

## Regular Paper

# Efficient Algorithm for $2 \times n$ Map Folding with a Box-pleated Crease Pattern

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**Abstract:** In this paper, we study a variation of the map folding problem. The input is a  $2 \times n$  map with a box-pleated crease pattern of size  $2 \times n$ . Precisely, viewing the crease pattern as a planar graph, its vertices and edges respectively form the subsets of the vertices set and edges set of the planar graph of the square and diagonal grid. The question is whether the map can be flat-folded or not. If the answer is yes, then what is the time complexity to make the decision? Our conclusion is that any locally flat-foldable  $2 \times n$  map with such a box-pleated crease pattern is globally flat-foldable. We present linear-time algorithms for both deciding the flat foldability and finding a feasible way of folding.

**Keywords:** map folding problem, flat-foldable, linear-time algorithm

## 1. Introduction

In computational origami, one of the most popular problems is the *flat-folding problem*, which asks whether a paper with a given crease pattern can be flat-folded [1]. A flat-folding refers to a mapping from the paper to its folded state satisfying that the image of each crease is a line segment with a dihedral angle measuring either  $\pi$  or  $-\pi$  and the image of each face is a congruent face. In the folded state, each face is also called a *layer*. The paper layers should satisfy certain conditions that inhibit the paper from penetrating itself. The subproblem obtained when we restrict the input to have a single vertex is called *single-vertex flat foldability* (or *local flat foldability of a vertex*). For the single-vertex flat foldability, a solution to the decision problem was proposed in Ref. [1]. When a general crease pattern has a global mountain–valley assignment for which every vertex follows the single-vertex flat foldability, we say that the entire crease pattern is *locally flat-foldable*. For the decision of local flat foldability of entire crease patterns, a linear-time algorithm was given in Ref. [2]. Here, we mention two major conditions concerning angles and assignments of mountain/valley foldings:

**Condition 1 (Kawasaki [3], Justin [4]):** For a flat-foldable vertex, the alternate angles between its adjacent creases must sum to  $\pi$ .

**Condition 2 (Maekawa [5], Justin [4]):** For a flat-foldable vertex, the numbers of related creases assigned to be mountains and valleys differ by  $\pm 2$ .

On the other hand, for a piece of paper with a general crease pattern, the *global flat foldability*, i.e., whether a flat-folded state really exists, is intractable. Since Bern and Hayes first showed

that the flat-folding problem is NP-hard in general [2], this problem has been widely investigated for many variants.

The map folding problem has been studied for almost 40 years as a simpler version. However, even in this restricted case, there remain many unsolved problems [6]. In the standard map folding problem, a map is defined by a rectangular sheet with a square grid pattern. The sheet is specified as an  $m \times n$  regular square grid. Its mountain-valley assignment is defined as a mapping from the collection of non-boundary edges of every square to the set  $\{M, V\}$ , where  $M$  and  $V$  refer to mountain and valley folds, respectively.

In Ref. [6], they primarily investigated the map folding problem on a *simple folding model*, where the crease pattern must be folded by a sequence of simple *simple folds* which rigidly rotates a subset of the paper about the supporting line of a subset of creases. In this simple model, they showed the weak NP-completeness of the map folding problem for both the maps in a rectangular shape with diagonal creases and the maps with the regular square grid pattern but on an orthogonal piece of paper. Recently, results regarding the hardness have been extended and strengthened to more general simple folding models in Ref. [7].

When we turn to the general folding model, the map folding problem asks if a feasible folded state is consistent with a given crease pattern with/without an  $MV$  assignment. This problem reflects different aspects than the simple folding model. In Ref. [8], it was claimed that the map folding problem for a map of size  $2 \times n$  with an  $MV$  assignment could be solved in  $O(n^9)$  time. For a map of size  $m \times n$ , a method to decide the validity of the overlap orders of layers in the final folded state is given in Ref. [9]. Note that, counterintuitively, this problem is quite complicated. **Figure 1** gives the minimal map unable to be flat-folded, which is of size  $2 \times 5$ . However, it is quite difficult to understand why the map could not be folded, even in practice.

The  $MV$  assignment appears to have a significant effect on the difficulty of the map folding problem. Conversely, it is trivial to

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fold any regular square grid without an  $MV$  assignment.

As an extension of the square grids in the map folding problem, in this paper, we consider the patterns whose creases are only on a subset of a square grid and the diagonals of the squares, which are named *box-pleating* in Ref. [10]. The box-pleated patterns are popularly used for constructing arbitrary polycubes. Their practical uses also involve transformational robotics and self-assembly. In Ref. [10], it was presented that deciding the flat foldability of box-pleated crease patterns is NP-hard no matter with or without an  $MV$  assignment when  $m$  and  $n$  are both restricted to be relatively large numbers, at least much larger than an  $m$  as 2 in the current paper.

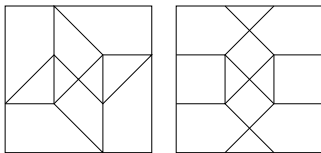
However, there may exist polynomial-time solutions for the flat foldability of maps with box-pleated crease patterns when the size of the maps is restricted. In this context, recently, maps of size  $1 \times n$  were investigated by the first two authors [11]. The considered maps are of size  $1 \times n$  with box-pleated patterns and without an  $MV$  assignment. They proved that every such map is globally flat-foldable as long as it is locally flat-foldable. On the other hand, for the maps with box-pleated patterns of size  $3 \times 3$ , instances which cannot be flat-folded exist even when the entire map is locally flat-foldable. Two representative patterns unable to be flat-folded are illustrated in Fig. 2 (exemplified in Ref. [12]). The pattern on the right is not locally flat-foldable. The pattern on the left is locally flat-foldable, and its inability to be folded is due to inevitable self-intersections.

In this paper, we investigate maps of size  $2 \times n$  with box-pleated crease patterns (Fig. 3) and with no  $MV$  assignment. The main theorem is as follows:

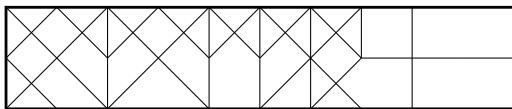
**Theorem 1.1.** *Let  $P$  be a map of size  $2 \times n$  with a box-pleated crease pattern, which involves creases on a subset of a square grid and the diagonals of the squares, but without an  $MV$  assignment. Then, any  $P$  satisfying local flat foldability can be flat-folded. Moreover, finding a method to fold  $P$  takes linear time.*



**Fig. 1** Minimal unfoldable map of size  $2 \times 5$ . The red solid line segments indicate mountains, whereas the blue dashed line segments indicate valleys.



**Fig. 2** (Left) Locally flat-foldable but not globally flat-foldable crease pattern. (Right) A crease pattern that is not locally flat foldable although each vertex individually can be flat folded.



**Fig. 3** Instance of a map of size  $2 \times 10$  without  $MV$  assignment. Bold lines indicate the border of  $P$ , and thin lines are creases in a crease pattern.

## 2. Terminology

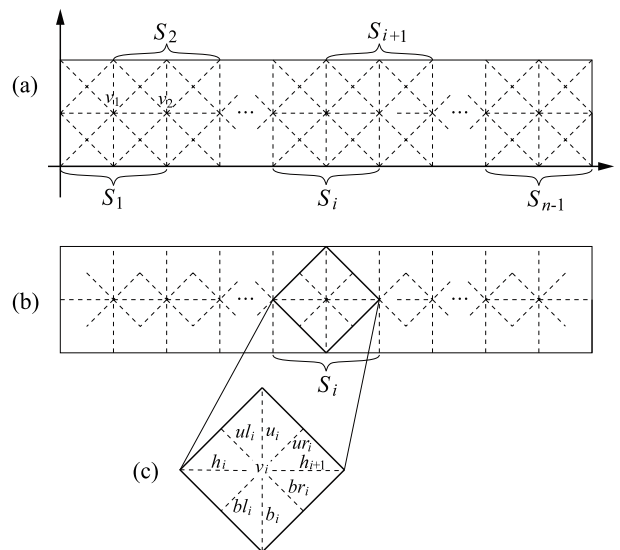
This section presents definitions of the terms used in this paper.

Our input is a map denoted by  $P$ , which is of size  $2 \times n$  with a box-pleated crease pattern. Its *box-pleated crease pattern* is defined as a subset of line segments in the square grid with diagonals other than the boundary of  $P$  (illustrated in Fig. 3). We distinguish between the front and the back of  $P$ . Following the terminology in Ref. [6], the line segments in the crease pattern are called *creases* and the endpoints of creases inside  $P$  are called *vertices*. Furthermore, a *mountain-valley assignment* (an  $MV$  assignment) decides the way that creases are supposed to be folded. In an  $MV$  assignment, every crease is assigned either a *mountain* (“ $M$ ”) or a *valley* (“ $V$ ”), which are denoted by red solid lines and blue dashed lines, respectively, in the illustrations of the present paper.

We suppose that there exists a right-handed Cartesian coordinate, the unit length of which is equal to the edge length of a unit square in  $P$ , and  $P$  is located entirely in the first quadrant with its lower-left corner located at the origin. Along the horizontal axis, a square of  $2 \times 2$  size centered at point  $(i, 1)$  ( $1 \leq i \leq n - 1, i \in \mathbb{N}$ ) is defined as the  $i$ th *section*, denoted by  $S_i$  with its center point denoted by  $v_i$  (see Fig. 4 (a)). Note that a pair of adjacent sections shares an area of size  $1 \times 2$ .

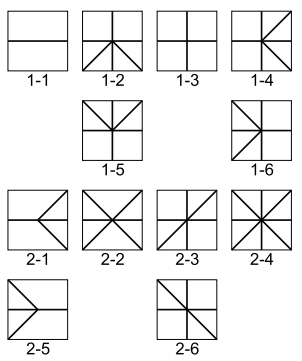
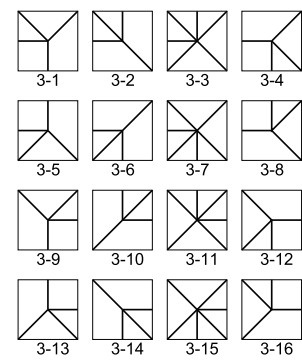
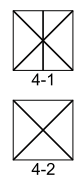
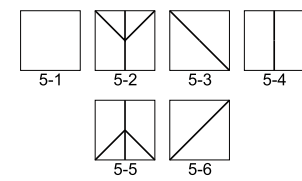
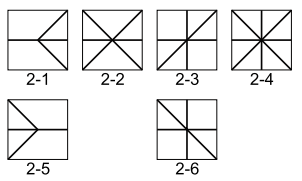
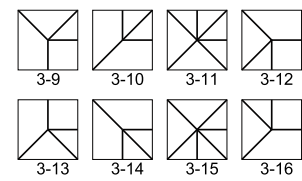

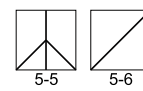
The creases incident to  $v_i$  within  $S_i$  are labeled  $h_i, u_i, b_i, ul_i, bl_i, ur_i,$  and  $br_i$ , as illustrated in Fig. 4 (c). Moreover,  $h_i$  is assigned to the left of  $v_i$ . In order to avoid duplicated labeling, the crease to the right of  $v_i$  is labeled  $h_{i+1}$ . The set of  $v_i$ s is denoted by  $\overline{V}_8$  because up to eight creases can exist around any  $v_i$ . Similarly, the set of remaining vertices (i.e., the vertices located at points  $(i - 1/2, 1/2)$  and  $(i - 1/2, 3/2)$  for  $i$  with  $1 \leq i \leq n$ ) is denoted by  $\overline{V}_4$  because each vertex has up to four incident creases. In order to prepare the discussion in the following sections, let  $C$  be the set of all creases incident to vertices in  $\overline{V}_8$ . Figure 4 (c) illustrates the creases in  $C$  around a single vertex.

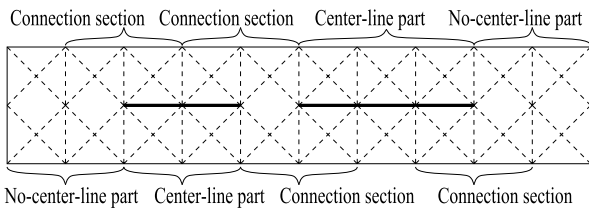
Here,  $P$  is separated into several parts regarding the horizontal creases, as illustrated in Fig. 5. Each part is classified into either



**Fig. 4** (a) Indexing of sections. (b) Creases in  $C$ . (c) Labels of creases incident to  $v_i$ .

**Table 1** Thirty-six possible patterns satisfying local flat foldability and associated operations.

Collection	Center-line part	Connection section	No-center-line part	
Pattern				
				
Operation	Direct folding			Combination folding



**Fig. 5** Separation with respect to the horizontal center line. The existence of creases at locations marked by dashed lines has no effect on the separation.

a *center-line part* or a *no-center-line part* according to the existence of horizontal creases. A center-line part is a part of consecutive sections each with two horizontal creases. A no-center-line part is a part of consecutive sections with no horizontal crease. A section joining two such parts is called a *connection section*, which shares its horizontal crease with a center-line part.

Our approach to fold  $P$  without self-intersection is to find an  $MV$  assignment which enables  $P$  to be folded into a zigzag form (in a rough sense). Specifically, the proposed approach can be described as follows. While folding  $P$  from  $S_1$  to  $S_{n-1}$  in order,  $S_{i+1}$  is always supposed to be folded to layers above  $S_i$  or to the same layer as  $S_i$ . We refer to this principle as the *overlap principle*. In the following discussion,  $P$  is postulated to be locally flat-foldable.

### 3. Outline of the Proof

Our proof is based primarily on the separation of  $P$ . A sufficient condition for flat foldability can be specified as the conjunction of three conditions as (1) no self-intersection appears in any part of  $P$ , (2) the overlap principle is satisfied in every part, and (3) the parts are able to be connected without violating the overlap principle. If these conditions can be simultaneously satisfied, then  $P$  is globally flat-foldable. In order to prove Theorem 1.1, we show that the proposed method of folding ensures the reachability of such a state as long as  $P$  is locally flat-foldable. Specifically, a general method to fold  $P$  into a zigzag form will be described in detail.

We classify 36 possible locally flat-foldable patterns in a single section with respect to  $C$  and provide two corresponding folding methods: *direct folding* (without unfolding) and *combination*

*folding* (a combination of folding and unfolding operations). The summary in **Table 1** concerns certain methods of folding corresponding to all these patterns. Reasonable  $MV$  assignments will be given step-by-step for every pattern and their combinations. Section 4.1 explains that we can limit the analysis to  $C$ , rather than analyzing all of the creases. In Sections 4.2 and 4.3, we explain how to specify the  $MV$  assignments for every center-line part and no-center-line part, respectively, in order to flat fold these parts under the overlap principle. In Section 4.4, we prove that every connection section can be effectively flat-folded to join a center-line part and a no-center-line part.

In conclusion,  $P$  can be flat-folded if and only if its vertices are all locally flat-foldable. The decision on whether  $P$  can be flat-folded is solvable in linear time by checking the vertices one by one locally. The linear-time algorithm provided in Section 5 will guide a flat-foldable  $P$  to its final flat-folded state by specifying the folding process.

### 4. Flat Foldability Realized in All Possible Patterns and Connections

We prove Theorem 1.1 in this section. Only the patterns with respect to  $C$ , namely the set of creases around vertices in  $\overline{V}_8$ , are considered in our proof. The reason for this is given at the beginning of Section 4.1. Then, methods to fold arbitrary center-line parts and no-center-line parts are described in Sections 4.2 and 4.3. Finally, all possibilities of their boundary sections are discussed in Section 4.4 in the form of exhausting their combinations, each of which encloses a connection section.

#### 4.1 Why Only $C$ is Considered

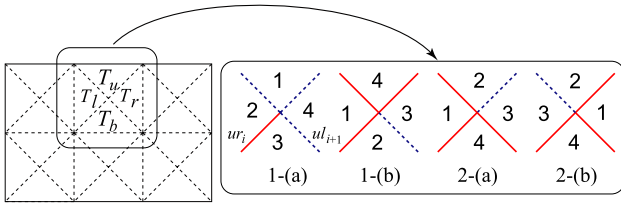
We consider only the creases in  $C$  in our proof. We explain the reason for this through a proof of the following proposition by showing that there always exists a valid  $MV$  assignment for the creases not in  $C$  and a corresponding feasible overlap of the layers, even after an  $MV$  assignment is defined on the creases in  $C$ .

**Proposition 4.1.** *The flat foldability of  $P$  can be decided only by the creases in set  $C$ .*

A  $1 \times 2$  area is shared by a pair of adjacent sections  $S_i$  and  $S_{i+1}$ . Here, we focus on a single square in the upper row (The bottom

**Table 2** *MV* assignments and the overlapping order in every folded state. The numbers in ascending order indicate the corresponding layers from bottom to top.

Case	1	2	3	4	5	6	7	8
MV assignment and overlapping order	1-1 	1-2    1-5 	1-3 	1-4    1-6 	2-1    2-5 	2-2 	2-3    2-6 	2-4 



**Fig. 6** Two possible cases for creases around a vertex in  $\bar{V}_4$ .

row can be handled in exactly the same manner). There are three possibilities, corresponding to zero, two, and four creases in the square. For the case of two creases, the local flat foldability forces the creases to be on the same line and with the same assignment. Thus, the entire assignment in the square can be decided by only the creases in  $C$ . The remaining case is such that four creases exist in the square, two of which are in  $C$  and the other two of which are not in  $C$ . We show that, once the assignment of the two creases in  $C$  is decided, the assignment of the remaining creases can always be decided while maintaining the overlap relation.

There exist eight locally flat-foldable  $MV$  assignments in the square. Only half of these assignments, in which  $ur_i$  is assigned to be a mountain, as illustrated in **Fig. 6**, are explained here. The other assignments can be obtained by interchanging the roles of  $M$  and  $V$ . Four triangles separated by creases are identified by labels  $T_r, T_u, T_l,$  and  $T_b$ , as in **Fig. 6**. The numbers shown in the triangle areas indicate the overlapping orders of these four triangles in the folded states when  $T_b$  faces the front up.

In order to maintain the overlap principle,  $T_r$  must be placed above  $T_l$ . When  $M$  and  $V$  are assigned to  $ur_i$  and  $ul_{i+1}$ , respectively, there exist two possible  $MV$  assignments in the square, as shown by 1-(a) and 1-(b) in **Fig. 6**. Since  $T_r$  is placed above  $T_l$  in both cases, we can choose either  $MV$  assignment. In another case, when  $M$  is assigned to both  $ur_i$  and  $ul_{i+1}$ , there also exist two possible  $MV$  assignments in the square, as shown in 2-(a) and 2-(b). In order to place  $T_r$  above  $T_l$ , we choose the assignment of 2-(a) in this case. When the  $MV$  assignments are reversed from the abovementioned cases, 2-(b) will be chosen.

Each end of  $1 \times 2$  size of  $P$  is also considered in the same manner. We can choose any of 1-(a), 1-(b), and 2-(a) according to the assignment of the crease in  $C$ .

The folds along these creases in a certain section do not cause intersection with other sections because this section is folded to different layers from other sections. Moreover, these folds do not cause self-intersections within a single section. Thus, the first condition can be satisfied.

Since the overlap principle is required for folding operations according to  $C$  and creases other than  $C$  would not violate the overlap principle, which agree with the latter two conditions, the correctness of Proposition 4.1 is claimed.

In the following sections, we discuss the flat foldability of different parts. Based on Proposition 4.1, we consider only the creases in  $C$  in the following sections.

**4.2 Folding a Center-line Part**

As the classification shown in Table 1, a center-line part is handled with direct folding, which is a sequence of only folding and no unfolding operations.

For each of the 12 possible patterns of a section, we show available  $MV$  assignments and the corresponding overlapping orders under the overlap principle in **Table 2** (We assumed that the area directly under the left horizontal crease faces the front up in the upper illustrations and the back of this area faces up in the lower illustrations). The folding of each section (except for the sections in Pattern 1-1 ) as indicated in Table 2, places its left section under its right section. The section in Pattern 1-1 puts the right half of its left adjacent section in the same layer as the left half of its right section.

Note that both ends of size  $1 \times 2$  of a center-line part are not discussed at present. Their folding operations are specified in Section 4.4 when the connection sections are analyzed.

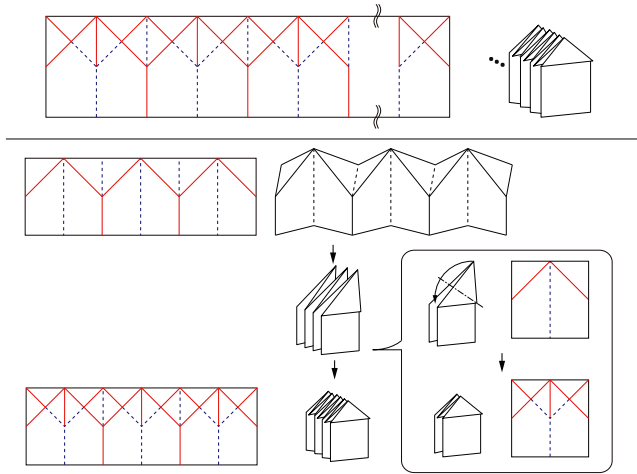
The above analysis is concluded considering the following Lemma.

**Lemma 4.2.** Any center-line part in  $P$  can be flat-folded in a zigzag manner.

*Proof.* The flat-folded state of a center-line part can be achieved by applying the above operations to the sections one by one from left to right. Specifically, each section should be folded as illustrated in the upper row in Table 2 when the area directly under the left horizontal crease faces the front up, otherwise, each section should be folded as shown in the lower row. □

**4.3 Folding a No-center-line Part**

This section considers the patterns in a no-center-line part. We use combination folding to handle these patterns, except for Pattern 4-1 and Pattern 4-2 . Patterns 4-1 and 4-2 are distinguished from the other patterns because folding either of these patterns can be seen as separating  $P$  into two individual parts without any interference with each other. This means that either part can be handled as a new independent map. We use combi-



**Fig. 7** The no-center-line parts and the method to fold them. The upper illustration shows an initial flat-folded state of a no-center-line part and its  $MV$  assignment, and the lower illustration shows how to fold the part.

nation folding instead of direct folding because the no-center-line part includes only vertical and diagonal creases, which are similar to the  $1 \times n$  patterns introduced in [11]. We extend their method of achieving the final flat-folded state via a sequence of folds and unfolds for application to the no-center-line part. Although there may exist other ways to flat fold this part, we use combination folding.

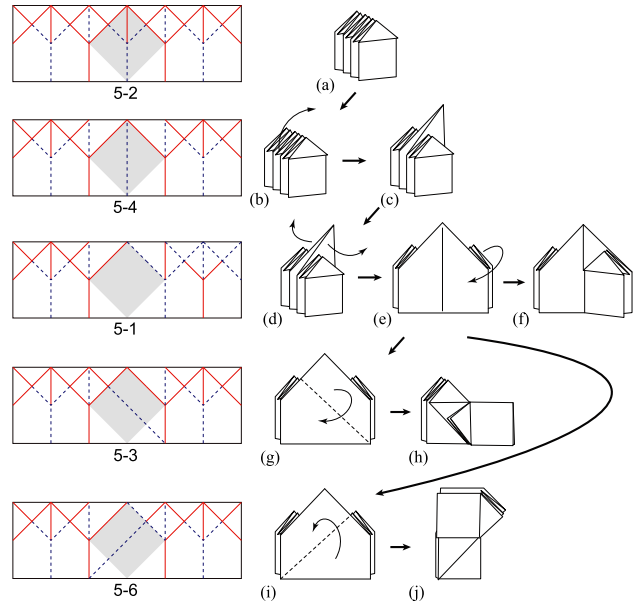
The combination folding that we use here initially folds each section as Pattern 5-2, and then modifies the sections to the intended patterns. By using Pattern 5-2 to define an intermediate state on the way to the final flat-folded state, the final zigzag form can be easily achieved.

Let  $\{S_i \mid a \leq i \leq b\}$  be a maximal no-center-line part without Pattern 4-1 or Pattern 4-2 for the same  $a$  and  $b$ . The basic idea is to fold every  $S_i$  initially as to fold Pattern 5-2 with two different  $MV$  assignments alternately with respect to  $i$  (see Fig. 7). The  $MV$  assignment ensures the initial flat-folded state, as shown in the upper illustration. The unfolds are then applied during two phases. The first phase adjusts the sections in  $\{S_i \mid a \leq i \leq b, i \text{ is odd}\}$  to their actual patterns, whereas the second phase adjusts sections in  $\{S_i \mid a \leq i \leq b, i \text{ is even}\}$ . Without loss of generality, we assume that both  $a$  and  $b$  are odd.

Now, we present a proposition to state the feasibility of the first phase. Only the individual sections have to be considered at this point because their adjacent sections are currently fixed to Pattern 5-2.

**Proposition 4.3.** *The flat-folded states of Patterns 5-1, 5-3, 5-4, 5-5, and 5-6 can always be obtained from the flat-folded state of Pattern 5-2 while maintaining the overlap principle.*

*Proof.* The adjustment from Pattern 5-2 to every other pattern is shown in Fig. 8 by introducing the  $MV$  assignments and the corresponding states side by side. A section in Pattern 5-4 only needs a single unfold within its upper half, while the adjustments to Pattern 5-3 and Pattern 5-6 involve unfolds along the vertical creases followed by folds along diagonal



**Fig. 8** Operations of combination foldings.

creases. Pattern 5-1 requires changing the  $MV$  assignment of the sections right to it because the unfolds along the vertical creases of this pattern would violate the overlap principle. This revision can be seen as an alternation of the  $MV$  assignments between sections with odd and even indexes.

The illustration for Pattern 5-5 is omitted because this pattern can be folded with the same operation of Pattern 5-4 and the succeeding folds along its two diagonal creases, as shown in Pattern 5-2.

The first phase ends when all sections in  $\{S_i \mid a \leq i \leq b, i \text{ is odd}\}$  are folded. The next objective is to fold the sections in  $\{S_i \mid a \leq i \leq b, i \text{ is even}\}$  without violating the overlap principle. For each individual, the folded states of patterns in adjacent sections have been modified from Pattern 5-2. Therefore, we have to consider all possible combinations of the six patterns in the area covered by the two sections adjacent to  $S_i$  (a  $2 \times 4$  portion with center point  $v_i$ ). In order to handle each  $S_i$ , we first adjust  $S_i$  to a tractable state according to  $S_{i-1}$  and  $S_{i+1}$  and then apply other folds.

Next, we give the second proposition for the no-center-line part and discuss all cases in detail in order to prove this proposition.

**Proposition 4.4.** *All possible combinations of two patterns in a no-center-line part can be flat-folded.*

*Proof.* This proposition is proven by categorizing all cases of possible combinations in a no-center-line part and listing their corresponding  $MV$  assignments, which lead to flat-folded states. In addition to the discussion on flat foldability, since we intend to fold the no-center-line part into a flat state which also obeys the overlap principle, the method of folding provided here is in fact an adjustment from the folded state obtained at the first phase to a flat state based on the real pattern under consideration of the overlap principle.

We classify 36 ( $6 \times 6$ ) possible combinations into 10 classes with respect to their performances appearing in the folding, as

**Table 4** Ten classes of combinations involved in the second phase of unfolding the no-center-line part.

Case	2	4	6	7	8	9	10
Patterns with MV assignment before and after the operation							

**Table 3** Ten classes of combinations involved in the second phase.

Case	Combinations of patterns ( $S_{i-1}, S_{i+1}$ )
1	$(\{5-1, 5-4\}, \{5-1, 5-4\})$
2	$(5-2, 5-2)$
3	$(5-2, \{5-1, 5-3, 5-4, 5-5, 5-6\}),$ $(\{5-1, 5-3, 5-4, 5-5, 5-6\}, 5-2)$
4	$(5-5, 5-5)$
5	$(5-5, \{5-1, 5-3, 5-4, 5-6\}),$ $(\{5-1, 5-3, 5-4, 5-6\}, 5-5)$
6	$(\{5-1, 5-4\}, 5-6), (5-6, \{5-1, 5-4\})$
7	$(\{5-1, 5-4\}, 5-3), (5-3, \{5-1, 5-4\})$
8	$(5-6, 5-3)$
9	$(5-3, 5-6)$
10	$(5-3, 5-3), (5-6, 5-6)$

listed in **Table 3**. For example, four combinations of (5-1, 5-4), (5-1, 5-1), (5-4, 5-1), and (5-4, 5-4) are categorized into Case 1. The combination of (5-2, 5-2) is categorized into Case 2, and so on. We give the corresponding *MV* assignment for each case on the basis of the overlap principle.

For cases other than Cases 1, 3, and 5, we define their corresponding folding in the following manner. First, revise  $S_i$  to the corresponding assignment, as illustrated in **Table 4**, and then apply the same unfolding operation as in the first phase. Note that the assignments in the upper and lower rows in Table 4 correspond to those before and after the revision, respectively. Furthermore, the illustrated assignments correspond to the case in which every leftmost area faces the front up. The assignments are supposed to be reversed when the leftmost area faces the back up.

The operations for Cases 1, 3, and 5 are as follows.

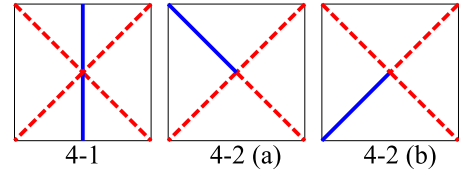
Case 1: As in the first stage, it is sufficient to perform a trivial unfolding operation on  $S_i$ , because neither part intersects the crease near  $v_i$ .

Case 3: Apply the same operation to Case 2. The only difference is that when there is no corresponding crease in another part, the diagonal crease in  $S_i$  will also be deleted.

Case 5: Apply the same operation as Case 4. Similar to case 3, sometimes unnecessary diagonal creases in  $S_i$  are also eliminated.

The proof for the proposition 4.4 is complete for the reason that all the cases are exhausted.  $\square$

Finally, Patterns 4-1 and 4-2 are supposed to be handled by direct folding. **Figure 9** illustrates the corresponding *MV* assignments for handling the case in which the leftmost triangle faces the front up. When the leftmost triangle faces the back up, we can simply interchange all of the *M*s and *V*s to adjust these values so as to maintain the overlap principle. In both cases, there exist two optional choices to fold Pattern 4-2 , with the only



**Fig. 9** *MV* assignments for Patterns 4-1 and 4-2 .

difference on which line directs the separation.

After locally folding a section in Pattern 4-1 or Pattern 4-2 ,  $P$  can be respectively regarded as two independent parts joined by vertical creases or two diagonal creases assigned the same and along the same line.

Now all of the corresponding handlings of the patterns involved in a no-center-line part are described in detail. We present Lemma 4.5 to reemphasize the flat foldability of a no-center-line part, although this is a trivial conclusion of Propositions 4.3 and 4.4.

**Lemma 4.5.** Any no-center-line part in  $P$  can be flat-folded independently.

#### 4.4 Folding a Connection Section

Every connection section has two different kinds of neighbors: one from a center-line part and the other from a no-center-line part. In this section, we first consider the possible combinations of two patterns from a connection section and a no-center-line part, except for Patterns 4-1 and 4-2 . We then consider the possible combinations of two patterns from a connection section and a center-line part. The corresponding combinatorial counts are 96 ( $16 \times 6$ ) and 192 ( $16 \times 12$ ), respectively. Because both numbers are large, eight and 20 combinations are chosen as representatives in **Tables 5** and **6** based on the symmetry and the initial state defined for the no-center-line part in Section 4.3. The details are given along with Propositions 4.6 and 4.7, as follows.

**Proposition 4.6.** Any two adjacent sections  $S_i$  and  $S_{i+1}$  belonging to a connection section and a no-center-line part, respectively, can always be folded.

*Proof.* Only the combinations involving Pattern 5-2 are taken into consideration. The reason is that the flat-folded state of the other five patterns can be obtained from the flat-folded state of Pattern 5-2 while maintaining the overlap principle (Proposition 4.3). The *MV* assignments and the flat-folded states for the possible combinations are given in Table 5, which completes the proof for Proposition 4.6.  $\square$

For another side of the connection section, Proposition 4.7 is given in the same manner.

**Table 5** MV assignments and overlapping orders in every folded state for the connections of a connection section and Pattern 5-2.

Case	1	2	3	4
MV assignment and overlap relation	<p>3-1 → 5-2</p>	<p>3-2 → 5-2</p>	<p>3-3 → 5-2</p>	<p>3-4 → 5-2</p>
	<p>3-2 → 5-2</p>	<p>3-6 → 5-2</p>	<p>3-7 → 5-2</p>	<p>3-8 → 5-2</p>

**Table 6** Twenty representative types of connections of a connection section and a center-line part.



Case	Patterns from the center-line part	Patterns from the connection section	MV assignment and overlap relationship for a typical example	
1			1-1 → 3-1	
			1-3 → 3-1	
			1-2 → 3-1	
			1-4 → 3-1	
			1-5 → 3-1	
			2-1 → 3-1	
			2-2 → 3-1	
			2-3 → 3-1	
			2-4 → 3-1	
			2-6 → 3-1	
	2			1-1 → 3-1
				1-2 → 3-1
			1-3 → 3-1	
			1-4 → 3-1	
			2-1 → 3-1	
			2-2 → 3-1	
			2-3 → 3-1	
			2-4 → 3-1	
			2-6 → 3-1	
3		All possible patterns		3-4 → 3-3
				3-8 → 3-3
				3-16 → 3-3
		It is easy to flat fold all the crease patterns following an assignment of 3-4 in this manner because there is no other crease involved in the left-half section.		


**Proposition 4.7.** Any two adjacent sections  $S_i$  and  $S_{i+1}$  belonging to a center-line part and a connection section, respectively, can always be folded.



*Proof.* Twenty representative combinations for a connection section with its left neighbor from a center-line part are given in Table 6. For completeness of the proof, instructions for the remaining possible combinations are given below. Without loss of generality, assume that  $S_i$  is the connection section to be considered, and the lower-left corner of the  $2 \times 4$  connection faces

the front up. First, for the unlisted patterns from a center-line part, the flat-folded states of the combinations can be obtained by folding these patterns in the manner described in Section 4.1, as illustrated in Table 2. Then, for the unlisted patterns from a connection section, we have the following.

For Case 1 in Table 6, all combinations involving Pattern 3-2 can be obtained by assigning its creases  $h_i, ul_i, u_i, br_i$  as  $h_i, ul_i, ur_i, b_i$  in Pattern 3-1 (change the assignments of  $b_i$  and

$ur_i$  if the lower-left corner faces the back up). For Patterns 3-5  and 3-6 , the assignments are specified as the mirrors of Patterns 3-1 and 3-2 with respect to the centerline.

Similarly, for Case 2, operations for Pattern 3-7  are achieved by mirroring the cases for Pattern 3-3 (change the assignments of  $u_i, ul_i, ur_i$ , and  $br_i$  if the lower-left corner faces the back up) with respect to the centerline.

For Case 3, we assign the given  $MV$  assignment in the table to Pattern 3-4 . Any pattern from a center-line part assigned as in Table 2 can then be easily connected to Pattern 3-4. The reason is that in the left half of Pattern 3-4, there exists no diagonal crease to cause intersection with its left neighbor section. The combinations involving Pattern 3-8  can also be handled by mirroring the cases of Pattern 3-4 with respect to the centerline.

The right half of a connection section is shared by a no-center-line part. In the  $MV$  assignments provided in Table 6, sometimes the assignments of creases shared by a connection section and no-center-line part are opposite to the initial assignments we provided for the no-center-line part in Section 4.3. In such cases, to keep the overlap principle, we apply the folding operation to the no-center-line part as if the back of it faces up. The possible operations are listed in Table 5.  $\square$

The symmetric cases, in which  $S_i$  and  $S_{i+1}$  exchange the parts to which they belong, can be performed with the same operations on their back. A summary of Propositions 4.6 and 4.7 is given in Lemma 4.8.

**Lemma 4.8.** *A connection section can always be flat-folded as the connection of a no-center-line part and a center-line part.*

Since all sections of  $P$  can be flat-folded because every part is folded into a zigzag pattern, no self-intersection would happen. Hence, any  $2 \times n$  map with a box-pleated pattern can be globally flat-folded as long as its local flat foldability is satisfied.

## 5. Linear-time Algorithm for Deciding the Flat Foldability of a Given $2 \times n$ Map

Section 4 presented the folding operations for all parts in  $P$  and proved the equivalence between its global flat foldability and its local flat foldability. Since linear time is needed in order to determine whether a given crease pattern is locally flat-foldable [2], the time complexity of deciding the global flat foldability of  $P$  is also linear.

In the remainder of this section, we prove the existence of a linear-time algorithm for finding a sequence of folding operations by giving Algorithm 1. We then produce the entire process on a specific  $2 \times 15$  map illustrated in Fig. 10 (a).

Algorithm 1 requires a series of sections  $\{S_i\}$  of a given map  $P$  as the input. The last element of  $\{S_i\}$  contains eight parameters, while every other element contains seven. This is because the shared horizontal creases are assigned to the right-side sections. These parameters indicate the existence and assignments of creases in each section. We denote the state of a crease by an integer in  $\{0, 1, 2, 3\}$  to define its non-existence, existence with no assignment, the assignment  $M$ , and the assignment  $V$ , respectively. The output is a sequence of folding operations. Since constant time is required in order to find the corresponding operation for each section during the computation, this algorithm returns a

**Input :** The section sequence  $\{S_i\}$  s.t.,  
 $S_i = (h_i, u_i, b_i, ul_i, bl_i, ur_i, br_i)$  with  $1 \leq i \leq n - 2$ ,  
 $S_{n-1} = (h_{n-1}, u_{n-1}, b_{n-1}, ul_{n-1},$   
 $bl_{n-1}, ur_{n-1}, br_{n-1}, h_n)$   
**Output:** A sequence of operations  $F$  for every section along the horizontal axis

**initialization**

$e \leftarrow (0, 0, 0, 0, 0, 0, 0, 0)$  // the present pattern

$P \leftarrow \emptyset$  // the sequence of parts

$F \leftarrow \emptyset$  // the sequence of operations

$i \leftarrow 1$

$n \leftarrow |\{S_i\}|$

**while**  $i \leq n - 2$  **do**

    update  $e$  as the parameters for  $S_i$

**if**  $e$  represents either Pattern 4-1 or Pattern 4-2 **then**  
         divide the present part into two parts with respect to  $e$ , and recall the operations involved in this while loop for both parts

**else**  
         decide the part to which part  $S_i$  belongs, append this part to  $P$

**end**

$i \leftarrow i + 1$

**end**

append  $S_{n-1}$  to  $P$

**foreach**  $p$  in  $P$  **do**

**if**  $p$  is a center-line part **then**

        append the corresponding direct folding operation to  $F$

**end**

**if**  $p$  is a no-center-line part **then**

        append the corresponding direct folding operations for Patterns 4-1 and 4-2 and the corresponding combination folding operation for other patterns to  $F$

**end**

**if**  $p$  is a connection section **then**

        append the corresponding operation of the connection to  $F$

**end**


**end**




**return**  $F$

**Algorithm 1:** Linear-time algorithm for computing the folding process of a given  $2 \times n$  map.

result in linear time. Thus, Theorem 1.1 is proven.

Next, we exemplify the folding process on the  $2 \times 15$  map illustrated in Fig. 10 (a).

The first traversal finds that  $S_8$  is in Pattern 4-1 . Thus, the map is separated into  $p_1$  and  $p_2$ , as illustrated in Fig. 10 (b).

Next,  $p_1$  is separated into three parts with respect to the centerline, namely, a center-line part, a connection section, and a no-center-line part in left-to-right order, as shown in Fig. 10 (c). The center-line part of  $p_1$  is composed by Patterns 1-5 , 1-4 , and 1-2 . An  $MV$  assignment is given to the section in Pattern 1-5 according to the assignment in the upper row of Table 2. For sections in Patterns 1-4 and 1-2, the  $MV$  assign-



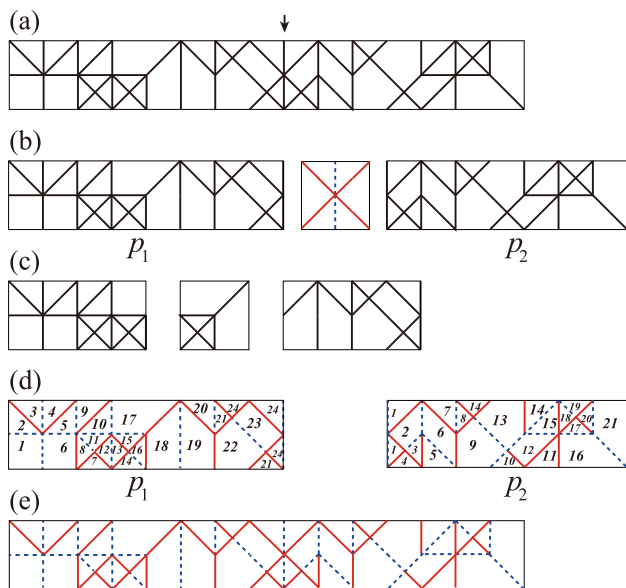





Fig. 10 Example of size  $2 \times 15$ .

ments are given according to the second row because their back are forced to face upward. Next, the connection section matches Case 1 in Table 6 and thus is folded similarly (1-5 () , 3-1 () ) by first changing Pattern 3-1 to Pattern 3-2 with its creases  $h_i, ul_i, u_i, br_i$  assigned in the same manner as  $h_i, ul_i, ur_i, b_i$  in Pattern 3-1, and, second, mirroring the pattern with respect to the centerline.

For the no-center-line part, we first fold the part into the initial state and then revise the creases in the third section, as shown by Pattern 5-3 () in Fig. 8. The final assignment of  $p_1$  is shown on the left-hand side of Fig. 10 (d). The numbers indicate overlapping orders.

For the remaining parts of the map,  $S_8$  is assigned as in Fig. 10 (e) in order to maintain the overlap principle, while the  $MV$  assignment and the overlapping order of  $p_2$  are shown in the right-hand side of Fig. 10 (d).

The entire  $MV$  assignment in Fig. 10 (e) is achieved by joining  $p_1$  and  $p_2$ . Then, this instance is folded into a final flat-folded state without self-intersection.

## 6. Conclusion and Future Work

A variation of map folding, deciding the global flat foldability of a map with a box-pleated crease pattern of size  $2 \times n$  without any  $MV$  assignment, is the focus of the present research. For such a map, we first presented our analysis of different sorts of vertices and then applied a classification of all possible patterns to match these patterns to two distinct operations, along with providing the corresponding  $MV$  assignments. We highlighted our method of categorization for a diversity of patterns. Above all, our most valuable finding is that a locally flat-foldable map of size  $2 \times n$  is also globally flat-foldable. In Section 5, we presented algorithms for both the decision problem and finding a feasible method of folding.

In approaching the topic of desirable future work, we want to mention that the authors of Ref. [12] reported that not every locally flat-foldable  $3 \times n$  map with a box-pleated pattern can be

globally flat-folded, but did not give a reason. We believe that further studies on the characteristics of foldable and unfoldable maps will be interesting for anyone who intends to comprehend maps with diagonal creases. Moreover, extensions of instances of this problem to those with fully defined  $MV$  assignments are expected.

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