]$ and $l_i = h - i + 1$ for $i \in [\lfloor\frac{h}{2}\rfloor + 2, h]$. If h is odd, then $w_{\lfloor\frac{h}{2}\rfloor+1}w_{\lfloor\frac{h}{2}\rfloor+2}$ is not contained in a shortest $o - w_i$ path for any $i \in [1, h]$. If h is even, then there are at least two shortest $o - w_{\lfloor\frac{h}{2}\rfloor+1}$ paths, one path contains $w_{\lfloor\frac{h}{2}\rfloor}w_{\lfloor\frac{h}{2}\rfloor+1}$ and another path contains $w_{\lfloor\frac{h}{2}\rfloor+1}w_{\lfloor\frac{h}{2}\rfloor+2}$. Therefore, we can choose $w_{\lfloor\frac{h}{2}\rfloor+1}w_{\lfloor\frac{h}{2}\rfloor+2}$ to be an edge that is not contained in a shortest $o - w_{\lfloor\frac{h}{2}\rfloor+1}$ path. Furthermore, $w_{\lfloor\frac{h}{2}\rfloor+1}w_{\lfloor\frac{h}{2}\rfloor+2}$ is not contained in a shortest $o - w_i$ path for any $i \in [1, h]$.

Case 2. $o \notin V(C_h)$

We first define the following sets.

– For odd h, let $W_{1,1}$ be a set of pairs of two vertices (w_i, w_j) such that $w_i, w_j \in V(C_h)$ and $l_i = l_j$ for distinct $i, j \in [1, \lfloor\frac{h}{2}\rfloor + 1]$, and $W_{1,2}$ be a set of pairs of two vertices (w_i, w_j) such that $w_i, w_j \in V(C_h)$ and $l_i = l_j$ for distinct $i, j \in \{1\} \cup [\lfloor\frac{h}{2}\rfloor + 2, h]$.

– For even h, let $W_{2,1}$ be a set of pairs of two vertices (w_i, w_j) such that $w_i, w_j \in V(C_h)$ and $l_i = l_j$ for distinct $i, j \in [1, \lfloor\frac{h}{2}\rfloor + 1]$, and $W_{2,2}$ be a set of pairs of two vertices (w_i, w_j) such that $w_i, w_j \in V(C_h)$ and $l_i = l_j$ for distinct $i, j \in \{1\} \cup [\lfloor\frac{h}{2}\rfloor + 1, h]$.

Hence, we have either Subcase 2.1 or Subcase 2.2 regardless of the parity of h as follows.

Subcase 2.1. $|W_{p,q}| \geq 1$ for some $q \in [1, 2]$

Choose a pair $(w_i, w_j) \in W_{p,q}$ so that $d_{C_h}(w_i, w_j)$ has the smallest value. Thus, $d_{C_h}(w_i, w_j)$ is 1 or 2, since if $d_{C_h}(w_i, w_j) \geq 3$, then there exists another pair $(w_{i'} w_{j'}) \in W_{p,q}$ such that $d_{C_h}(w_{i'}, w_{j'}) < d_{C_h}(w_i, w_j)$, contradicts the assumption.

If $d_{C_h}(w_i, w_j) = 1$, then w_iw_j is not contained in a shortest $o - w_i$ path and a shortest $o - w_j$ path. Furthermore, w_iw_j is not contained in a shortest $o - w_i$ path for any $i \in [1, h]$.

If $d_{C_h}(w_i, w_j) = 2$, then there exists $w_k \in V(C_h)$ such that $w_iw_k, w_kw_j \in E(C_h)$ and $l_k = l_i + 1$. Hence, there are at least two shortest $o - w_k$ paths, one path contains w_iw_k and another path contains w_kw_j. By using a similar argument as Case 1 for even h, we can choose w_iw_k to be an edge that is not contained in a shortest $o - w_i$ path for any $i \in [1, h]$.

Subcase 2.2. $|W_{p,q}| = 0$ for all $q \in [1, 2]$

Since $|W_{p,q}| = 0$ for all $q \in [1, 2]$, $l_i = l_1 + i - 1$ for $i \in [1, \lfloor\frac{h}{2}\rfloor + 1]$ and $l_i = l_1 + h - i + 1$ for $i \in [\lfloor\frac{h}{2}\rfloor + 2, h]$. Thus, by using a similar argument as Case 1, $w_{\lfloor\frac{h}{2}\rfloor+1}w_{\lfloor\frac{h}{2}\rfloor+2}$ is not contained in a shortest $o - w_i$ path for any $i \in [1, h]$. □

The next theorem shows characterization of connected graphs H with $srx_3(T_n \rhd_o H) = |E(T_n \rhd_o H)|$.

Theorem 2.2. *Let n and m be two integers at least* 3. *Let T_n be a tree of order n, H be a connected graph of order m, and o be an arbitrary vertex of H. Then $srx_3(T_n \rhd_o H) = |E(T_n \rhd_o H)|$ if and only if H is a tree.*

Proof. If H is a tree, then $srx_3(T_n \rhd_o H) = |E(T_n \rhd_o H)|$ by Theorem 2.1.

Conversely, suppose H is a graph with $srx_3(T_n \rhd_o H) = |E(T_n \rhd_o H)|$ but not a tree. Thus, H must contain cycles. Let $h \geq 3$ be the girth of H. Let C_h be a cycle of order h in H. We relabel vertices of H such that $V(C_h) = \{w_1, w_2, \ldots, w_h\}$, $E(C_h) = \{w_iw_{i+1} : i \in [1, h]$ and $w_{h+1} = w_1\}$, and $d(o, w_1) \leq d(o, w_i)$ for all $i \in [2, h]$.

If $o \in V(C_h)$, then $o = w_1$. Thus, $w_{\lfloor\frac{h}{2}\rfloor+1}w_{\lfloor\frac{h}{2}\rfloor+2}$ is not contained in a shortest $o - w_i$ path for all $i \in [1, h]$ by Observation 2.1. Therefore, by assigning color 1 to the edges $v_i^{\lfloor\frac{h}{2}\rfloor+1}v_i^{\lfloor\frac{h}{2}\rfloor+2}$ for all $i \in [1, n]$ and colors $2, 3, \ldots, |E(T_n \rhd_o H)| - n + 1$ to the remaining $|E(T_n \rhd_o H)| - n$ edges of $T_n \rhd_o H$, we can find a rainbow Steiner S-tree for every set S of three vertices of $T_n \rhd_o H$. Hence, $srx_3(T_n \rhd_o H) \leq |E(T_n \rhd_o H)| - n + 1$, a contradiction.

If $o \notin V(C_h)$, then observe that any choice of vertex o makes the cycle C_h satisfy either Subcase 2.1 or Subcase 2.2 as given in Observation 2.1. If Subcase 2.1 holds, then there exists an edge of C_h, say e, which is not contained in a shortest $o - w_i$ path for all $i \in [1, h]$ by Observation 2.1. Thus, by using a similar edge-coloring as case $o \in V(C_h)$, we will obtain a contradiction. If Subcase 2.2 holds, then $w_{\lfloor\frac{h}{2}\rfloor+1}w_{\lfloor\frac{h}{2}\rfloor+2}$ is not contained in a shortest $o - w_i$ path for all $i \in [1, h]$. Thus, by using a similar edge-coloring as case $o \in V(C_h)$, we will obtain a contradiction. □

Following Theorem 2.2, an immediate question arises: What is the sharp upper bound for $srx_3(T_n \rhd_o H)$ where $srx_3(T_n \rhd_o H) \neq |E(T_n \rhd_o H)|$? The answer of this question is given in Theorem 2.3. Before we proceed to this theorem, we first verify the following lemma.

Lemma 2.1. *Let H_1 and H_2 be connected graphs which admit strong* 3*-rainbow colorings c_1 and c_2, respectively, so that $c_1(E(H_1)) \cap c_2(E(H_2)) = \emptyset$. Then for any two vertices $u \in V(H_1)$ and $v \in V(H_2)$, the edge-coloring of the edge-colored graph G obtained from H_1 and H_2 by identifying u and v is a strong* 3*-rainbow coloring of G.*

Proof. By the assumption, for any subset S with $|S| = 3$ which is contained in $V(H_1)$ or $V(H_2)$, there exists a rainbow Steiner S-tree. Thus, without loss of generality, we may assume that $S \cap V(H_1) = \{x_1\}$ and $S \cap V(H_2) = \{x_2, x_3\}$. Observe that there exist a rainbow $u - x_1$ geodesic T_1 in H_1 and a rainbow Steiner $\{v, x_2, x_3\}$-tree T_2 in H_2. Since $u = v$ and $c_1(E(H_1)) \cap c_2(E(H_2)) = \emptyset$, the tree $T = T_1 \cup T_2$ is a rainbow Steiner S-tree. □

Theorem 2.3. *Let n and m be two integers at least* 3. *Let T_n be a tree of order n, H be a connected graph of order m, and o be an arbitrary vertex of H. Then*

$$srx_3(T_n \rhd_o H) \leq n\,(srx_3(H)) + n - 1.$$

Proof. We define an edge-coloring $c : E(T_n \rhd_o H) \to [1, n\,(srx_3(H)) + n - 1]$ as follows.

i. Assign colors $1, 2, \ldots, n - 1$ to the edges of T_n.

ii. For each $i \in [1, n]$, assign $srx_3(H)$ colors which are not used for $E(T_n)$ to the edges of H^i, so that each edge-coloring of H^i is a strong 3-rainbow coloring and $c(E(H^i)) \cap c(E(H^j)) = \emptyset$ for all $j \in [1, n]$ with $i \neq j$.

By the definition and using Lemma 2.1 repeatedly, the edge-coloring c is clearly a strong 3-rainbow coloring of $T_n \rhd_o H$. Thus, the theorem holds. □

Now, let us prove the sharpness of the upper bound in Theorem 2.3. Let m be an integer with $m \geq 3$. A *ladder* L_m is a

Cartesian product of a P_m and a P_2, where P_m is a path of order m. Let $V(L_m) = \{w_i : i \in [1, 2m]\}$ and $E(L_m) = \{w_i w_{i+1} : i \in [1, m-1] \cup [m+1, 2m-1]\} \cup \{w_i w_{i+m} : i \in [1, m]\}$. A *triangular ladder* [16] of order $2m$, denoted by TL_m, is a graph obtained from L_m by adding the edges $w_i w_{i+m+1}$ for $i \in [1, m-1]$. In the following theorem, we determine the strong 3-rainbow index of TL_m.

Theorem 2.4. *For $m \geq 3$, let TL_m be a triangular ladder of order $2m$. Then $srx_3(TL_m) = m$.*

Proof. It is easy to check that $sdiam_3(TL_m) = m$. Hence, $srx_3(TL_m) \geq m$ by Eq. (1). Next, we show that $srx_3(TL_m) \leq m$ by defining a strong 3-rainbow coloring $c : E(TL_m) \to [1, m]$ which can be obtained by assigning colors i to the edges $w_i w_{i+1}$ and $w_{i+m} w_{i+m+1}$ for $i \in [1, m-1]$ and color m to the edges $w_i w_{i+m}$ for $i \in [1, m]$ and $w_i w_{i+m+1}$ for $i \in [1, m-1]$. Now, we show that c is a strong 3-rainbow coloring of TL_m. Let S be a set of three vertices of TL_m. By symmetry, we consider two cases.

Case 1. $S = \{w_i, w_j, w_k\}$ for $i, j, k \in [1, m]$ with $i < j < k$

Then a tree T with $E(T) = \{w_l w_{l+1} : l \in [i, k-1]\}$ is a rainbow Steiner S-tree.

Case 2. $S = \{w_i, w_j, w_k\}$ for $i, j \in [1, m]$, $i < j$, and $k \in [m+1, 2m]$

If $k < i + m$, then a tree T with $E(T) = \{w_k w_{k-m}\} \cup \{w_l w_{l+1} : l \in [k-m, j-1]\}$ is a rainbow Steiner S-tree. If $i + m \leq k \leq j + m$, then a tree T with $E(T) = \{w_k w_{k-m}\} \cup \{w_l v_{l+1} : l \in [i, j-1]\}$ is a rainbow Steiner S-tree. If $k > j + m$, then a tree T with $E(T) = \{w_l w_{l+1} : l \in [i, k-m-2]\} \cup \{w_k w_{k-m-1}\}$ is a rainbow Steiner S-tree. □

Figure 1 gives an example of a strong 3-rainbow coloring of TL_5.

The *degree* $d_G(v)$ of a vertex v in G is the number of neighbours of v. The next theorem shows that $srx_3(T_n \rhd_o TL_m)$ attains the upper bound in Theorem 2.3.

Theorem 2.5. *Let n and m be two integers at least 3. Let T_n be a tree of order n, TL_m be a triangular ladder of order $2m$, and o be a vertex of TL_m with $d_{TL_m}(o) = 3$. Then $srx_3(T_n \rhd_o TL_m) = nm + n - 1$.*

Proof. By using Theorems 2.3 and 2.4, $srx_3(T_n \rhd_o TL_m) \leq nm + n - 1$.

Note that w_1 and w_{2m} are two vertices of TL_m which have degree 3. By symmetry, we consider $o = w_1$. Suppose that $srx_3(T_n \rhd_o L_m) \leq n + nm - 2$. Then there exists a strong 3-rainbow coloring $c : E(T_n \rhd_o TL_m) \to [1, n + nm - 2]$. For each $i \in [1, n]$, let $A_i = \{v_i^p v_i^{p+1} : p \in [1, m-1]\} \cup \{v_i^1 v_i^{1+m}\}$. The following properties hold.

(A1) $c(A_i) \cap c(E(T_n)) = \emptyset$ for all $i \in [1, n]$

Suppose that there exist $e \in A_i$ for some $i \in [1, n]$ and $f \in E(T_n)$ such that $c(e) = c(f)$. Let $e = xy$ and $f = uv$, and assume that $d(v_i^1, u) < d(v_i^1, v)$. Observe that the rainbow Steiner $\{x, y, v\}$-tree must contains edges e and f, but $c(e) = c(f)$, a contradiction.

(A2) $c(A_i) \cap c(A_j) = \emptyset$ for all $i, j \in [1, n]$ with $i \neq j$

Suppose that there exists $e \in A_i$ and $f \in A_j$ for some $i, j \in [1, n]$, $i \neq j$, such that $c(e) = c(f)$. Let $e = xy$ and $f = uv$, and assume that $d(v_j^1, u) < d(v_j^1, v)$. By using a similar argument as in the proof of (A1), we will obtain a contradiction.

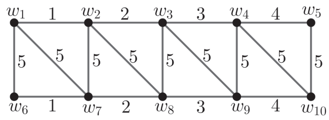

Fig. 1 A strong 3-rainbow coloring of TL_5.

Note that $|c(A_i)| = m$ for each $i \in [1, n]$. Hence, $\sum_{i=1}^{n} |c(A_i)| \geq nm$ by (A2). It follows by (A1) that $|c(E(T_n))| \leq n - 2$, contradicts Eq. (2). Thus, $srx_3(T_n \rhd_o TL_m) \geq nm + n - 1$. □

3. The Strong 3-rainbow Index of $T_n \rhd_o H$ for Some Connected Graphs H

The value of $srx_3(T_n \rhd_o H)$ is not only affected by the structure or the size of $T_n \rhd_o H$, but also can be affected by the choice of vertex $o \in V(H)$. In this section, we provide some graphs $T_n \rhd_o H$ whose srx_3 is affected or not affected by the choice of vertex o.

3.1 The Strong 3-rainbow Index of $T_n \rhd_o W_m$

Let m be an integer with $m \geq 3$. A *wheel* W_m of order $m + 1$ is a graph constructed by joining a vertex to every vertex of a cycle C_m. Let $V(W_m) = \{w_i : i \in [1, m+1]\}$ such that $E(W_m) = \{w_1 w_i : i \in [2, m+1]\} \cup \{w_i w_{i+1} : i \in [2, m+1]$ and $w_{m+2} = w_2\}$. The vertex w_1 is called the *center vertex* of W_m. For each $i \in [2, m+1]$, edge $w_1 w_i$ is called the *spoke* of W_m. In Ref. [2], the first and second authors studied the srx_3 of W_m. The results are given in Lemma 3.1 and Theorem 3.1. To make it easier for the readers, we also provide the proof of these results.

Lemma 3.1. *[2] For $m \geq 4$, let W_m be a wheel of order $m + 1$ which admits a strong 3-rainbow coloring. Then any color is assigned to at most two spokes $w_1 w_i$ and $w_1 w_j$ where $w_i w_j \in E(W_m)$.*

Proof. Suppose that there are three spokes of W_m, $w_1 w_i$, $w_1 w_j$, and $w_1 w_k$, which are colored the same. Without loss of generality, assume that $d_{C_m}(w_i, w_j) \geq 2$. Observe that the rainbow Steiner $\{w_1, w_i, w_j\}$-tree must contain spokes $w_1 w_i$ and $w_1 w_j$, but these two spokes are colored the same, a contradiction. □

Theorem 3.1. *[2] For $m \geq 3$, let W_m be a wheel of order $m + 1$. Then*

$$srx_3(W_m) = \begin{cases} 3, & \text{for } m = 4; \\ \lceil \frac{m}{2} \rceil, & \text{otherwise.} \end{cases}$$

Proof. For $m = 3$, $sdiam_3(W_3) = 2$. Thus, $srx_3(W_3) \geq 2$ by Eq. (1). Now, we show that $srx_3(W_3) \leq 2$ by defining a strong 3-rainbow coloring of W_3 as shown in **Fig. 2**.

For $m = 4$, suppose that $srx_3(W_4) \leq 2$. Then there exists a strong 3-rainbow coloring $c : E(W_4) \to [1, 2]$. Observe that we need at least two colors to color all spokes of W_4 by Lemma 3.1. Thus, without loss of generality, let $c(w_1 w_2) = c(w_1 w_3) = 1$ and $c(w_1 w_4) = c(w_1 w_5) = 2$. By considering $\{w_1, w_2, w_3\}$, $\{w_2, w_3, w_4\}$, and $\{w_3, w_4, w_5\}$, successively, $c(w_2 w_3) = 2$, $c(w_3 w_4) = 1$, and $c(w_4 w_5) = 2$. However, there is no rainbow Steiner $\{w_1, w_4, w_5\}$-tree, a contradiction. Thus, $srx_3(W_4) \geq 3$. Next, we show that $srx_3(W_4) \leq 3$ by defining a strong 3-rainbow coloring of W_4 as shown in Fig. 2.

Let $m \geq 5$. Thus, $srx_3(W_m) \geq \lceil \frac{m}{2} \rceil$ by Lemma 3.1. Next, we show that $srx_3(W_m) \leq \lceil \frac{m}{2} \rceil$ by defining a strong 3-rainbow coloring $c : E(W_m) \to [1, \lceil \frac{m}{2} \rceil]$ as follows.

i. Assign colors $\lfloor \frac{i}{2} \rfloor$ to the spokes $w_1 w_i$ for $i \in [2, m+1]$.

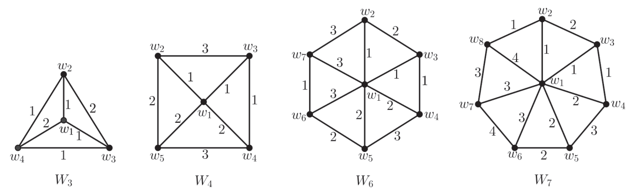

Fig. 2 Strong 3-rainbow colorings of W_3, W_4, W_6, and W_7.

ii. Define $c(w_iw_{i+1}) = c(w_1w_{i+2})$ for even $i \in [2, m+1]$ and $c(w_iw_{i+1}) = c(w_1w_i)$ for odd $i \in [2, m+1]$.

Now, we show that c is a strong 3-rainbow coloring of W_m. Let S be a set of three vertices of W_m. Let $i, j, k \in [2, m+1]$ with $i \neq j$, $i \neq k$, and $j \neq k$. We consider two cases.

Case 1. The vertices of S belong to the cycle C_m

Without loss of generality, let $S = \{w_i, w_j, w_k\}$. If $d(S) = 2$, then a path of length 2 which contains all vertices of S is a rainbow Steiner S-tree. If i is even ($i \neq m+1$ if m is odd), $j = i+1$, and $k = i+3$, then a tree T with $E(T) = \{w_iw_{i+1}, w_{i+1}w_{i+2}, w_{i+2}w_{i+3}\}$ is a rainbow Steiner S-tree. If i is even, $j = i+1$, and $k \geq i+4$ or $k \leq i-2$ (or $i = m+1$ if m is odd, $j = 2$, and $k = 4$), then a tree T with $E(T) = \{w_1w_i, w_iw_j, w_1w_k\}$ is a rainbow Steiner S-tree. For other values of i, j, and k, a tree T with $E(T) = \{w_1w_i, w_1w_j, w_1w_k\}$ is a rainbow Steiner S-tree.

Case 2. Two vertices of S belong to the cycle C_m

Without loss of generality, let $S = \{w_1, w_i, w_j\}$. If i is even and $j = i+1$, then a tree T with $E(T) = \{w_1w_i, w_iw_{i+1}\}$ is a rainbow Steiner S-tree. For other values of i and j, a tree T with $E(T) = \{w_1w_i, w_1w_j\}$ is a rainbow Steiner S-tree. □

Our first result in this subsection is the $srx_3(T_n \rhd_o W_m)$ where o is the center vertex of W_m.

Theorem 3.2. *Let n and m be two integers with $n \geq 3$ and $m \geq 4$. Let T_n be a tree of order n, W_m be a wheel of order $m+1$, and o be the center vertex of W_m. Then $srx_3(T_n \rhd_o W_m) = n\lceil \frac{m}{2} \rceil + n - 1$.*

Proof. Let c be a strong 3-rainbow coloring of $T_n \rhd_o W_m$. First, we verify two properties.

(B1) $c(v_i^1 v_i^p) \notin c(E(T_n))$ for all $i \in [1, n]$ and $p \in [2, m+1]$

Suppose that there exist $v_i^1 v_i^p \in E(W_m^i)$ for some $i \in [1, n]$ and $p \in [2, m+1]$ and $f \in E(T_n)$ such that $c(v_i^1 v_i^p) = c(f)$. Let $f = uv$ and assume that $d(v_i^1, u) < d(v_i^1, v)$. Observe that the rainbow Steiner $\{v_i^1, v_i^p, v\}$-tree must contain edges $v_i^1 v_i^p$ and f, but $c(v_i^1 v_i^p) = c(f)$, a contradiction.

(B2) $c(v_i^1 v_i^p) \neq c(v_j^1 v_j^q)$ for all $i, j \in [1, n]$, $i \neq j$, and $p, q \in [2, m+1]$

By considering $\{v_i^1, v_i^p, v_j^q\}$ for all $i, j \in [1, n]$, $i \neq j$, and $p, q \in [2, m+1]$, it is clear that $c(v_i^1 v_i^p) \neq c(v_j^1 v_j^q)$.

Thus, by using Eq. (2), Lemma 3.1, (B1), and (B2), $srx_3(T_n \rhd_o W_m) \geq n\lceil \frac{m}{2} \rceil + n - 1$.

Next, we prove the upper bound. For $m \geq 5$, $srx_3(T_n \rhd_o W_m) \leq n\lceil \frac{m}{2} \rceil + n - 1$ by Theorems 2.3 and 3.1. For $m = 4$, we show that $srx_3(T_n \rhd_o W_4) \leq 3n - 1$ by defining a strong 3-rainbow coloring $c : E(T_n \rhd_o W_4) \to [1, 3n-1]$ as follows.

i. Assign colors $1, 2, \ldots, n-1$ to the edges of T_n.

ii. For each $i \in [1, n]$, assign colors $n+2(i-1)$ to the edges $v_i^1 v_i^2$, $v_i^1 v_i^3$, and $v_i^4 v_i^5$, and colors $n+1+2(i-1)$ to the edges $v_i^1 v_i^4$, $v_i^1 v_i^5$, and $v_i^2 v_i^3$.

iii. Assign colors $n+2i$ to the edges $v_i^3 v_i^4$ and $v_i^5 v_i^2$ for $i \in [1, n-1]$ and color n to the edges $v_n^3 v_n^4$ and $v_n^5 v_n^2$.

By the coloring above, it is easy to find a rainbow Steiner S-tree for every set S of three vertices of $T_n \rhd_o W_4$. □

Following the theorem above, we obtain that $srx_3(T_n \rhd_o W_m)$ (o is the center vertex of W_m) attains the upper bound in Theorem 2.3 for $m \geq 5$.

Now, consider graphs $T_n \rhd_o W_m$ where o is not the center vertex of W_m. Without loss of generality, we may assume that $o = w_2$. This assumption applies until the end of this subsection. First, we verify the following observation.

Observation 3.1. *Let n and m be two integers at least 3. Let $o = w_2 \in V(W_m)$. If c is a strong 3-rainbow coloring of $T_n \rhd_o W_m$, then*

(i) $c(v_i^2 v_i^p) \notin c(E(T_n))$ for all $i \in [1, n]$ and $p \in \{1, 3, m+1\}$;
(ii) $c(v_i^1 v_i^p) \notin c(E(T_n))$ for all $i \in [1, n]$ and $p \in [4, m]$;
(iii) $c(v_i^2 v_i^3) \neq c(v_i^2 v_i^{m+1})$ for all $i \in [1, n]$ and $m \geq 4$;
(iv) $c(v_i^2 v_i^p) \neq c(v_j^2 v_j^q)$ for all $i, j \in [1, n]$, $i \neq j$, and $p, q \in \{1, 3, m+1\}$;
(v) $c(v_i^1 v_i^2) \neq c(v_j^1 v_j^p)$ for all $i, j \in [1, n]$ and $p \in [5, m-1]$; and
(vi) $c(v_i^1 v_i^p) \neq c(v_j^1 v_j^q)$ for all $i, j \in [1, n]$, $i \neq j$, and $p, q \in [5, m-1]$.

Proof. We distinguish several cases.

(i) Let $f = uv$ be an arbitrary edge of T_n and assume that $d(v_i^2, u) < d(v_i^2, v)$. By considering $\{v_i^2, v_i^p, v\}$ for $p \in \{1, 3, m+1\}$, $c(v_i^2 v_i^p) \neq c(f)$. Furthermore, $c(v_i^2 v_i^p) \notin c(E(T_n))$.

(ii) An argument similar to that used in the proof of (i) will verify that $c(v_i^1 v_i^p) \notin c(E(T_n))$ for all $i \in [1, n]$ and $p \in [4, m]$.

(iii) By considering $\{v_i^2, v_i^3, v_i^{m+1}\}$ for all $i \in [1, n]$, it is clear that $c(v_i^2 v_i^3) \neq c(v_i^2 v_i^{m+1})$.

(iv) By considering $\{v_i^2, v_i^p, v_j^q\}$ for all $i, j \in [1, n]$, $i \neq j$, and $p, q \in \{1, 3, m+1\}$, $c(v_i^2 v_i^p) \neq c(v_j^2 v_j^q)$.

(v) By considering $\{v_i^1, v_i^2, v_j^p\}$ for all $i, j \in [1, n]$ and $p \in [5, m-1]$, $c(v_i^1 v_i^2) \neq c(v_j^1 v_j^p)$.

(vi) By considering $\{v_i^2, v_i^p, v_j^q\}$ for all $i, j \in [1, n]$, $i \neq j$, and $p, q \in [5, m-1]$, $c(v_i^1 v_i^p) \neq c(v_j^1 v_j^q)$. □

Theorem 3.3. *Let n and m be two integers at least 3. Let T_n be a tree of order n, W_m be a wheel of order $m+1$, and o is not the center vertex of W_m. Then*

$$srx_3(T_n \rhd_o W_m) = \begin{cases} 2n+1, & \text{for } m = 3; \\ 3n, & \text{for } m \in [4, 5]; \\ \lceil \frac{m-5}{2} \rceil n + 2n, & \text{for even } m \geq 6; \\ \lceil \frac{m-5}{2} \rceil n + 2n + 1, & \text{for odd } m \geq 6. \end{cases}$$

Proof. Recall that we assume $o = w_2$. We consider three cases.

Case 1. $m = 3$

Suppose that $srx_3(T_n \rhd_o W_3) \leq 2n$. Then there exists a strong 3-rainbow coloring $c : E(T_n \rhd_o W_3) \to [1, 2n]$. By using Eq. (2) and Observation 3.1 (i) and (iv), we need at least $2n-1$ distinct colors to color edges of T_n and edges $v_i^1 v_i^2$ for all $i \in [1, n]$. This implies we have at most one color left, say color a. Next, consider edge $v_1^2 v_1^3$. By using Observation 3.1 (i) and (iv), $c(v_1^2 v_1^3) \in \{c(v_1^1 v_1^2), a\}$. If $c(v_1^2 v_1^3) = a$, then $c(v_2^2 v_2^3) = c(v_2^1 v_2^2)$ by Observation 3.1 (i) and (iv). By considering $\{v_2^1, v_2^3, v_i^1\}$ for all $i \in [1, n]$ with $i \neq 2$, $c(v_2^1 v_2^3) \notin c(E(T_n)) \cup \{c(v_i^1 v_i^2)\}$. This forces $c(v_2^1 v_2^3) = a$. However, there is no rainbow Steiner $\{v_2^1, v_2^3, v_1^3\}$-tree, a contradiction. Thus, $c(v_1^2 v_1^3) = c(v_1^1 v_1^2)$. Similarly, $c(v_1^2 v_1^4) = c(v_1^1 v_1^2)$. Now, we have $c(v_1^1v_1^2) = c(v_1^2v_1^3) = c(v_1^2v_1^4)$. By considering $\{v_1^1, v_1^p, v_i^1\}$ and $\{v_1^3, v_1^4, v_i^1\}$ for all $i \in [2,n]$ and $p \in [3,4]$, we obtain $c(v_1^1v_1^3) = c(v_1^1v_1^4) = c(v_1^3v_1^4) = a$. However, there is no rainbow Steiner $\{v_1^1, v_1^3, v_1^4\}$-tree, a contradiction. Thus, $srx_3(T_n \rhd_o W_3) \geq 2n+1$.

Next, we show that $srx_3(T_n \rhd_o W_3) \leq 2n+1$ by defining a strong 3-rainbow coloring $c : E(T_n \rhd_o W_3) \to [1, 2n+1]$. We first assign colors $1, 2, \ldots, n-1$ to the edges of T_n. For each $i \in [1,n]$, assign colors $i+n-1$ to the edges $v_i^2v_i^p$ for $p \in \{1,3,4\}$, color $2n$ to the edges $v_i^1v_i^p$ for $p \in [3,4]$, and color $2n+1$ to the edges $v_i^3v_i^4$. By this coloring, it is easy to show that for every set S of three vertices of $T_n \rhd_o W_3$, there exists a rainbow Steiner S-tree. **Figure 3** gives an example of a strong 3-rainbow coloring of $P_4 \rhd_o W_3$.

Case 2. $m \in [4,5]$

Suppose that $srx_3(T_n \rhd_o W_m) \leq 3n-1$. Then there exists a strong 3-rainbow coloring $c : E(T_n \rhd_o W_m) \to [1, 3n-1]$. By using Eq. (2) and Observation 3.1 (i), (iii), and (iv), we need at least $3n-1$ distinct colors to color edges of T_n and edges $v_i^2v_i^3$ and $v_i^2v_i^{m+1}$ for all $i \in [1,n]$. This implies we have used all available colors. For further steps, let $i \in [2,n]$, $p \in [4,m]$, and $q \in \{3, m+1\}$. Observe that the rainbow Steiner $\{v_1^1, v_1^p, v_i^q\}$-tree must contains edges $v_1^1v_1^2$, $v_1^1v_1^p$, and $v_i^2v_i^q$, which means $\{c(v_1^1v_1^2), c(v_1^1v_1^p)\} \nsubseteq c(E(T_n)) \cup \{c(v_i^2v_i^q)\}$. This forces $\{c(v_1^1v_1^2), c(v_1^1v_1^p)\} \subseteq \{c(v_1^2v_1^3), c(v_1^2v_1^{m+1})\}$, where $c(v_1^1v_1^2) \neq c(v_1^1v_1^p)$. If $c(v_1^1v_1^2) = c(v_1^2v_1^3)$ and $c(v_1^1v_1^p) = c(v_1^2v_1^{m+1})$, then consider $\{v_1^1, v_1^3, v_i^q\}$. We obtain that $c(v_1^1v_1^3) \notin c(E(T_n)) \cup \{c(v_1^1v_1^2), c(v_i^2v_i^q)\}$, implying that $c(v_1^1v_1^3) = c(v_1^2v_1^{m+1})$. By considering $\{v_1^3, v_1^4, v_i^q\}$, we also have $c(v_1^3v_1^4) = c(v_1^2v_1^{m+1})$. However, there is no rainbow Steiner $\{v_1^1, v_1^3, v_1^4\}$-tree since $c(v_1^1v_1^3) = c(v_1^1v_1^4) = c(v_1^3v_1^4)$, a contradiction. Similarly, if $c(v_1^1v_1^2) = c(v_1^2v_1^{m+1})$ and $c(v_1^1v_1^p) = c(v_1^2v_1^3)$, then there is no rainbow Steiner $\{v_1^1, v_1^m, v_1^{m+1}\}$-tree, a contradiction. Thus, $srx_3(T_n \rhd_o W_m) \geq 3n$.

Next, we show that $srx_3(T_n \rhd_o W_m) \leq 3n$ by defining a strong 3-rainbow coloring $c : E(T_n \rhd_o W_m) \to [1, 3n]$. We first assign colors $1, 2, \ldots, n-1$ to the edges of T_n. For $m = 4$ and each $i \in [1,n]$, assign colors $n + 2(i-1)$ to the edges $v_i^1v_i^2$, $v_i^1v_i^3$, $v_i^3v_i^4$, and $v_i^5v_i^2$, colors $n+1+2(i-1)$ to the edges $v_i^1v_i^4$, $v_i^1v_i^5$, and $v_i^2v_i^3$, and color $3n$ to the edges $v_i^4v_i^5$. For $m = 5$ and each $i \in [1,n]$, assign colors $n+2(i-1)$ to the edges $v_i^1v_i^2$, $v_i^2v_i^3$, and $v_i^5v_i^6$, colors $n+1+2(i-1)$ to the edges $v_i^1v_i^3$, $v_i^1v_i^4$, $v_i^4v_i^5$, and $v_i^6v_i^2$, and color $3n$ to the edges $v_i^1v_i^5$, $v_i^1v_i^6$, and $v_i^3v_i^4$. By this coloring, it is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $T_n \rhd_o W_m$. **Figure 4** gives examples of strong 3-rainbow colorings of $P_4 \rhd_o W_4$ and $P_4 \rhd_o W_5$.

Case 3. $m \geq 6$

For odd m, suppose that $srx_3(T_n \rhd_o W_m) \leq \lceil \frac{m-5}{2} \rceil n + 2n$. Then there exists a strong 3-rainbow coloring $c : E(T_n \rhd_o W_m) \to [1, \lceil \frac{m-5}{2} \rceil n + 2n]$. By symmetry, we consider the following two subcases.

- There exists a fixed $i \in [1,n]$ such that $c(v_i^1v_i^4) = c(v_i^1v_i^5)$

  Without loss of generality, let $i = 1$. First, consider spokes $v_1^1v_1^p$ for $p \in \{2\} \cup [6, m-1]$. By using Lemma 3.1, these spokes can not be colored with $c(v_1^1v_1^4)$ and $c(v_1^1v_1^2) \neq c(v_1^1v_1^q)$ for all $q \in [6, m-1]$. Thus, we need at least $\lceil \frac{m-6}{2} \rceil + 2 = \lceil \frac{m-5}{2} \rceil + 2$ (since m is odd) distinct colors to color spokes $v_1^1v_1^p$ for all $p \in \{2\} \cup [4, m-1]$. This implies we have at most $\lceil \frac{m-5}{2} \rceil n + 2n - \left(\lceil \frac{m-5}{2} \rceil + 2\right) = \left(\lceil \frac{m-5}{2} \rceil + 2\right)(n-1)$ colors left. Next, consider all edges of T_n and spokes $v_i^1v_i^p$ for all $i \in [2,n]$ and $p \in \{2\} \cup [5, m-1]$. By using Observation 3.1, we need at least $\left(\lceil \frac{m-5}{2} \rceil + 2\right)(n-1)$ new distinct colors to color these edges, implying that we have used all remaining colors. This forces for each $i \in [2,n]$, we use exactly $\lceil \frac{m-5}{2} \rceil$ colors to color spokes $v_i^1v_i^p$ for all $p \in [5, m-1]$, where every color is assigned to exactly two spokes. Now, consider spoke $v_2^1v_2^4$. By using Lemma 3.1 and Observation 3.1 (ii), $c(v_2^1v_2^4) \notin c(E(T_n))$ and $c(v_2^1v_2^4) \neq c(v_2^1v_2^p)$ for all $p \in \{2\} \cup [5, m-1]$. This forces $c(v_2^1v_2^4) = c(v_j^1v_j^p)$ for some $j \in [1,n]$ with $j \neq 2$ and $p \in \{2\} \cup [5, m-1]$. However, there is no rainbow Steiner $\{v_2^1, v_2^4, v_j^q\}$-tree for $q \in [5, m-1]$ since the tree must contain spokes $v_2^1v_2^4$, $v_j^1v_j^2$, and $v_j^1v_j^q$, a contradiction.

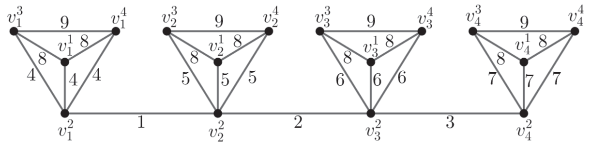

Fig. 3 A strong 3-rainbow coloring of $P_4 \rhd_o W_3$.

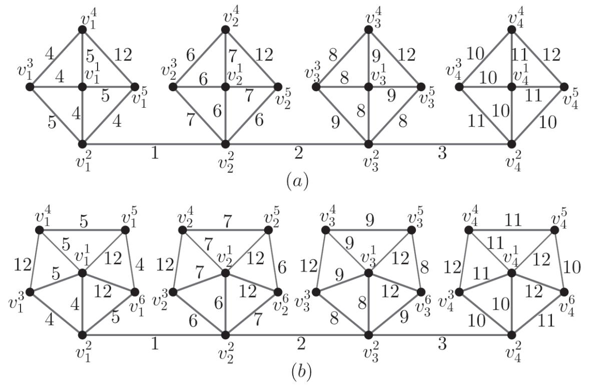

Fig. 4 Strong 3-rainbow colorings of (a) $P_4 \rhd_o W_4$ and (b) $P_4 \rhd_o W_5$.

The subcase above implies the following subcase.

- $c(v_i^1v_i^4) \neq c(v_i^1v_i^5)$ and $c(v_i^1v_i^m) \neq c(v_i^1v_i^{m-1})$ for all $i \in [1,n]$

  By using Eq. (2) and Observation 3.1, we need at least $\lceil \frac{m-5}{2} \rceil n + 2n - 1$ distinct colors to color edges of T_n and spokes $v_i^1v_i^p$ for all $i \in [1,n]$ and $p \in \{2\} \cup [5, m-1]$. This implies we have at most one color left, say color a. Next, consider spoke $v_1^1v_1^4$. It follows by Lemma 3.1 and Observation 3.1 (ii) that $c(v_1^1v_1^4) \notin c(E(T_n))$ and $c(v_1^1v_1^4) \neq c(v_1^1v_1^p)$ for all $p \in \{2\} \cup [6, m-1]$. This forces $c(v_1^1v_1^4) = a$ or $c(v_1^1v_1^4) = c(v_j^1v_j^p)$ for some $j \in [2,n]$ and $p \in \{2\} \cup [5, m-1]$. If $c(v_1^1v_1^4) = c(v_j^1v_j^p)$, then there is no rainbow Steiner $\{v_1^1, v_1^4, v_j^q\}$-tree for $q \in [5, m-1]$ since the tree must contain spokes $v_1^1v_1^4$, $v_j^1v_j^2$, and $v_j^1v_j^q$. Hence, $c(v_1^1v_1^4) = a$. Similarly, $c(v_1^1v_1^m) = a$. Therefore, $c(v_1^1v_1^4) = c(v_1^1v_1^m) = a$, contradicts Lemma 3.1.

Thus, $srx_3(T_n \rhd_o W_m) \geq \lceil \frac{m-5}{2} \rceil n + 2n + 1$ for odd m. Similarly, we can also prove the lower bound for even m.

Now, we prove the upper bound. Let $x = \lceil \frac{m-5}{2} \rceil + 1$. For even m, we define an edge-coloring $c : E(T_n \rhd_o W_m) \to [1, \lceil \frac{m-5}{2} \rceil n + 2n]$ as follows.

i. Assign colors $1, 2, \ldots, n-1$ to the edges of T_n.
ii. For each $i \in [1,n]$, assign colors $\lfloor \frac{p}{2} \rfloor + n - 1 + x(i-1)$ to the

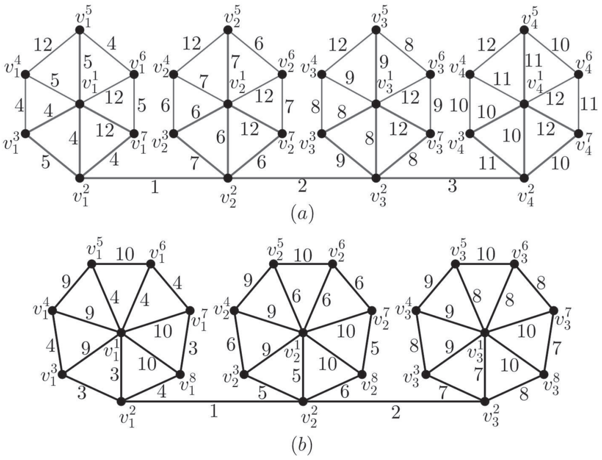

Fig. 5 Strong 3-rainbow colorings of (a) $P_4 \rhd_o W_6$ and (b) $P_3 \rhd_o W_7$.

spokes $v_i^1 v_i^p$ for $p \in [2, m-1]$ and color $n + xn$ to the spokes $v_i^1 v_i^m$ and $v_i^1 v_i^{m+1}$.

iii. For $m = 6$ and each $i \in [1, n]$, define $c(v_i^p v_i^{p+1}) = c(v_i^1 v_i^{p+2})$ for $p \in \{2, 4\}$, $c(v_i^6 v_i^7) = c(v_i^2 v_i^3)$, and $c(v_i^p v_i^{p+1}) = c(v_i^1 v_i^2)$ for $p \in \{3, 5, 7\}$.

iv. For $m \geq 8$ and each $i \in [1, n]$, define $c(v_i^p v_i^{p+1}) = c(v_i^1 v_i^{p+2})$ for even $p \in [2, m]$, $c(v_i^p v_i^{p+1}) = c(v_i^1 v_i^p)$ for odd $p \in [3, m-1]$, and $c(v_i^{m+1} v_i^2) = c(v_i^{m-1} v_i^m)$.

For odd m, we define an edge-coloring $c : E(T_n \rhd_o W_m) \to [1, \lceil \frac{m-5}{2} \rceil n + 2n + 1]$ as follows.

i. Assign colors $1, 2, \ldots, n-1$ to the edges of T_n.

ii. For each $i \in [1, n]$, assign color $n + x(i-1)$ to the spokes $v_i^1 v_i^2$, colors $\lceil \frac{p}{2} \rceil + n - 2 + x(i-1)$ to the spokes $v_i^1 v_i^p$ for $p \in [5, m-1]$, color $n + xn$ to the spokes $v_i^1 v_i^3$ and $v_i^1 v_i^4$, and color $n + xn + 1$ to the spokes $v_i^1 v_i^m$ and $v_i^1 v_i^{m+1}$.

iii. For each $i \in [1, n]$, define $c(v_i^p v_i^{p+1}) = c(v_i^1 v_i^p)$ for even $p \in [2, m-1]$, $c(v_i^{m+1} v_i^2) = n + 1 + x(i-1)$, and $c(v_i^p v_i^{p+1}) = c(v_i^1 v_i^{p+2})$ for odd $p \in [3, m]$.

By the colorings above, it is not hard to show that there exists a rainbow Steiner S-tree for every set S of three vertices of $T_n \rhd_o W_m$. **Figure 5** gives examples of strong 3-rainbow colorings of $P_4 \rhd_o W_6$ and $P_3 \rhd_o W_7$. □

Following Theorems 3.2 and 3.3, the choice of vertex $o \in V(W_m)$ affects the value of $srx_3(T_n \rhd_o W_m)$. However, there are some graphs H such that $srx_3(T_n \rhd_o H)$ is the same for any choice of vertex $o \in V(H)$. For example, $srx_3(T_n \rhd_o T_m) = |E(T_n \rhd_o T_m)|$ for any vertex $o \in V(T_m)$ by Theorem 2.1. Our next two results also show that the choice of vertex $o \in V(H)$ does not effect the value of $srx_3(T_n \rhd_o H)$, where H is a ladder or a cycle.

3.2 The Strong 3-rainbow Index of $T_n \rhd_o L_m$

Before we proceed to the main result, we first provide the following theorem which has been studied in Ref. [2]. To make it easier for the readers, we also provide the proof of this theorem.

Theorem 3.4. *[2] For $m \geq 3$, let L_m be a ladder of order $2m$. Then $srx_3(L_m) = m$.*

Proof. Since $sdiam_3(L_m) = m$, $srx_3(L_m) \geq m$ by Eq. (1). Now, we show that $srx_3(L_m) \leq m$ by defining a strong 3-rainbow coloring $c : E(L_m) \to [1, m]$. This coloring can be obtained by assigning colors i to the edges $w_i w_{i+1}$ and $w_{i+m} w_{i+m+1}$ for $i \in [1, m-1]$ and color m to the edges $w_i w_{i+m}$ for $i \in [1, m]$. By using a similar argument as in the proof of Theorem 2.4, we can show that there exists a rainbow Steiner S-tree for every set S of three vertices of L_m. □

Now, we determine the strong 3-rainbow index of $T_n \rhd_o L_m$.

Theorem 3.5. *Let n and m be two integers at least 3. Let T_n be a tree of order n, L_m be a ladder of order $2m$, and o be an arbitrary vertex of L_m. Then $srx_3(T_n \rhd_o L_m) = nm + n - 1$.*

Proof. By using Theorems 2.3 and 3.4, $srx_3(T_n \rhd_o L_m) \leq nm + n - 1$.

Without loss of generality, let $o = w_s$ for some $s \in [1, m]$. For each $i \in [1, n]$, let $A_i = \{v_i^p v_i^{p+1} : p \in [1, m-1]\} \cup \{v_i^s v_i^{s+m}\}$. Let c be a strong 3-rainbow coloring of $T_n \rhd_o L_m$. By using a similar argument as in the proof of (A1) and (A2) in Theorem 2.5, we obtain the following properties.

(C1) $c(A_i) \cap c(E(T_n)) = \emptyset$ for all $i \in [1, n]$

(C2) $c(A_i) \cap c(A_j) = \emptyset$ for all $i, j \in [1, n]$ with $i \neq j$

Note that $|c(A_i)| \geq m$ for each $i \in [1, n]$. Hence, $srx_3(T_n \rhd_o L_m) \geq nm + n - 1$ by Eq. (2), (C1), and (C2). □

Following the theorem above, we obtain that $srx_3(T_n \rhd_o L_m)$ attains the upper bound in Theorem 2.3. Recall that $sdiam_3(T_n \rhd_o H)$ is the natural lower bound for $srx_3(T_n \rhd_o H)$ by Eq. (1). Consider graphs $P_n \rhd_o L_m$ where $o \in V(L_m)$ with $d_{L_m}(o) = 2$. We can check that $sdiam_3(P_n \rhd_o L_m) = 3m + n - 1$ for $n \geq 3$. Hence, $srx_3(P_n \rhd_o L_m) = sdiam_3(P_n \rhd_o L_m)$ for $n = 3$ by Theorem 3.5.

3.3 The Strong 3-rainbow Index of $T_n \rhd_o C_m$

For $m \geq 3$, let $V(C_m) = \{w_i : i \in [1, m]\}$ such that $E(C_m) = \{w_i w_{i+1} : i \in [1, m]$ and $w_{m+1} = w_1\}$. Consider graphs $T_n \rhd_o C_m$ where o is an arbitrary vertex of C_m. Without loss of generality, we may assume that $o = w_1$. This assumption applies until the end of this subsection. First, we verify the following observations which will be used to prove the lower bound for $srx_3(T_n \rhd_o C_m)$.

Observation 3.2. *For $m = 7$ or $m \geq 9$, if c is a strong 3-rainbow coloring of C_m, then no edge of C_m is colored the same.*

Proof. Suppose that there are two edges of C_m, say $v_1 v_2$ and $v_p v_{p+1}$ for some $p \in [2, m]$, which are colored the same. Note that $d_{C_m}(v_1, v_p) \leq \lfloor \frac{m}{2} \rfloor$. Hence, we only consider when $1 \leq p - 1 \leq \lfloor \frac{m}{2} \rfloor$. Observe that the rainbow Steiner $\{v_1, v_{\lceil \frac{p+1}{2} \rceil}, v_{p+1}\}$-tree is a tree T with $E(T) = \{v_l v_{l+1} : l \in [1, p]\}$ where no edge of the tree is colored the same, but $c(v_1 v_2) = c(v_p v_{p+1})$, a contradiction. □

Observation 3.3. *For $m \geq 4$, let c be a strong 3-rainbow coloring of $T_n \rhd_o C_m$. If $e \in E(C_m^i)$ for each $i \in [1, n]$, then $c(e) \notin c(E(T_n))$.*

Proof. Suppose that $c(e) \in c(E(T_n))$. Then there exists $f \in E(T_n)$ such that $c(e) = c(f)$. Let $e = xy$ and $f = uv$, and assume that $d(v_i^1, u) < d(v_i^1, v)$. Observe that the rainbow Steiner $\{x, y, v\}$-tree must contain edges e and f, but $c(e) = c(f)$, a contradiction. □

Observation 3.4. *Let m be an odd integer at least 3. Let c be a strong 3-rainbow coloring of $T_n \rhd_o C_m$. For each $i \in [1, n]$, let $A_i = E(C_m^i) \setminus \{v_i^{\lceil \frac{m}{2} \rceil} v_i^{\lceil \frac{m}{2} \rceil + 1}\}$. Then $c(A_i) \cap c(A_j) = \emptyset$ for all $i, j \in [1, n]$ with $i \neq j$.*

Proof. By considering $\{v_i^1, v_i^p, v_j^q\}$ for all $i, j \in [1, n]$, $i \neq j$, and $p, q \in [\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 1]$, $c(A_i) \cap c(A_j) = \emptyset$. □

Observation 3.5. *Let m be an even integer at least 4. Let c be a strong 3-rainbow coloring of $T_n \rhd_o C_m$. For each $i \in [1, n]$, let*

$A_i = E(C_m^i) \setminus \{v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}, v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2}\}$. *Then* $c(A_i) \cap c(A_j) = \emptyset$ *for all* $i, j \in [1, n]$ *with* $i \neq j$.

Proof. By considering $\{v_i^1, v_i^p, v_j^q\}$ for all $i, j \in [1, n]$, $i \neq j$, and $p, q \in \{\frac{m}{2}, \frac{m}{2} + 2\}$, $c(A_i) \cap c(A_j) = \emptyset$. □

Observation 3.6. *Let* m *be an even integer at least* 10. *Let* c *be a strong* 3*-rainbow coloring of* $T_2 \rhd_o C_m$. *Then at least three colors are needed to color edges* $v_1^{\frac{m}{2}} v_1^{\frac{m}{2}+1}$, $v_1^{\frac{m}{2}+1} v_1^{\frac{m}{2}+2}$, $v_2^{\frac{m}{2}} v_2^{\frac{m}{2}+1}$, *and* $v_2^{\frac{m}{2}+1} v_2^{\frac{m}{2}+2}$ *in* $T_2 \rhd_o C_m$.

Proof. Observe that the rainbow Steiner $\{v_1^{\frac{m}{2}}, v_1^{\frac{m}{2}+2}, v_2^{\frac{m}{2}+1}\}$-tree must contains edges $v_1^{\frac{m}{2}} v_1^{\frac{m}{2}+1}$, $v_1^{\frac{m}{2}+1} v_1^{\frac{m}{2}+2}$, and either $v_2^{\frac{m}{2}} v_2^{\frac{m}{2}+1}$ or $v_2^{\frac{m}{2}+1} v_2^{\frac{m}{2}+2}$. Hence, we need at least three colors to color these four edges in $T_2 \rhd_o C_m$. □

Observation 3.7. *Let* m *be an even integer at least* 10. *Let* c *be a strong* 3*-rainbow coloring of* $T_n \rhd_o C_m$. *For each* $i \in [1, n]$, *let* $c(v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}) = a_i$ *and* $c(v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2}) = b_i$. *Then* $\{a_i, b_i\} \neq \{a_j, b_j\}$ *for all* $i, j \in [1, n]$ *with* $i \neq j$.

Proof. An argument similar to that used in the proof of Observation 3.6 will verify that $\{a_i, b_i\} \neq \{a_j, b_j\}$ for all $i, j \in [1, n]$ with $i \neq j$. □

Observation 3.8. *Let* n *and* r *be two integers at least* 3 *and* m *be an even integer at least* 10. *Let* r *be the minimum number such that* $n \leq \frac{r(r-1)}{2}$. *If* c *is a strong* 3*-rainbow coloring of* $T_n \rhd_o C_m$, *then* r *is the minimum number of colors needed to color edges* $v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}$ *and* $v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2}$ *for all* $i \in [1, n]$.

Proof. Suppose that $r - 1$ is the maximum number of colors needed to color edges $v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}$ and $v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2}$ for all $i \in [1, n]$. Following Observation 3.7, we have at most $\binom{r-1}{2}$ color pairs to color all pairs of two edges $\{v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}, v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2}\}$ for all $i \in [1, n]$, where $\binom{r-1}{2}$ is the number of combinations of $r - 1$ colors taken 2 at a time. Note that $\binom{r-1}{2} = \frac{(r-1)!}{2!(r-3)!} = \frac{(r-1)(r-2)}{2}$. However, $n > \frac{(r-1)(r-2)}{2}$, this forces there are at least two pairs of two edges $\{v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}, v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2}\}$ and $\{v_j^{\frac{m}{2}} v_j^{\frac{m}{2}+1}, v_j^{\frac{m}{2}+1} v_j^{\frac{m}{2}+2}\}$ for some $i, j \in [1, n]$, $i \neq j$, such that $\{c(v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}), c(v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2})\} = \{c(v_j^{\frac{m}{2}} v_j^{\frac{m}{2}+1}), c(v_j^{\frac{m}{2}+1} v_j^{\frac{m}{2}+2})\}$, contradicts Observation 3.7. □

Now, we determine the strong 3-rainbow index of $T_n \rhd_o C_m$.

Theorem 3.6. *Let* n, m, *and* r *be three integers at least* 3. *Let* T_n *be a tree of order* n, C_m *be a cycle of order* m, *and* o *be an arbitrary vertex of* C_m. *Let* r *be the minimum number such that* $n \leq \frac{r(r-1)}{2}$. *Then*

$$srx_3(T_n \rhd_o C_m) = \begin{cases} 2n, & \text{for } m = 3; \\ 3n - 1, & \text{for } m = 4; \\ n(m-2), & \text{for } m \in \{5, 6, 8\}; \\ nm, & \text{for odd } m \geq 7; \\ nm - n + r - 1, & \text{for even } m \geq 10. \end{cases}$$

Proof. Recall that we assume $o = w_1$. We distinguish several cases based on m.

Case 1. m is odd

Subcase 1.1. $m = 3$

Suppose that $srx_3(T_n \rhd_o C_3) \leq 2n - 1$. Then there exists a strong 3-rainbow coloring $c : E(T_n \rhd_o C_3) \to [1, 2n - 1]$. For an arbitrary $i \in [1, n]$, consider edge $v_i^1 v_i^2$. Let $f = uv$ be an arbitrary edge of T_n and assume that $d(v_i^1, u) < d(v_i^1, v)$. By considering $\{v_i^1, v_i^2, v\}$, $c(v_i^1 v_i^2) \neq c(f)$. Furthermore, $c(v_i^1 v_i^2) \notin c(E(T_n))$.

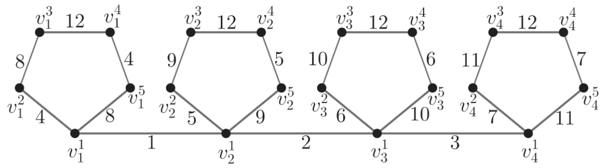

Fig. 6 A strong 3-rainbow coloring of $P_4 \rhd_o C_5$.

It follows by Eq. (2) and Observation 3.4 that we need at least $2n - 1$ distinct colors to color edges of T_n and edges $v_i^1 v_i^2$ for all $i \in [1, n]$. Next, observe that the rainbow Steiner $\{v_1^2, v_1^3, v_i^2\}$-tree for all $i \in [2, n]$ can be obtained by identifying vertex v_1^1 in a rainbow Steiner $\{v_1^1, v_1^2, v_1^3\}$-tree and a rainbow $v_1^1 - v_i^2$ geodesic. Hence, no edge of Steiner $\{v_1^1, v_1^2, v_1^3\}$-tree is colored with $c(v_i^1 v_i^2)$ and colors from $c(E(T_n))$. It means we only have one color, that is $c(v_1^1 v_1^2)$, to color two edges in Steiner $\{v_1^1, v_1^2, v_1^3\}$-tree, which is impossible. Thus, $srx_3(T_n \rhd_o C_3) \geq 2n$.

Next, we show that $srx_3(T_n \rhd_o C_3) \leq 2n$ by defining a strong 3-rainbow coloring $c : E(T_n \rhd_o C_3) \to [1, 2n]$. We first assign colors $1, 2, \ldots, n - 1$ to the edges of T_n. For each $i \in [1, n]$, assign colors $i + n - 1$ to the edges $v_i^1 v_i^2$ and $v_i^3 v_i^1$ and color $2n$ to the edges $v_i^2 v_i^3$. By this coloring, it is easy to find a rainbow Steiner S-tree for every set S of three vertices of $T_n \rhd_o C_3$.

Subcase 1.2. $m = 5$

By using a similar argument as in the proof of lower bound for $m = 3$, it is easy to show that $srx_3(T_n \rhd_o C_5) \geq 3n$. Now, we show that $srx_3(T_n \rhd_o C_5) \leq 3n$ by defining a strong 3-rainbow coloring $c : E(T_n \rhd_o C_5) \to [1, 3n]$. We first assign colors $1, 2, \ldots, n - 1$ to the edges of T_n. For each $i \in [1, n]$, we assign colors $i + n - 1$ to the edges $v_i^1 v_i^2$ and $v_i^4 v_i^5$, colors $i + 2n - 1$ to the edges $v_i^2 v_i^3$ and $v_i^5 v_i^1$, and color $3n$ to the edges $v_i^3 v_i^4$. By this coloring, it is easy to find a rainbow Steiner S-tree for every set S of three vertices of $T_n \rhd_o C_5$. **Figure 6** gives an example of a strong 3-rainbow coloring of $P_4 \rhd_o C_5$.

Subcase 1.3. $m \geq 7$

Suppose that $srx_3(T_n \rhd_o C_m) \leq nm - 1$. Then there exists a strong 3-rainbow coloring $c : E(T_n \rhd_o C_m) \to [1, nm - 1]$. For each $i \in [1, n]$, let $A_i = E(C_m^i) \setminus \{v_i^{\lceil \frac{m}{2} \rceil} v_i^{\lceil \frac{m}{2} \rceil + 1}\}$. By using Eq. (2), Observations 3.2, 3.3, and 3.4, we need at least $nm - 1$ distinct colors to color all edges of $T_n \rhd_o C_m$ except edges $v_i^{\lceil \frac{m}{2} \rceil} v_i^{\lceil \frac{m}{2} \rceil + 1}$ for all $i \in [1, n]$, which means we have used all available colors. Next, consider $\{v_1^{\lceil \frac{m}{2} \rceil}, v_1^{\lceil \frac{m}{2} \rceil + 1}, v_i^p\}$ for all $i \in [2, n]$ and $p \in \{\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 1\}$. Note that edge $v_1^{\lceil \frac{m}{2} \rceil} v_1^{\lceil \frac{m}{2} \rceil + 1}$ should be contained in the rainbow Steiner $\{v_1^{\lceil \frac{m}{2} \rceil}, v_1^{\lceil \frac{m}{2} \rceil + 1}, v_i^p\}$-tree, which implies this edge cannot be colored with colors from $c(E(T_n))$ and $c(A_i)$ for all $i \in [2, n]$. By using Observation 3.2, edge $v_1^{\lfloor \frac{m}{2} \rfloor} v_1^{\lfloor \frac{m}{2} \rfloor + 1}$ also cannot be colored with colors from $c(A_1)$. It means we need one new distinct color to color this edge, which is impossible. Thus, $srx_3(T_n \rhd_o C_m) \geq nm$.

Next, we show that $srx_3(T_n \rhd_o C_m) \leq nm$ by defining a strong 3-rainbow coloring $c : E(T_n \rhd_o C_m) \to [1, nm]$ as follows.

i. Assign colors $1, 2, \ldots, n - 1$ to the edges of T_n.
ii. Assign color n to the edges $v_i^{\lceil \frac{m}{2} \rceil} v_i^{\lceil \frac{m}{2} \rceil + 1}$ for all $i \in [1, n]$.
iii. Assign colors $n + 1, n + 2, \ldots, nm - 1, nm$ to the remaining $nm - n$ edges of $T_n \rhd_o C_m$.

By the coloring above, we obtain that all edges of $T_n \rhd_o C_m$ have distinct colors except edges $v_i^{\lceil \frac{m}{2} \rceil} v_i^{\lceil \frac{m}{2} \rceil + 1}$ for all $i \in [1, n]$, that

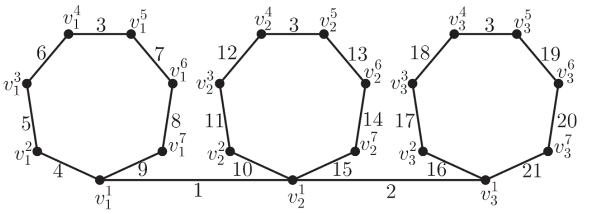

Fig. 7 A strong 3-rainbow coloring of $P_3 \rhd_o C_7$.

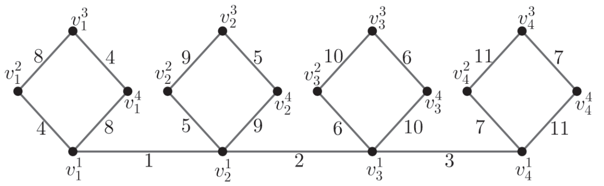

Fig. 8 A strong 3-rainbow coloring of $P_4 \rhd_o C_4$.

is $c(v_i^{\lceil \frac{m}{2} \rceil} v_i^{\lceil \frac{m}{2} \rceil+1}) = c(v_j^{\lceil \frac{m}{2} \rceil} v_j^{\lceil \frac{m}{2} \rceil+1})$ for all $i, j \in [1, n]$ with $i \neq j$. Hence, it is not hard to check that this coloring is a strong 3-rainbow coloring of $T_n \rhd_o C_m$. **Figure 7** gives an example of a strong 3-rainbow coloring of $P_3 \rhd_o C_7$.

Case 2. m is even

Subcase 2.1. m = 4

Let c be a strong 3-rainbow coloring of $T_n \rhd_o C_4$. Since $c(v_i^1 v_i^2) \neq c(v_i^4 v_i^1)$ for each $i \in [1, n]$, it follows by Eq. (2), Observation 3.3 and 3.5 that $srx_3(T_n \rhd_o C_4) \geq 3n - 1$.

Next, we show that $srx_3(T_n \rhd_o C_4) \leq 3n-1$ by defining a strong 3-rainbow coloring $c : E(T_n \rhd_o C_4) \to [1, 3n - 1]$. We first assign colors $1, 2, \ldots, n-1$ to the edges of T_n. For each $i \in [1, n]$, assign colors $i+n-1$ to the edges $v_i^1 v_i^2$ and $v_i^3 v_i^4$ and colors $i+2n-1$ to the edges $v_i^2 v_i^3$ and $v_i^4 v_i^1$. By this coloring, it is easy to show that there exists a rainbow Steiner S-tree for every set S of three vertices of $T_n \rhd_o C_4$. **Figure 8** gives an example of a strong 3-rainbow coloring of $P_4 \rhd_o C_4$.

Subcase 2.2. m = 6

Suppose that $srx_3(T_n \rhd_o C_6) \leq 4n-1$. Then there exists a strong 3-rainbow coloring $c : E(T_n \rhd_o C_6) \to [1, 4n - 1]$. By considering $\{v_i^1, v_i^3, v_i^6\}$ for each $i \in [1, n]$, it is clear that no edge of path $v_i^3 v_i^2 v_i^1 v_i^6$ is colored the same. It follows by Eq. (2), Observation 3.3 and 3.5 that we need at least $4n - 1$ distinct colors to color edges of T_n and edges $v_i^1 v_i^2$, $v_i^2 v_i^3$, and $v_i^6 v_i^1$ for all $i \in [1, n]$. Next for all $i \in [2, n]$ and $p \in \{3, 6\}$, consider $\{v_1^3, v_1^5, v_i^p\}$. By identifying vertex v_1^1 in a rainbow Steiner $\{v_1^1, v_1^3, v_1^5\}$-tree and a rainbow $v_1^1 - v_i^p$ geodesic, we obtain the rainbow Steiner $\{v_1^3, v_1^5, v_i^p\}$-tree. Hence, no edge of Steiner $\{v_1^1, v_1^3, v_1^5\}$-tree is colored with $c(v_i^1 v_i^2)$, $c(v_i^2 v_i^3)$, $c(v_i^6 v_i^1)$, and colors from $c(E(T_n))$. It means we only have three colors, $c(v_1^1 v_1^2)$, $c(v_1^2 v_1^3)$, and $c(v_1^6 v_1^1)$, to color four edges in Steiner $\{v_1^1, v_1^3, v_1^5\}$-tree, which is impossible. Thus, $srx_3(T_n \rhd_o C_6) \geq 4n$.

Next, we show that $srx_3(T_n \rhd_o C_6) \leq 4n$ by defining a strong 3-rainbow coloring $c : E(T_n \rhd_o C_6) \to [1, 4n]$. We first assign colors $1, 2, \ldots, n - 1$ to the edges of T_n. For each $i \in [1, n]$, assign colors $i + n - 1$ to the edges $v_i^1 v_i^2$, colors $i + 2n - 1$ to the edges $v_i^2 v_i^3$ and $v_i^5 v_i^6$, colors $i + 3n - 1$ to the edges $v_i^3 v_i^4$ and $v_i^6 v_i^1$, and color $4n$ to the edges $v_i^4 v_i^5$. By this coloring, it is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $T_n \rhd_o C_6$. **Figure 9** gives an example of a strong 3-rainbow coloring of $P_4 \rhd_o C_6$.

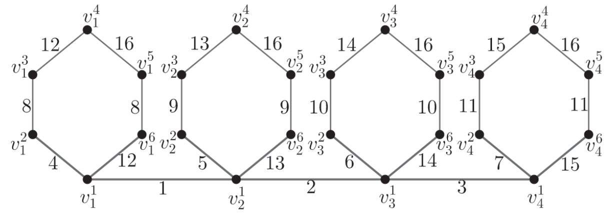

Fig. 9 A strong 3-rainbow coloring of $P_4 \rhd_o C_6$.

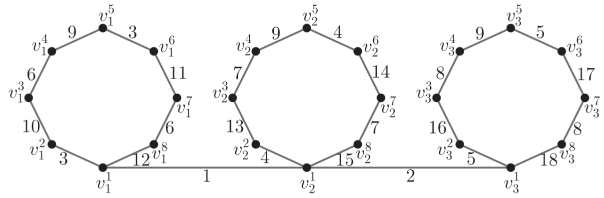

Fig. 10 A strong 3-rainbow coloring of $P_3 \rhd_o C_8$.

Subcase 2.3. m = 8

Suppose that $srx_3(T_n \rhd_o C_8) \leq 6n-1$. Then there exists a strong 3-rainbow coloring $c : E(T_n \rhd_o C_8) \to [1, 6n - 1]$. For each $i \in [1, n]$, by considering $\{v_i^1, v_i^3, v_i^7\}$, no edge of path $v_i^3 v_i^2 v_i^1 v_i^8 v_i^7$ is colored the same. It follows by Eq. (2), Observations 3.3 and 3.5 that we need at least $5n - 1$ distinct colors to color edges of T_n and edges $v_i^1 v_i^2$, $v_i^2 v_i^3$, $v_i^7 v_i^8$, and $v_i^8 v_i^1$ for all $i \in [1, n]$. This implies we have at most n colors left. Let A be the set of these n colors. Next, for an arbitrary $i \in [1, n]$, consider edges $v_i^3 v_i^4$ and $v_i^6 v_i^7$. It is easy to prove that $c(v_i^3 v_i^4) \notin \{c(v_i^1 v_i^2), c(v_i^2 v_i^3), c(v_i^8 v_i^1)\}$ and $c(v_i^6 v_i^7) \notin \{c(v_i^1 v_i^2), c(v_i^7 v_i^8), c(v_i^8 v_i^1)\}$. Then by considering $\{v_i^1, v_i^p, v_j^q\}$ for all $j \in [1, n]$, $j \neq i$, and $p, q \in \{4, 6\}$, it follows by Observations 3.3 and 3.5 that $c(v_i^3 v_i^4) \in \{c(v_i^7 v_i^8)\} \cup A$ and $c(v_i^6 v_i^7) \in \{c(v_i^2 v_i^3)\} \cup A$, with condition, $c(v_i^3 v_i^4) = c(v_i^7 v_i^8)$ if and only if $c(v_i^6 v_i^7) \neq c(v_i^2 v_i^3)$. It means we need n new distinct colors to color edges $v_i^3 v_i^4$ and $v_i^6 v_i^7$ for all $i \in [1, n]$. Hence, we have used all remaining colors. Without loss of generality, let $i = 1$. If $c(v_1^3 v_1^4) = c(v_1^7 v_1^8)$ and $c(v_1^6 v_1^7) \in A$, then consider $\{v_1^3, v_1^5, v_j^p\}$ for all $j \in [2, n]$ and $p \in \{4, 6\}$. Since $c(v_1^4 v_1^5) \notin c(E(T_n))$ by Observation 3.3, this forces $c(v_1^4 v_1^5) \in \{c(v_1^6 v_1^7), c(v_1^7 v_1^8), c(v_1^8 v_1^1)\}$. But $c(v_1^4 v_1^5) \notin \{c(v_1^6 v_1^7), c(v_1^7 v_1^8)\}$, which implies $c(v_1^4 v_1^5) = c(v_1^8 v_1^1)$. However, there is no rainbow Steiner $\{v_1^3, v_1^5, v_1^8\}$-tree, a contradiction. Similarly, if $c(v_1^3 v_1^4) \in A$ and $c(v_1^6 v_1^7) = c(v_1^2 v_1^3)$, then we will obtain a contradiction by considering $\{v_1^5, v_1^7, v_j^p\}$ for all $j \in [2, n]$ and $p \in \{4, 6\}$. Thus, $srx_3(T_n \rhd_o C_8) \geq 6n$.

Next, we show that $srx_3(T_n \rhd_o C_8) \leq 6n$ by defining a strong 3-rainbow coloring $c : E(T_n \rhd_o C_8) \to [1, 6n]$ as follows.

i. Assign colors $1, 2, \ldots, n - 1$ to the edges of T_n.
ii. For each $i \in [1, n]$, assign colors $i + n - 1$ to the edges $v_i^1 v_i^2$ and $v_i^5 v_i^6$, colors $i + 2n - 1$ to the edges $v_i^3 v_i^4$ and $v_i^7 v_i^8$, and color $3n$ to the edges $v_i^4 v_i^5$.
iii. Assign colors $3n + 1, 3n + 2 \ldots, 6n - 1, 6n$ to the remaining $3n$ edges of $T_n \rhd_o C_8$.

By the coloring above, it is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $T_n \rhd_o C_8$. **Figure 10** gives an example of a strong 3-rainbow coloring of $P_3 \rhd_o C_8$.

Subcase 2.4. m ≥ 10

Let c be a strong 3-rainbow coloring of $T_n \rhd_o C_m$. For each $i \in [1, n]$, let $A_i = E(C_m^i) \setminus \{v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}, v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2}\}$. Observe that for

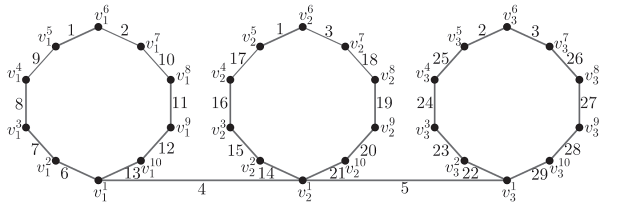

Fig. 11 A strong 3-rainbow coloring of $P_3 \rhd_o C_{10}$.

an arbitrary $i \in [1, n]$, by using Observation 3.2 and considering $\{v_i^{\frac{m}{2}}, v_i^{\frac{m}{2}+2}, v_j^p\}$ for all $j \in [1, n]$, $j \neq i$, and $p \in \{\frac{m}{2}, \frac{m}{2}+2\}$, edges $v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}$ and $v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2}$ cannot be colored with colors from $c(A_j)$ for all $j \in [1, n]$. Hence, by using Eq. (2), Observations 3.2, 3.3, 3.5, and 3.8, we need at least $(m-2)n+r+n-1 = nm-n+r-1$ distinct colors to color all edges of $T_n \rhd_o C_m$. Thus, $srx_3(T_n \rhd_o C_m) \geq nm - n + r - 1$.

Next, we show $srx_3(T_n \rhd_o C_m) \leq nm - n + r - 1$. We define an edge-coloring $c : E(T_n \rhd_o C_m) \to [1, nm - n + r - 1]$ as follows.

i. Assign a list of combinations of r colors taken 2 at a time to all pairs of two edges $\{v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}, v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2}\}$ for all $i \in [1, n]$, so that $\{c(v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}), c(v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2})\} \neq \{c(v_j^{\frac{m}{2}} v_j^{\frac{m}{2}+1}), c(v_j^{\frac{m}{2}+1} v_j^{\frac{m}{2}+2})\}$ for all $i, j \in [1, n]$ with $i \neq j$.

ii. Assign colors $r+1, r+2, \ldots, r+n-1$ to the edges of T_n.

iii. Assign colors $r+n, r+n+1, r+n+2, \ldots, r+nm-n-1$ to the remaining $(m-2)n$ edges of $T_n \rhd_o C_m$.

By the coloring above, it is not hard to show that c is a strong 3-rainbow coloring of $T_n \rhd_o C_m$. **Figure 11** gives an example of a strong 3-rainbow coloring of $P_3 \rhd_o C_{10}$. □

It is easy to check that $srx_3(C_4) = 2$. Thus, following Theorem 3.6, we obtain that $srx_3(T_n \rhd_o C_4)$ attains the upper bound in Theorem 2.3.

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