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Continuous Flattening of Multi-layered Pyramids with Rigid Radial Edges

Kazuki Matsubara[1,a)] Chie Nara[2,b)]

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Abstract: Several methods described in the literature have proved that any convex pyramid can be continuously flattened. Recently, the problem of continuous flattening of polyhedra having divisions, i.e., polyhedra in which some of the edges are incident to three or more faces, has been proposed. However, for such multi-layered structures, continuous flattening motions are unknown. In this study, under the assumption that every radial edge is rigid, we prove that a continuous flattening motion exists for a pyramid with a convex base. Moreover, in a similar manner, we demonstrate that a continuous flattening motion exists for a multi-layered pyramid having a common convex base, with each apex having a common perpendicular foot. Finally, we illustrate an example of a multi-layered pyramid with a non-convex base that cannot be continuously flattened while maintaining the rigidity of the radial edges.

Keywords: Pyramid, continuous flattening, rigid radial edge, multi-layered pyramid

1. Introduction

We use the term *polyhedron* for a polyhedral surface in three-dimensional Euclidean space $\mathbb{R}^3$ that is permitted to touch itself but without self-crossing. Furthermore, polyhedra of higher genus are allowed here.

Flat folding of a polyhedron refers to its folding by creases, without self-crossing, into a multi-layered flat folded state with a finite number of creases. It is known that any polyhedron of genus zero has a flat folded state [2], [6], [8]. The original problem of *continuous* flattening of polyhedra was proposed in Ref. [5], and the existence of a continuous motion has been demonstrated in Refs. [1], [8] for any convex polyhedron. However, the existence of a continuous motion from the surface down to a flat folded state for any polyhedron is still an open problem.

Recently, the second author proposed the problem of flattening multi-layered structures [12] and provided flattening states for some of them. However, for such structures, continuous flattening motions are unknown.

Now, we define the multi-layered structure discussed in this paper.

Definition 1. *Let $\Gamma = \Gamma_n = B_1B_2 \cdots B_n$ be an n-gon and O be a point in the interior of Γ_n. Let $A_1, \ldots, A_k$ be points on the line passing through O and orthogonal to Γ. Then, we call the set of $\triangle A_iB_jB_{j+1}$ $(1 \leq i \leq k, 1 \leq j \leq n)$ a multi-layered pyramid, which we denote by $\mathcal{P} = \mathcal{P}(\Gamma_n; A_1, \ldots, A_k)$, where $B_{n+1} = B_1$. The edges A_iB_j and B_jB_{j+1} are called a radial edge and a horizontally aligned edge, respectively, and A_i and Γ are called the apexes and the base of $\mathcal{P}$, respectively* (**Fig. 1**).

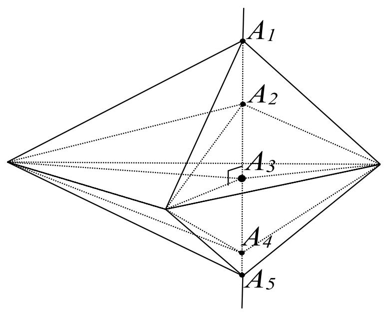

Fig. 1 Multi-layered pyramid $\mathcal{P}(\Gamma_3; A_1, A_2, A_3, A_4, A_5)$ with $A_3 = O$.

Every multi-layered pyramid $\mathcal{P}(\Gamma; A_1, \ldots, A_k)$ can be considered in $\mathbb{R}^3$ with every apex on the z-axis and the base on the xy-plane. We consider such multi-layered pyramids in $\mathbb{R}^3$ throughout the paper. Note that some A_i may coincide with the origin O, that is, $A_i = O$, in which case we consider the base to consist of the triangular faces $\triangle A_iB_jB_{j+1}$ $(1 \leq j \leq n)$.

An important constraint on continuous flattening is the Bellows theorem [3]: the volume of any polyhedron with rigid faces is invariant even if the polyhedron is flexible. Flattening a polyhedron necessarily changes the volume (from a nonzero value to zero), which implies that some faces cannot be rigid; i.e., their shapes are continuously changed by infinitely many moving creases. This also implies that any multi-layered pyramid in which all its edges are rigid cannot be continuously flattened. In Ref. [13], the authors focused on the rigidity of radial edges of the same length during the flattening motion. In this study, we assume that all radial edges (not necessary of the same length) of a multi-layered pyramid are rigid and that horizontally aligned edges can be folded. Additionally, we demonstrate that a continuous flattening motion exists for multi-layered pyramids, as depicted in **Fig. 2**.

1 Chuo Gakuin University, Abiko, Chiba 270–1196, Japan
2 Meiji Institute for Advanced Study of Mathematical Sciences, Nakano, Tokyo 164–8525, Japan
a) additivestructure@gmail.com
b) cnara@jeans.ocn.ne.jp

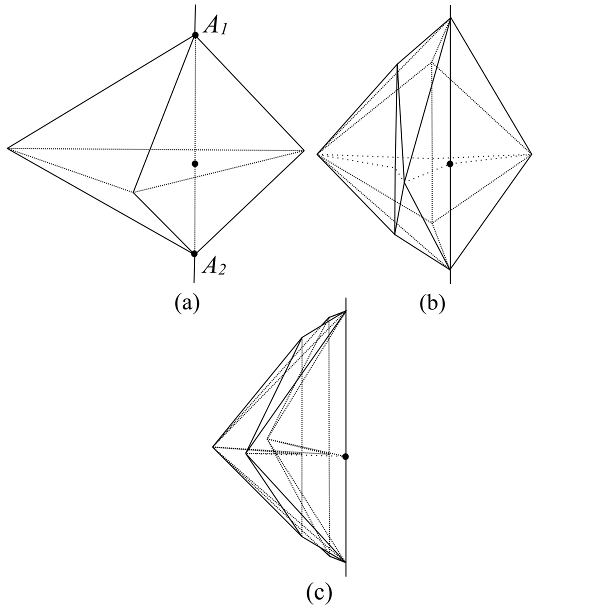

Fig. 2 (a) Multi-layered pyramid $\mathcal{P}(\Gamma_3; A_1, A_2)$, (b) halfway-folded state, and (c) flat folded state.

There are many ways to continuously flatten polyhedra; see Refs. [1], [4], [7], [8], [9], [10], [11], [12], [13]. To the best of the authors' knowledge, a continuous flattening motion for a multi-layered structure has not been described in the literature. The main results in this paper concern the existence and non-existence of a continuous flattening motion for a multi-layered pyramid with rigid radial edges. The statements of the results follow.

Theorem 1. *There exists a continuous flattening motion for any multi-layered pyramid with a convex base and rigid radial edges.*

Theorem 2. *There exists a multi-layered pyramid with a non-convex base and rigid radial edges that cannot be continuously flattened.*

In this paper, we consider the continuous flattening of a multi-layered pyramid $\mathcal{P}$ at time t from 0 to 2. The position of any point $P \in \mathcal{P}$ at time t is denoted by $P(t)$, where we omit (0) at time 0. In particular, our flattening method for $\mathcal{P}$ can be divided into two parts: (1) lifting the apexes ($0 \le t \le 1$) and (2) rotating the horizontally aligned edges ($1 \le t \le 2$). We call a structure consisting of the edges of $\mathcal{P}$ the 1-*skeleton* of $\mathcal{P}$. In Sections 2 and 3, we consider the motion of the 1-skeleton of $\mathcal{P}$ when $\mathcal{P}$ is the simplest multi-layered pyramid in some sense. We show that two processes (lifting the apexes and rotating the horizontally aligned edges) produce a continuous flattening motion of the 1-skeleton. In Section 4, the folded state of each face of $\mathcal{P}$, which depends on the folded state of the 1-skeleton, is precisely discussed. In Section 5, the existence of a continuous flattening motion for any multi-layered pyramid with a convex base is proved. In Section 6, we present an example of a multi-layered pyramid with a non-convex base for which no continuous flattening motions exist if all the radial edges are rigid during the motion.

2. Lifting the Apexes

Let $\mathcal{P}(\Gamma_n; A_1, A_2)$ be a pyramid with $\Gamma_n = B_1 \cdots B_n$ on the xy-plane, A_1 on the positive z-axis, and $A_2 = O$ (see **Fig. 3**). In

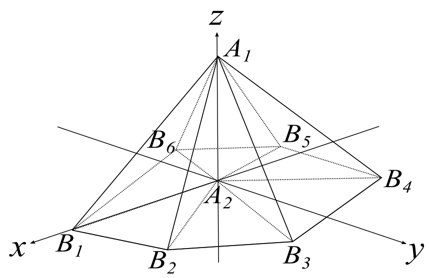

Fig. 3 Pyramid with 12 radial edges.

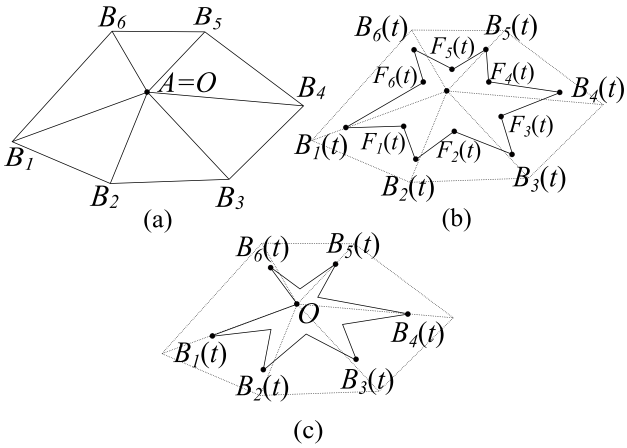

Fig. 4 (a) $\mathcal{P}(\Gamma_6, A)$, (b) halfway-folded state of the horizontally aligned edges, and (c) folded state at the end of the lifting motion ($t = 1$) with $|B_6(1)O| + |B_1(1)O| = |B_1B_6|$.

this section, we define a continuous motion of the 1-skeleton of $\mathcal{P}(\Gamma_n; A_1, A_2)$ such that the radial edges are rigid and the horizontally aligned edges, which move on the xy-plane, are folded continuously to synchronize with the motion of the apexes.

First, we focus on $\mathcal{P}(\Gamma_n; A)$, which has radial edges B_jO ($1 \le j \le n$) and apex $A = O$ (see **Fig. 4** (a)). We move A on the positive z-axis and the horizontally aligned edges on the xy-plane, each of which may be folded into two connected line segments at each t ($0 \le t \le 1$) (see Fig. 4 (b)).

Let d be a real number with $0 \le d \le \min_{1\le j\le n} |B_jO|$. We further restrict d, which is defined as the distance from O to the perimeter of Γ_n. Let

$$A(t) = (0, 0, td),\ 0 \le t \le 1$$

and let

$$l_j(t) = \sqrt{|B_jO|^2 - (td)^2},\ 1 \le j \le n, 0 \le t \le 1.$$

Because $l_j(0) + l_{j+1}(0) = |B_jO| + |B_{j+1}O| > |B_jB_{j+1}|$ for $1 \le j \le n$, where $B_{n+1} = B_1$, we can assume that d satisfies the following conditions:

(C1) For some j ($1 \le j \le n$)

$$l_j(1) + l_{j+1}(1) = |B_jB_{j+1}|.$$

(C2) For any t ($0 \le t < 1$) and any j ($1 \le j \le n$)

$$l_j(t) + l_{j+1}(t) > |B_jB_{j+1}|.$$

Note that we stop the lifting of A when the condition (C1) is satisfied by some j ($1 \le j \le n$). Then, we fix j as any of such j, and set $t = 1$ at that time. Let H_j be the foot of the perpendicular line from A (= O) to the edge B_jB_{j+1}. Then, since

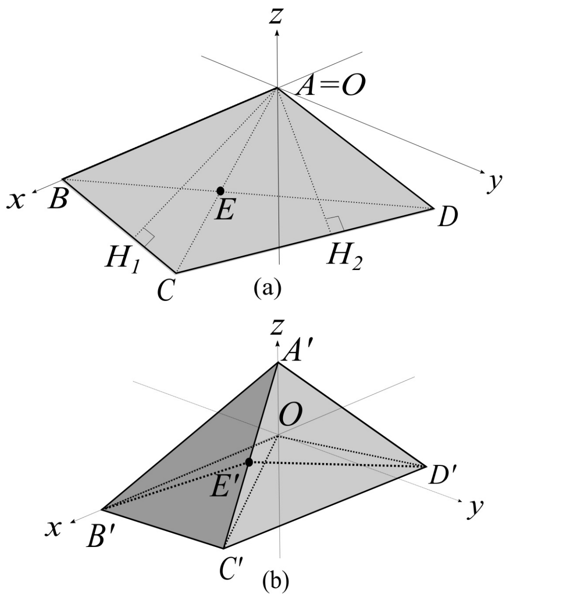

Fig. 5 (a) Tetragon $ABCD$ and (b) folded state of $ABCD$.

$\angle A(1)OB_j(1) = \angle A(1)OB_{j+1}(1) = 90^\circ$ holds, $|AH_j| = |A(1)O| = d$ holds and AH_j is one of the shortest perpendicular line segments from A to the straight lines $B_{j'}B_{j'+1}$ $(1 \le j' \le n)$.

Now, the positions of $B_j(t)$ $(1 \le j \le n)$ in the lifting motion are determined in the following two lemmas.

Lemma 1. *Let $ABCD$ be a tetragon on the xy-plane consisting of two rigid triangles $\triangle ABC$ and $\triangle ACD$, that is, $ABCD$ can be folded along the line segment AC. We assume that $A = O$ and $\angle ACB + \angle ACD < 180^\circ$ (see* ***Fig. 5*** *(a)). Let $h = \min(|AH_1|, |AH_2|)$, where H_1 and H_2 are the feet of the perpendicular lines from A to BC and CD, respectively. If we continuously lift A on the positive z-axis from $A_0 = O$ to $A_1 = (0, 0, h)$ and move the horizontally aligned edges on the xy-plane, then we have*

$$\angle B'C'D' < \angle BCD$$

during the motion, where A' $(\neq O)$, B', C', and D' are the points corresponding to A, B, C, and D, respectively, in a folded state of $ABCD$ (see Fig. 5 *(b)).*

Proof. Let E be an intersection of the straight lines AC and BD, and E' be the point corresponding to E on the straight line $A'C'$. Then, $A' \neq O$ implies that E' is not on the xy-plane. Hence, we have $\angle B'E'D' < 180^\circ$ and $|B'D'| < |B'E'| + |E'D'| = |BD|$. Since $\angle ACB + \angle ACD < 180^\circ$ holds and both BC and CD are rigid, by the law of cosines, it follows that $\angle B'C'D' < \angle BCD$. □

Lemma 2. *Let $\mathcal{P}(\Gamma_n; A)$ be the convex n-gon on the xy-plane with a convex polygon $\Gamma_n = B_1 \cdots B_n$ with $n \ge 3$ and $A = O$. Then, for any position of $A(t)$ $(0 \le t \le 1)$, there exist $B_1(t), \ldots, B_n(t)$ on the xy-plane such that all radial edges are rigid, that is, $|A(t)B_j(t)| = |AB_j|$, and $|B_j(t)B_{j+1}(t)| \le |B_jB_{j+1}|$ for any t $(0 \le t \le 1)$. Moreover, $OB_j(t)$ $(1 \le j \le n)$ are in anticlockwise order about O for $0 \le t \le 1$.*

Proof. We show that we can determine the positions of $B_j(t)$ on the circle with center O and radius $l_j(t) = \sqrt{|B_jO|^2 - (td)^2}$ that satisfy the equation

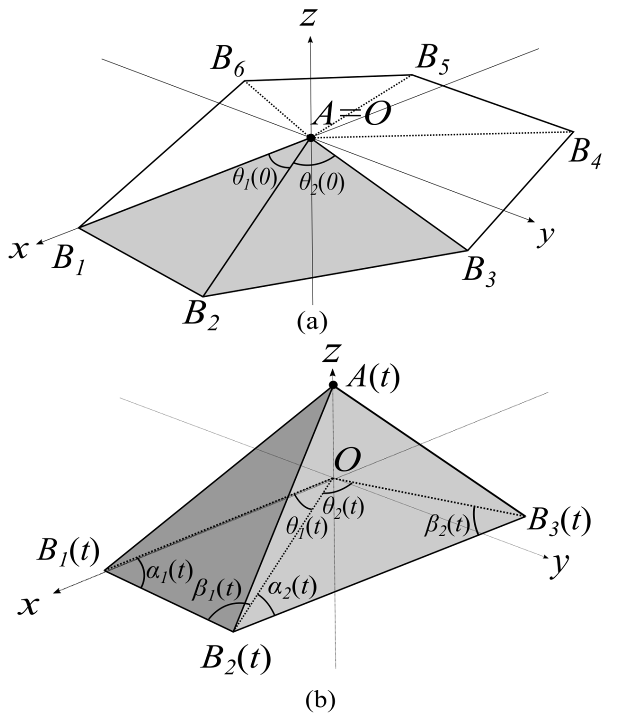

Fig. 6 (a) Angles $\theta_1(0)$, $\theta_2(0)$ and (b) angles $\theta_i(t), \alpha_i(t)$ and $\beta_i(t)$ with $i = 1, 2$.

$$\sum_{j=1}^{n} \angle B_j(t)OB_{j+1}(t) = 360^\circ \tag{1}$$

for any t $(0 \le t \le 1)$. We focus on two consecutive faces $\triangle B_jOB_{j+1}$ and $\triangle B_{j+1}OB_{j+2}$ for each fixed j $(1 \le j \le n)$. Then, we move B_j, B_{j+1}, and B_{j+2} such that B_jB_{j+1} and $B_{j+1}B_{j+2}$ are rigid, synchronizing with the lifting motion of A. When A moves from O to $(0, 0, d)$ for $0 \le t \le 1$, d is the length of the shortest perpendicular line segment from A to B_jB_{j+1} $(1 \le j \le n)$, that is, the distance from O to the perimeter of Γ_n. Hence, $\triangle AB_jB_{j+1}$ and $\triangle AB_{j+1}B_{j+2}$ can move from $t = 0$ to $t = 1$ as in Lemma 1.

We define the angles of $\triangle OB_j(t)B_{j+1}(t)$ as follows:

$$\theta_j(t) = \angle B_j(t)OB_{j+1}(t),\ \alpha_j(t) = \angle OB_j(t)B_{j+1}(t)$$

and

$$\beta_j(t) = \angle OB_{j+1}(t)B_j(t)$$

(see **Fig. 6**). By Lemma 1 the following holds.

$$\beta_j(t) + \alpha_{j+1}(t) = \angle B_j(t)B_{j+1}B(t)_{j+2}(t) < \angle B_jB_{j+1}B_{j+2}$$

holds for any j $(1 \le j \le n)$ and any t $(0 < t \le 1)$. Hence,

$$\sum_{j=1}^{n}(\alpha_j(t) + \beta_j(t)) < \sum_{j=1}^{n} \angle B_jB_{j+1}B_{j+2} = 180^\circ \times (n-2).$$

Thus, we have

$$\sum_{j=1}^{n} \theta_j(t) = \sum_{j=1}^{n}(180^\circ - (\alpha_j(t) + \beta_j(t))) > 360^\circ.$$

Now, $\angle B_j(t)OB_{j+1}(t)$ can be changed to any angle less than or

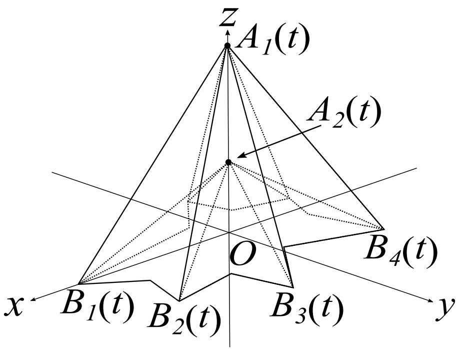

Fig. 7 Folded state of a 1-skeleton of a pyramid during the lifting motion.

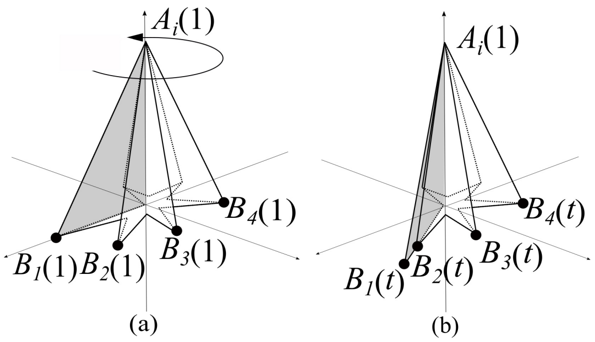

Fig. 8 Rotating motion of $OB_j(1)$ toward $OB_n(1)$.

equal to $\theta_j(t)$ by folding B_jB_{j+1} at any point, which is determined and denoted by $F_j(t)$ later. Hence, by folding B_jB_{j+1} if necessary, we can choose $\angle B_j(t)OB_{j+1}(t) \le \theta_j(t)$ such that the equality (1) and $|B_j(t)B_{j+1}(t)| \le |B_jB_{j+1}|$ hold $(1 \le j \le n, 0 \le t \le 1)$. □

Note that $B_1(t), \ldots, B_n(t)$ in Lemma 2 can be determined continuously with respect to t $(0 \le t \le 1)$. In contrast, the following remark is important for the proof of Theorem 2 in Section 6.

Remark 1. *In Lemmas 1 and 2, the assumption $\angle ACB + \angle ACD < 180°$ for ABCD played a key role. However, we cannot extend it to all of the non-convex tetragons, because both $\angle OC'B' > \angle OCB$ and $\angle OC'D' > \angle OCD$ may occur, that is, $\angle OC'B' + \angle OC'D' > \angle OCB + \angle OCD$ may hold. Hence, in a continuously flattening motion of a non-convex n-gon $\mathcal{P}(\Gamma_n; A)$ with rigid radial edges, there may not exist $B_1(t), \ldots, B_n(t)$ satisfying the equation (1) even for small $t > 0$.*

Note that it is not necessary for any $B_j(t)$ to lie on the line segment B_jO. Now, for $B_j(t)$ and $B_{j+1}(t)$ $(1 \le j \le n, 0 \le t \le 1)$, we can uniquely determine a folding point $F_j(t)$ on the xy-plane satisfying the following conditions (Fig. 4 (b)):

$$\angle B_j(t)OF_j(t) = \angle B_{j+1}(t)OF_j(t), \tag{2}$$

$$|B_j(t)F_j(t)| + |B_{j+1}(t)F_j(t)| = |B_jB_{j+1}|, \tag{3}$$

$$\angle B_j(t)F_j(t)O + \angle B_{j+1}(t)F_j(t)O \ge 180°. \tag{4}$$

Now, we consider the lifting motion for the radial edges of the pyramid $\mathcal{P}(\Gamma_n; A_1, A_2)$ with $A_2 = O$. Recall that $A_2(t) = (0, 0, td)$ for $0 \le t \le 1$. We define the motion of A_1 for $0 \le t \le 1$. For any fixed t $(0 \le t \le 1)$, $|B_jA_1|^2 - |B_j(t)O|^2$ is common for all j's with $1 \le j \le n$, because

$$\begin{aligned}|B_jA_1|^2 - |B_j(t)O|^2 &= |B_jA_1|^2 - (|B_jO|^2 - (td)^2)\\ &= |OA_1|^2 + (td)^2.\end{aligned}$$

Define $A_1(t) = (0, 0, \sqrt{|OA_1|^2 + (td)^2})$ for $0 \le t \le 1$ (see **Fig. 7**).

We can now state the following lemma.

Lemma 3. *For $\mathcal{P}(\Gamma_n; A_1, A_2)$ with $A_2 = O$, let $A_1(t) = (0, 0, \sqrt{|OA_1|^2 + (td)^2})$ and $A_2(t) = (0, 0, td)$ $(0 \le t \le 1)$. There exist positions $B_j(t)$ $(1 \le j \le n, 0 \le t \le 1)$ such that all radial edges are rigid and $|B_j(t)B_{j+1}(t)| \le |B_jB_{j+1}|$. Moreover, we can set the line segments $B_j(t)O$ $(1 \le j \le n)$ in anticlockwise order about O.*

By Lemma 3, we can set $A_i(t) = (0, 0, \sqrt{|OA_i|^2 + (td)^2})$ for each i $(1 \le i \le k)$ during the lifting motion $(0 \le t \le 1)$. Note that, for each of the folding points $F_j(t)$ $(1 \le j \le n)$, the corresponding point on the edge B_jB_{j+1} may stay or continuously move on the edge. When some of the points $F_j(t)$ reach O at the first time, i.e., $t = 1$, we proceed to the next step of rotation.

3. Rotating the Horizontally Aligned Edges

In this section, for $1 \le t \le 2$, we determine a rotating motion of $\mathcal{P}(\Gamma_n; A_i)$ for each i $(1 \le i \le k)$ after the lifting motion $(0 \le t \le 1)$. During the rotating motion, the apex $A_i(1)$ is fixed, and each of the bottom vertices $B_j(1)$ moves on a circle with center O and radius $|OB_j(1)|$. Furthermore, for $1 \le t \le 2$, we can uniquely determine a folding point $F_j(t)$ on the xy-plane satisfying the conditions (2)–(4). Then, every horizontally aligned edge moves on the xy-plane such that each edge B_jB_{j+1} is folded at $F_j(t)$.

The rotating motion is illustrated in **Fig. 8**. Without loss of generality, we can assume that $F_n(t)$ reaches O at the end of the lifting motion; that is, $F_n(1) = O$. Then, for $1 \le t \le 2$, we rotate $B_j(1)$ $(1 \le j \le n-1)$ about the z-axis by the angle $\sum_{j \le j' \le n-1} \angle B_{j'}(1)OB_{j'+1}(1)$ until it reaches the line $OB_n(1)$. Thus, the 1-skeleton of the pyramid is flattened on the plane including the triangle $A_i(1)OB_n(1)$.

4. Folded States of Faces

In this section, for the motion of the 1-skeleton of $\mathcal{P} = \mathcal{P}(\Gamma_n; A_1, \ldots, A_k)$, the existence of a folded state of the triangular faces is discussed. We focus on any of the fixed triangular faces of $\mathcal{P}$.

Lemma 4. *Let $\mathcal{P} = \mathcal{P}(\Gamma_n; A)$ with a convex n-gon Γ. We choose any j $(1 \le j \le n)$ and t $(0 \le t \le 2)$, and fix them. Let the folding point $F = F_j(t) = (x, y, 0)$ and denote by F^* the corresponding point of F on B_jB_{j+1}. There exist R and $R^* \in \triangle AB_jB_{j+1}$ such that $F^*R^* \perp B_jB_{j+1}$, $|AR^*| = |A(t)R|$, and $R = (x, y, |R^*F^*|)$ (**Fig. 9**).*

Proof. Let $B'_{j+1}(t)$ be a point on the xy-plane with

$$\triangle A(t)B_j(t)B'_{j+1}(t) \equiv \triangle AB_jB_{j+1}$$

in non-reflective order, and F' be the point on $B_j(t)B'_{j+1}(t)$ with $|B_j(t)F'| = |B_jF^*|$ (Fig. 9 (a)). Because F is inside $\triangle OB_j(t)B_{j+1}(t)$, it is seen that F is a point specified by rotating F' around $B_j(t)$ toward the line segment $OB_j(t)$. Hence, $|OF| \le |OF'|$ and

$$|A(t)F| \le |A(t)F'| = |AF^*|. \tag{5}$$

Now, let l be the line perpendicular to the xy-plane passing

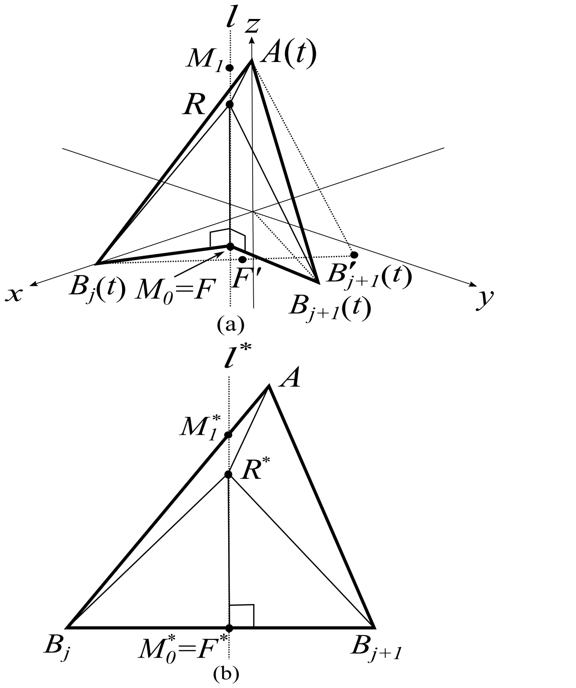

Fig. 9 (a) $\triangle A(t)B_j(t)B_{j+1}(t)$ at time t ($0 \le t \le 2$) and (b) $\triangle AB_jB_{j+1}$ at time 0.

through F, and l^* be the line on the plane including $\triangle AB_jB_{j+1}$, perpendicular to B_jB_{j+1} passing through F^* (Fig. 9 (b)). Then, l^* intersects the edges AB_j or AB_{j+1}. Without loss pf generality, we can assume that l^* intersects the edge AB_j. Let $M_0^* = F^*$, $M_1^* = l^* \cap AB_j$, $M_0 = F$, and $M_1 = (x, y, |M_0^*M_1^*|)$.

Because $\triangle B_jF^*M_1^* \equiv \triangle B_j(t)FM_1$ and

$$|B_jM_1^*| + |AM_1^*| = |AB_j| = |A(t)B_j(t)| \le |B_j(t)M_1| + |A(t)M_1|,$$

it follows that

$$|AM_1^*| \le |A(t)M_1| \tag{6}$$

holds.

Let M and M^* be moving points on the line segments M_0M_1 and $M_0^*M_1^*$ with $|M_0M| = |M_0^*M^*|$, respectively, where they move continuously. Then, it is shown by Eqs. (5), (6), and the continuity of real numbers that there exist points R and $R^* \in F^*M_1^*$ such that $F^*R^* \perp B_jB_{j+1}$, $|A(t)R| = |AR^*|$, and $R = (x, y, |F^*R^*|)$. □

Lemma 4 implies that the 1-skeleton of $\mathcal{P}(\Gamma_n; A_1, \ldots, A_k)$ has a folded state of each of the triangular faces $\triangle A_iB_jB_{j+1}$ in $\mathbb{R}^3$ with creases A_iR_{ij}, B_jR_{ij}, $B_{j+1}R_{ij}$, and F_jR_{ij}, where F_j and R_{ij} correspond to the points F and R in Lemma 4. Moreover, we must avoid collisions between the folded states of the faces. For each $1 \le j \le n$, an orthogonal projection of $\triangle A_iB_jB_{j+1}$ ($1 \le i \le k$) to the xy-plane is $\triangle OB_jB_{j+1}$. Hence, $\triangle A_iB_jB_{j+1}$ and $\triangle A_{i'}B_{j'}B_{j'+1}$ ($1 \le i \le i' \le k$, $1 \le j < j' \le n$) do not experience collisions. Next, for each j ($1 \le j \le n$), we show that the relative interiors of $A_i(t)R_{ij}$ ($1 \le i \le k$) are disjoint at any time t ($0 \le t \le 2$).

We choose any j and fix it ($1 \le j \le n$). Additionally, we rewrite R_i instead of R_{ij} for the sake of simplicity.

Lemma 5. *For any* $1 \le i < i' \le k$,

$$|OA_i(t)| \le |OA_{i'}(t)| \quad \text{if and only if} \quad |FR_i| \le |FR_{i'}|,$$

where F, R_i *and* $R_{i'}$ *are points corresponding to* F *and* R *in Lemma* 4*, respectively.*

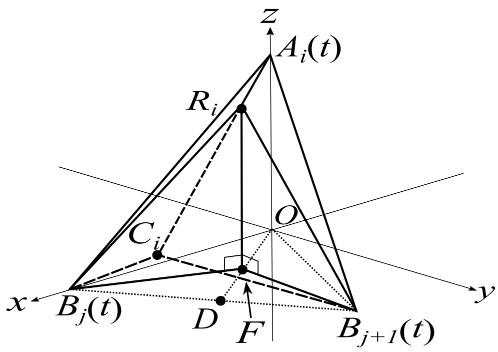

Fig. 10 Point C_i on the line A_iR_i.

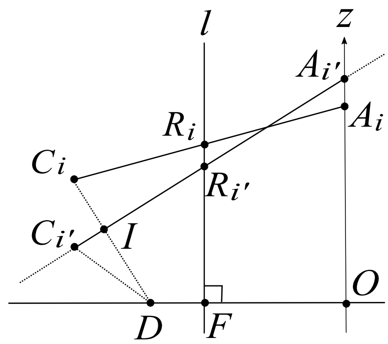

Fig. 11 Positional relationship among points on the plane H when $|OA_i| \le |OA_{i'}|$ and $|FR_i| \ge |FR_{i'}|$.

Proof. We can assume without loss of generality that $A_i(t)$ and $A_{i'}(t)$ lie on the positive z-axis. Let Π be the plane including the four points O, $A_i(t)$, $A_{i'}(t)$, and F, and let D be the intersection of Π and the line segment $B_j(t)B_{j+1}(t)$ (**Fig. 10**). Let C_i and $C_{i'}$ be points on the straight lines $A_i(t)R_i$ and $A_{i'}(t)R_{i'}$, respectively, such that

$$|B_j(t)C_i| + |B_{j+1}(t)C_i| = |B_j(t)F| + |B_{j+1}(t)F| = |B_jB_{j+1}|$$

and

$$|B_j(t)C_{i'}| + |B_{j+1}(t)C_{i'}| = |B_j(t)F| + |B_{j+1}(t)F| = |B_jB_{j+1}|.$$

We assume that $|OA_i| \le |OA_{i'}|$ and $|FR_i| \ge |FR_{i'}|$ hold (**Fig. 11**). Then, there exists an intersection I of the straight line $A_{i'}R_{i'}$ and the line segment C_iD. Since F^* (corresponds to F^* in Lemma 4) is on the edge B_jB_{j+1} of $\triangle A_iB_jB_{j+1}$ and the point $C_{i'}$ corresponds to F^* on the line $A_{i'}(t)R_{i'}$, we have

$$\begin{aligned} |B_j(t)I| + |B_{j+1}(t)I| &> |B_j(t)C_{i'}| + |B_{j+1}(t)C_{i'}| \\ &= |B_jB_{j+1}|. \end{aligned} \tag{7}$$

In contrast, by $\angle B_j(t)IB_{j+1}(t) > \angle B_j(t)C_iB_{j+1}(t)$,

$$|B_j(t)I| + |B_{j+1}(t)I| < |B_j(t)C_i| + |B_{j+1}(t)C_i| = |B_jB_{j+1}|.$$

This contradicts (7). □

Lemmas 4 and 5 play an important role in Section 5.

5. Continuous Flattening Motions of Multi-layered Pyramids

One of our main results in this paper is the existence of a continuous flattening motion for a multi-layered pyramid. Here, every apex of the multi-layered pyramid is on the z-axis, where every z coordinate of the apexes can take any real number (including

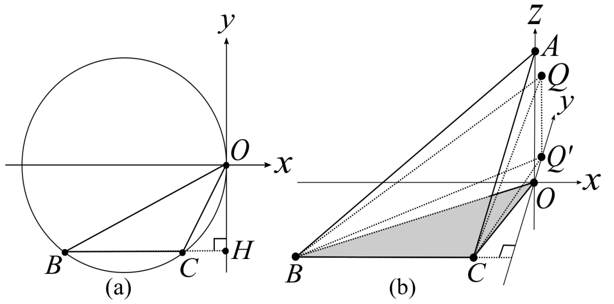

Fig. 12 (a) Circumcircle of $\triangle OBC$ and (b) $\triangle ABC$ and $\triangle QBC$.

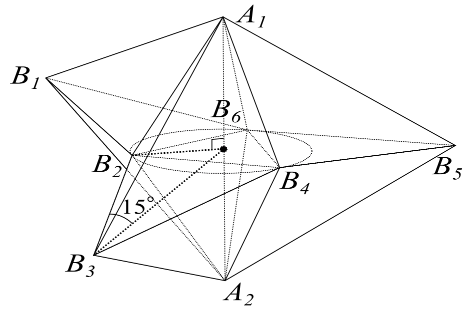

Fig. 13 Multi-layered pyramid $\mathcal{P}(\Gamma; A_1, A_2)$ with a star hexagonal base.

negative real numbers).

Now, Theorem 1 can be proved as follows.

Proof of Theorem 1. Let $\mathcal{P}(\Gamma_n; A_1, \ldots, A_k)$ be a multi-layered pyramid with $\Gamma_n = B_1 \cdots B_n$. Because Γ_n is convex and A_i $(1 \leq i \leq k)$ are on the z-axis, Lemmas 2 through 5 determine the positions of $A_i(t)$, $B_j(t)$, $F_j(t)$, and $R_{ij}(t)$ $(1 \leq i \leq k,\ 1 \leq j \leq n,\ 0 \leq t \leq 2)$, each of which is continuous for $0 \leq t \leq 2$. Hence, each $\mathcal{P}(\Gamma_n; A_i)$ can be continuously flattened, satisfying all the required conditions. Moreover, the motions of $\mathcal{P}(\Gamma; A_i)$ $(1 \leq i \leq k)$ synchronize with one another without collisions. □

6. Non-existence of Continuous Flattening

We show that the condition of convexity is necessary for Theorem 1. The following lemma plays a key role in the proof of Theorem 2.

Lemma 6. *Let BC be a line segment on the xy-plane satisfying the following two conditions:* (*1*) *BC is parallel to the x-axis, and* (*2*) *the circumcircle of $\triangle OBC$ is tangent to the y-axis* (***Fig. 12*** (*a*)). *Let A be any point on the z-axis, and let Q be a point obtained by rotating A about the line BC. Q' denotes the orthogonal projection of Q on the xy-plane. Then,*

$$\angle BOC > \angle BQ'C \tag{8}$$

(Fig. 12 (*b*)).

Proof. Because BC is orthogonal to the y-axis, Q' must be on the y-axis. Hence, Q' is outside the circumcircle of $\triangle OBC$. Thus, it follows that $\angle BOC > \angle BQ'C$. □

Note that, since the alternate segment theorem shows $\angle OBC = \angle COH$ in Fig. 12 (a), it is seen that on the xy-plane, a segment BC parallel to the x-axis satisfies the condition (2) in Lemma 6 if and only if

$$2\angle OBC = 90^\circ - \angle BOC. \tag{9}$$

Hence, we can apply Lemma 6 to $\triangle OBC$ satisfying (9) on the xy-plane and any point A on the z-axis; then, we have the inequality (8). Furthermore, we can uniquely determine $\angle BQ'C$ from the form of $\triangle ABC$ and the z-coordinate of Q in Lemma 6.

We prove Theorem 2 by illustrating an example.

Proof of Theorem 2. Let $\Gamma = B_1B_2B_3B_4B_5B_6$ be a star hexagon such that

$$\angle B_2B_3B_4 = \angle B_4B_5B_6 = \angle B_6B_1B_2 = 30^\circ$$

and

$$\angle B_1B_2B_3 = \angle B_3B_4B_5 = \angle B_5B_6B_1 = 210^\circ$$

(**Fig. 13**). We consider $\mathcal{P}(\Gamma; A_1, A_2)$ such that $A_1 = (0, 0, 1)$, $A_2 = (0, 0, -1)$, and $\angle B_jOB_{j+1} = 60^\circ$ $(1 \leq j \leq 6)$.

Let us suppose that there exists a continuous flattening motion of $\mathcal{P}(\Gamma; A_1, A_2)$ with rigid radial edges for time t from 0 to 1. Since we cannot stretch any edge during the flattening motion, $|B_j(t)B_{j+1}(t)| \leq |B_jB_{j+1}|$ must hold for any t $(0 \leq t \leq 1)$.

If A_1 and A_2 are fixed at the initial positions, then $|OB_j(t)|$ cannot change and $\angle B_j(t)OB_{j+1}(t) \leq 60^\circ$ holds for any j $(1 \leq j \leq 6)$. Hence, $\sum_{j=1}^{6} \angle B_j(t)OB_{j+1}(t) = 360^\circ$ implies that both $\angle B_j(t)OB_{j+1}(t) = 60^\circ$ and $|B_j(t)B_{j+1}(t)| = |B_jB_{j+1}|$ $(1 \leq j \leq 6)$ hold for $0 \leq t \leq 1$, that is, $\mathcal{P}(\Gamma; A_1, A_2)$ is rigid. Thus, without loss of generality, at the beginning of any motion of $\mathcal{P}(\Gamma; A_1, A_2)$ we can assume that A_1 and A_2 are either approaching or leaving each other on the z-axis. Moreover, because of the rigidity of all radial edges, we can also assume that $B_1, \ldots, B_6$ continuously move on the xy-plane synchronizing with the motions of A_1 and A_2 such that $B_1(t), \ldots, B_6(t)$ are in anticlockwise order about O and satisfy $\sum_{j=1}^{6} \angle B_j(t)OB_{j+1}(t) = 360^\circ$.

Let us consider six triangles $\triangle OB_jB_{j+1}$ $(1 \leq j \leq 6)$. Since

$$2\angle OB_{2j-1}B_{2j} = 30^\circ,$$
$$90^\circ - \angle B_{2j-1}OB_{2j} = 90^\circ - 60^\circ = 30^\circ$$

and

$$2\angle OB_{2j+1}B_{2j} = 30^\circ,$$
$$90^\circ - \angle B_{2j+1}OB_{2j} = 90^\circ - 60^\circ = 30^\circ$$

for $1 \leq j \leq 3$, the triangles $\triangle OB_jB_{j+1}$ $(1 \leq j \leq 6)$ satisfy the condition (9), where $B_7 = B_1$. Hence, Lemma 6 can be applied for each of the triangles $\triangle A_1B_jB_{j+1}$ $(1 \leq j \leq 6)$, which are congruent to one another.

We fix $t > 0$ at the beginning of the motion of $\mathcal{P}(\Gamma; A_1, A_2)$. We show that $\angle B_j(t)OB_{j+1}(t) < 60^\circ$ holds for any j $(1 \leq j \leq 6)$. Then, we choose any j and fix it $(1 \leq j \leq 6)$. Now, we consider $\angle B_j(t)OB_{j+1}(t)$ in two cases: $|B_j(t)B_{j+1}(t)| = |B_jB_{j+1}|$ and $|B_j(t)B_{j+1}(t)| < |B_jB_{j+1}|$.

(Case I) If $|B_j(t)B_{j+1}(t)| = |B_jB_{j+1}|$ holds, then $\triangle A_1(t)B_j(t)B_{j+1}(t) \equiv \triangle A_1B_jB_{j+1}$ holds and we can uniquely determine $\angle B_j(t)OB_{j+1}(t)$ from the position of $A_1(t)$. Put $\theta(t) = \angle B_j(t)OB_{j+1}(t)$. Lemma 6 shows that $\theta(t) < \angle B_jOB_{j+1} = 60^\circ$, because the z-coordinate of $A(t)$ is not equal to the one of A.

(Case II) Even if $|B_j(t)B_{j+1}(t)| < |B_jB_{j+1}|$ holds, $|OB_j(t)|$ and $|OB_{j+1}|$ exhibit the same values as those in Case I. Hence, by the law of cosines, with $\triangle OB_j(t)B_{j+1}(t)$, it follows that $\angle B_j(t)OB_{j+1}(t) < \theta(t) < 60^\circ$.

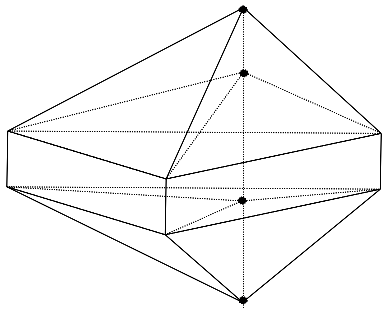

Fig. 14 Two multi-layered pyramids and a triangular prism.

Since all of the triangles $\triangle A_1(t)B_j(t)B_{j+1}(t)$ $(1 \leq j \leq 6)$ are congruent to one another, we have $\sum_{j=1}^{6} \angle B_j(t)OB_{j+1}(t) \leq 6\theta(t) < 360°$, which is a contradiction. Hence, a continuous flattening motion cannot exist for $\mathcal{P}(\Gamma; A_1, A_2)$. □

Remark 2. *By the condition* (9)*, as examples for the proof of Theorem 2, multi-layered pyramids* $\mathcal{P} = \mathcal{P}(\Gamma_{2n}; A_1, A_2)$ *with a star* $2n$*-gonal base* $(n \geq 3)$ *can be obtained more generally such that*

$$\angle OB_{2j-1}B_{2j-2} = \angle OB_{2j-1}B_{2j} = 45° - \frac{90°}{n},\ 1 \leq j \leq n$$

and

$$\angle OB_{2j}B_{2j-1} = \angle OB_{2j}B_{2j+1} = 135° - \frac{90°}{n},\ 1 \leq j \leq n,$$

where $B_0 = B_{2n}$ *and* $B_1 = B_{2n+1}$.

Remark 3. *Our method can be applied for more general types of multi-layered structures, for example, a structure consisting of two multi-layered pyramids and a prism orthogonal to the xy-plane, as shown in* ***Fig. 14****. The triangular prism can be flattened according to the motions of horizontally aligned edges obtained with our method for the top and bottom multi-layered pyramids. For such structures, we can provide a continuous flattening motion if the apexes do not collide with one another during the lifting motions.*

However, it appears to be difficult to show the existence (*or non-existence*) *of a continuous flattening motion for a given multi-layered pyramid with a non-convex base. We intend to investigate this problem in the future.*

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Kazuki Matsubara received his M.S. and Ph.D. degrees from Hiroshima University in 2005 and 2014, respectively.

Chie Nara received her B.A., M.S., and Ph.D. degrees from Ochanomizu University. She served as a professor at Tokai University before taking up the current research position at Meiji University.