Electronic Preprint for Journal of Information Processing Vol.28

Regular Paper

On Domination Number of Triangulated Discs

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Received: December 30, 2019, Accepted: September 10, 2020

Abstract: Let G be a 3-connected triangulated disc such that the boundary cycle C of the outer face is an induced cycle of G and $G - C$ is a tree. In this paper we prove that $\gamma(G) \leq \frac{n+2}{4}$, which gives a partial solution for the conjecture that the same inequality holds for any 3-conneced triangulated disc. We also show related conjectures.

Keywords: dominating set, domination number, planar graph, triangulated disc

1. Introduction

For a graph $G = (V(G), E(G))$ and $v \in V(G)$, let $N_G(v)$ denote the set of all the vertices which are adjacent to v in G, and let $N_G[v] = \{v\} \cup N_G(v)$. For $S \subset V(G)$, let $N_G[S] = \bigcup_{v \in S} N_G[v]$, and let $\langle S \rangle_G$ denote the induced subgraph of G induced by S. For $S \subset V(G)$ and $v \in V(G)$, we say S *dominates* v if $v \in N_G[S]$. If $D \subset V(G)$ dominates all the vertices of G, then D is said to be a *dominating set* of G. The *domination number* of G, denoted $\gamma(G)$, is defined as the minimum cardinality of a dominating set of G. A plane graph G is said to be a *triangulated disc* if G is 2-connected and all its faces are triangles except for the outer (infinite) face. The boundary cycle of the outer face of G is said to be the *outer cycle* of G and is denoted $C(G)$. $G - C(G)$ is said to be an *inner subgraph* of G and is denoted $In(G)$. An *l-coloring* is a function $f : V(G) \to \{1, ..., l\}$. An l-coloring f is *proper* if $f(u) \neq f(v)$ for each edge $uv \in E(G)$. If G is l-colored and $v \in V(G)$ is dominated by the set of all the vertices of color i ($i = 1, 2, ..., l$), then we say *v is dominated by color i*.

In 1996, Matheson and Tarjan [2] proved that any triangulated disc G with n vertices satisfies $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$. They also conjectured that $\gamma(G) \leq \lfloor \frac{n}{4} \rfloor$ for every n-vertex maximal planar graph G with sufficiently large n. Note that we need two vertices to dominate the six vertices of the octahedron graph, and there also exists a 11-vertex maximal planar graph with $\gamma(G) = 3 > \lfloor \frac{11}{4} \rfloor$ (**Fig. 1**), therefore we cannot omit the condition that n is sufficiently large. In 2010, King and Pelsmajer [7] proved that the conjecture of Matheson and Tarjan holds for maximal planar graphs with a maximum degree 6. In 2013, Campos and Wakabayashi [1] and Tokunaga [3] independently proved $\gamma(G) \leq \lfloor \frac{n+t}{4} \rfloor$ for each n-vertex outerplanar graph G with $n \geq 3$ having t vertices of degree 2. In 2016, Li, Zhu, Shao, and Xu improved the upper bound in Refs. [1], [3] by showing $\gamma(G) \leq \frac{n+k}{4}$, where k is the number of pairs of consecutive 2-degree vertices with a distance of at least 3 on the outer cycle.

In Ref. [3], the author gave a simple proof by showing that G has a proper 4-coloring such that each vertex except those with degree two is dominated by all the four colors, and a similar method is also applied to other related problems [4], [5]. Moreover, the author conjectured as follows.

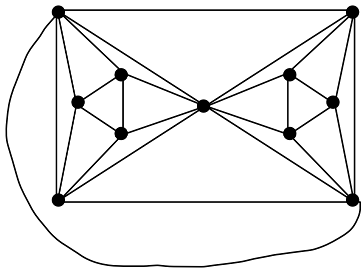

Fig. 1 Maximal planar graph with 11 vertices and domination number 3.

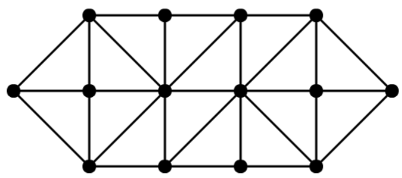

Fig. 2 3-connected triangulated disc with 14 vertices and domination number $\frac{14+2}{4} = 4$.

Conjecture 1 *Suppose G is a 3-connected n-vertex triangulated disc, then* $\gamma(G) \leq \lfloor \frac{n+2}{4} \rfloor$.

Figure 2 shows that the upper bound in Conjecture 1 is sharp. Note that the inner subgraph of the graph in Fig. 2 is a path. There are many graphs satisfying the equality in Conjecture 1 whose inner subgraphs are trees. In this paper, we prove the following theorem.

Theorem 1 *Suppose G is an n-vertex triangulated disc such that $In(G)$ is a tree and $C(G)$ is an induced cycle of G, then* $\gamma(G) \leq \lfloor \frac{n+2}{4} \rfloor$.

2. Proof of Theorem 1

To prove Theorem 1, we show the following lemma.

Lemma 1 *Suppose G is an n-vertex triangulated disc such that $In(G)$ is a tree and $C(G)$ is an induced cycle of G, and let v be a vertex of $C(G)$ with* $\deg_G(v) = 3$. *Then, $G - v$ has a proper*

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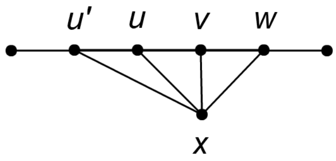

Fig. 3 Case 1.

4-coloring f such that each vertex of $G - v$ is dominated by all the four colors except the vertices of $N_G(v)$.

To prove Lemma 1, let us introduce the following notation. For an l-coloring f of graph G and $v \in V(G)$, let

$$\bar{f}(v) = \{1, 2, \cdots, l\} - \bigcup_{v' \in N_G[v]} \{f(v')\},$$

and let

$$f^*(v) = \begin{cases} 0 & \text{when } \bar{f}(v) = \emptyset, \\ i & \text{when } \bar{f}(v) = \{i\}, \\ \infty & \text{otherwise.} \end{cases}$$

Note that if G is a triangulated disk and f is a proper 4-coloring of G, $|\bar{f}(v)| \leq 1$ holds for each $v \in V(G)$, which implies $f^*(v) \in \{0, 1, 2, 3, 4\}$.

Proof of Lemma 1. Let G and v be as in Lemma 1. Let $C = C(G)$, $T = In(G)$ and $N_G(v) = \{u, w, x\}$. Since C is an induced cycle of G, we may assume $N_C(v) = \{u, w\}$ and x is the unique vertex of T which is adjacent to v. We use induction on $n = |V(G)|$. Since the statement of Lemma 1 clearly holds for K_4, we assume $n \geq 5$. In view of $\deg_G(u) \geq 3$ and $\deg_G(w) \geq 3$, there are two cases as follows.

Case 1. $\deg_G(u) = 3$ or $\deg_G(w) = 3$.

We may assume $\deg_G(u) = 3$ without loss of generality. Let u' be the vertex of $N_C(u)$ satisfying $u' \neq v$, and let $G' = G - v + uw$. Since $In(G') = In(G)$ is a tree and $C(G') = C - v + uw$ is an induced cycle of G', G' satisfies the assumption of Lemma 1. Thus the induction hypothesis, G' has proper 4-coloring f' such that each vertex of $G' - u$ is dominated by all the four colors except u', x, w. Here we define f as follows; If $f'^*(u') \neq 0$, then let $f(u) = f'^*(u')$, and if $f'^*(u') = 0$, then let $f(u)$ be any value different from $f'(u')$ and $f'(x)$. Furthermore, let $f(y) = f'(y)$ for $y \neq u$. Then, f satisfies the conclusion of Lemma 1.

Case 2. $\deg_G(u) \geq 4$ and $\deg_G(w) \geq 4$.

We divide this case into two subcases in view of $\deg_T(x)$.

Subcase 2.1. $\deg_T(x) = 1$

Let x' be the unique vertex of T which is adjacent to x, and let $G' = G - v$. By the assumption of Case 2 and Subcase 2.1, $\deg_{G'}(x) = 3$. Further, since $In(G') = In(G) - x$ is a tree and $C(G') = C - v + ux + xw$ is an induced cycle of G', G' satisfies the assumption of Lemma 1. Therefore by induction hypothesis, $G' - x$ has proper 4-coloring f' such that each vertex of $G' - x$ is dominated by all the four colors except u, x', w. Here we define 4-coloring f as follows; If $f'^*(x') \neq 0$, then let $f(x) = f'^*(x')$. If $f'^*(x') = 0$, then let $f(x)$ be any value different from $f'(u), f'(x')$ and $f'(w)$. Moreover, let $f(y) = f'(y)$ for $y \neq x$. Then, f satisfies the conclusion of Lemma 1.

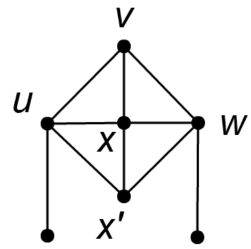

Fig. 4 Subcase 2.1.

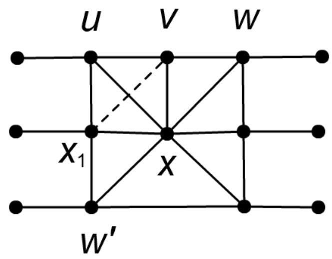

Fig. 5 Subcase 2.2.

Subcase 2.2. $\deg_T(x) \geq 2$.

Let x_1 be the unique vertex of $V(T) \cap N_G(u) \cap N_G(x)$, and let w' be the vertex of $N_G(x_1) \cap N_G(x)$ satisfying $w' \neq u$. Let T_1 be a component of $T - x$ containing x_1 and let $T_2 = T - T_1$. Also, let $G_1 = \langle N_G[V(T_1)] \rangle_G$ and $G_2 = \langle N_G[V(T_2)] \rangle_G - u + x_1 v$. Since T_1, T_2 are trees and $C(G_1), C(G_2)$ are induced cycles of G_1, G_2, respectively, both G_1 and G_2 satisfy the assumption of Lemma 1. Thus by induction hypothesis, $G_1 - x$ has a proper 4-coloring f_1 such that each vertex of $G_1 - x$ is dominated by all the four colors except u, x_1, w', and $G_2 - v$ has a proper 4-coloring f_2 such that each vertex of $G_2 - v$ is dominated by all the four colors except x_1, x, w. Let $j \in \{1, 2, 3, 4\} - \{f_1(u), f_1(x_1), f_1(w')\}$, and let

$$k = \begin{cases} j & \text{when } f_1^*(x_1) = 0 \\ f_1^*(x_1) & \text{when } f_1^*(x_1) \neq 0. \end{cases}$$

We can make $f_1(y) = f_2(y)$ for $y \in V(G_1 - x) \cap V(G_2 - v) = \{x_1, w'\}$ and $f_2(x) = k$ by exchanging colors. Now let $f(y) = f_1(y)$ for $y \in (V(G_1) - x)$ and let $f(y) = f_2(y)$ for $y \in (V(G_2) - v)$, then f satisfies the conclusion of Lemma 1. □

Proof of Theorem 1. Let G, v, f be as in Lemma 1 and let u, w, x be as in the proof of Lemma 1. Let G' be the $(n + 2)$-vertex graph such that $V(G') = V(G) \cup \{p, q\}$ and $E(G') = E(G) \cup \{pu, pv, pw, qu, qv, qw\}$. Further, we give a 4-coloring f' of G' satisfying $f'(y) = f(y)$ for $y \in V(G) - v$ and $\{f'(x), f'(v), f'(p), f'(q)\} = \{1, 2, 3, 4\}$. Then, each vertex of $V(G)$ is dominated by all the four colors, and hence we may assume $S = \{v \in V(G') \mid f'(v) = 1\}$ satisfies $|S| \leq \lfloor \frac{n+2}{4} \rfloor$ without loss of generality. Finally, if we let

$$S' = \begin{cases} S & \text{when } S \cap \{p, q\} = \emptyset, \\ S - p + v & \text{when } p \in S, \\ S - q + v & \text{when } q \in S, \end{cases}$$

then, S' is a dominating set of G satisfying $|S'| \leq \lfloor \frac{n+2}{4} \rfloor$ □

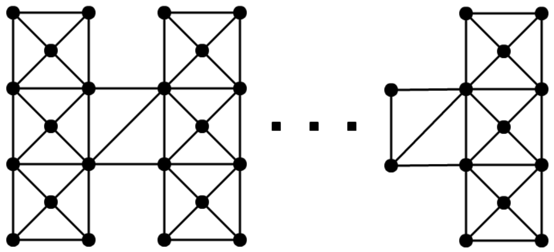

Fig. 6 Triangulated disc G with $\delta(G) = 3$ and $\gamma(G) = \frac{3}{11}|V(G)|$.

3. Other Conjectures

If we weaken the assumption of 3-connectivity in Conjecture 1 to $\delta(G) \geq 3$, then the upper bound in Conjecture 1 appears to change as follows.

Conjecture 2 *Suppose G is an n-vertex triangulated disc satisfying* $\delta(G) \geq 3$*, then* $\gamma(G) \leq \lfloor \frac{3}{11} n \rfloor$.

Figure 6 shows that the upper bound in Conjecture 2 cannot be improved.

Though there is still a gap between Conjecture 1 and Theorem 1, if the following conjecture is true, then Conjecture 1 holds for 4-connected maximal planar graphs.

Conjecture 3 *Suppose G is a 4-connected n-vertex maximal planar graph. Then* $V(G)$ *can be divided into* S_1, S_2 *such that* $\langle S_1 \rangle_G$, $\langle S_2 \rangle_G$ *are a maximal outerplanar graph and a tree, respectively.*

Note that if we delete all the edges connecting two vertices of S_1 in the above conjecture, we get a graph satisfying the assumption of Theorem 1.

Acknowledgments The author would like to thank the referees for their many helpful comments and suggestions.

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