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Pantographs and Phase Transitions for the Boundedness of Orbits

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Abstract: We investigate the phase transition of a dynamical system generating a possibly infinite orbit of points. The points of the orbit are generated according to the following basic operation. Given a positive real number a, called the expansion factor, and two points p, q at Euclidean distance $|pq|$ we determine the unique point p' on the straight line passing through p and q which is antipodal to the point p with respect to q and at a Euclidean distance $a|pq|$ from q. The operation on points previously defined is denoted by $p \Rightarrow_{a,q} p'$. Let $\mathbf{a} := a_0, a_1, \ldots, a_{n-1}$ be arbitrary but fixed positive real numbers and $\mathbf{q} := q_0, q_1, \ldots, q_{n-1}$, be n (anchor) points. An orbit consisting of an infinite sequence $p_0, p_1, \ldots, p_m, \ldots$ of points in the plane is generated by using the anchor points as follows. The orbit is initiated with an arbitrary point $p_0 := p$ and for all integers $m \geq 0$, satisfies $p_m \Rightarrow_{a_{m \bmod n}, q_{m \bmod n}} p_{m+1}$ so that $p_{m+1} := (p_m)'$. The resulting sequence of points is called the $(\mathbf{a}, \mathbf{q})$-orbit of p. For any starting point p and any pair $(\mathbf{a}, \mathbf{q})$ we characterize the boundedness of $(\mathbf{a}, \mathbf{q})$-orbits. Namely, we show that there is a phase transition concerning the boundedness of the resulting $(\mathbf{a}, \mathbf{q})$-orbit which depends on whether the product $a_0 a_1 \cdots a_{n-1}$ of the expansion factors is less or larger than one. We also characterize the behaviour of the orbits when $a_0 a_1 \cdots a_{n-1} = 1$. The "boundedness" phase transition phenomenon described above is shown to be valid for any dimension $d = 1, 2, 3$ in Euclidean space. In addition, we propose variants of this approach for generating orbits on convex polygons, and propose several open problems corresponding to phase transition phenomena.

Keywords: Anchor Points, Boundedness, Line, Orbit, Phase Transition, Point

1. Introduction

We study a simple dynamical system defined by a procedure which is generating possibly infinite orbits of points by making use of an abitrary but fixed set of anchor points placed in Euclidean space. We investigate and characterize the boundedness of the orbits thus generated.

1.1 Orbits from Anchor Points

Before describing the problem we begin with some preliminary definitions and explanations of related concepts. Although the points and orbits generated may be in any dimension $d = 1, 2, 3$ in Euclidean space, for the sake of simplicity in the discussion below we assume $d = 2$. We indicate later how to transfer the methodology to all d.

Given a positive real number a, called the *expansion factor*, and two points p, q at Euclidean distance $|pq|$ we determine the unique point p' on the straight line passing through the points p and q (see **Fig. 1**) which is antipodal to the point p with respect to q and at a Euclidean distance $a|pq|$ from q (throughout this paper the notation $|\cdot|$ is used to define Euclidean distance).

Definition 1 The previously defined *antipodal* operation between points p and q with expansion factor $a > 0$ is denoted by $p \Rightarrow_{a,q} p'$.

Fig. 1 The basic antipodal operation $p \Rightarrow_{a,q} p'$ mapping a point p to a unique point p' for generating orbits, where a is the expansion factor at q and $|qp'| = a|pq|$.

Suppose that n points $\mathbf{q} = (q_0, q_1, \ldots, q_{n-1})$, called *anchors* are located in arbitrary but fixed positions in the plane.

The generated orbits consist of points forming trajectories in the plane. Assume that each point q_i is associated with a positive real number a_i called the *expansion factor* of q_i. Let $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ be the sequence of expansion factors. The consecutive points of the orbit are generated according to the basic antipodal operation described above as follows.

Definition 2 Let p be any (starting) point in the plane. The $(\mathbf{a}, \mathbf{q})$-orbit of p with respect to the sequence $\mathbf{q}$ of anchors consists of a possibly infinite sequence $p_0 := p, p_1, \ldots, p_m, \ldots$ of points which is generated by using the antipodal operation on the anchor points $q_0, q_1, \ldots, q_{n-1}$ so that for all $m \geq 0$ the point p_{m+1} is antipodal to the point p_m with respect to the anchor point $q_{m \bmod n}$ whose respective expansion factor is $a_{m \bmod n}$ (see **Fig. 2**).

The points of the orbit are generated by making use of the sequence of anchors $\mathbf{q}$ in the given order and repeats in that order "cyclically" in that after using the anchor q_{n-1} it starts over with

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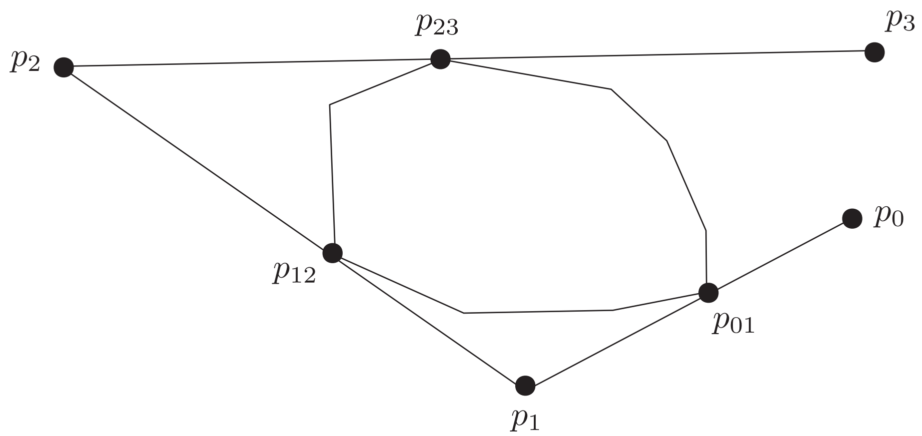

Fig. 3 The first three points p_0, p_1, p_2, p_3 in a a-forward outer billiard (orbit) for a bounded convex polygon with starting point p_0. By reversing direction p_2, p_1, p_0 this can also be considered a a-backward orbit.

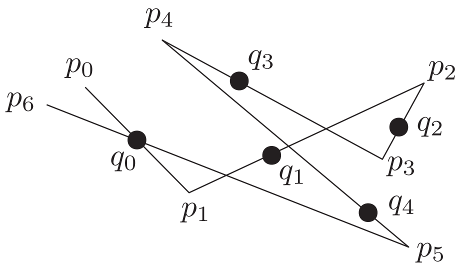

Fig. 2 The first seven points $p_0, p_1, p_2, p_3, p_4, p_5, p_6$ in a $(\mathbf{a}, \mathbf{q})$-orbit for the unit segments with five anchor points q_0, q_1, q_2, q_3, q_4 in the plane and respective expansion factors a_0, a_1, a_2, a_3, a_4. The sequence of points p_i continues indefinitely.

point q_0. The point p_{m+1} is called the antipodal point of p_m so that it satisfies $p_m \Rightarrow_{a_{m \bmod n}, q_{m \bmod n}} p_{m+1}$. We use the simpler notation $p_i \Rightarrow p_{i+1}$, when the implied expansion factor $a_{i \bmod n}$ can be implied easily from the context. Iterating this operation there results an infinite sequence $p_0 \Rightarrow p_1 \Rightarrow p_2 \Rightarrow \cdots$ (which may traverse a given point more than once) called the $(\mathbf{a}, \mathbf{q})$-orbit of p_0 with respect to the pair $(\mathbf{a}, \mathbf{q})$, while p_0 is called the starting point of the resulting orbit (see Fig. 2).

Definition 3 The sequence of points thus generated from the starting point $p := p_0$, are the anchor points $\mathbf{q}$ and the corresponding expansion factors $\mathbf{a}$ is denoted by $O_p(\mathbf{a}, \mathbf{q})$.

Unless it is necessary, mention of $\mathbf{q}$ may be omitted when this is easily implied from the context and when all the a_is are equal we then use the simpler notation $O_p(a)$, where we assume that $a_0 := a = \cdots = a_{n-1}$. Notice that if the anchor points are colinear and the starting point lies on this common line then the resulting orbit will also lie in its entirety on this line but otherwise in general the orbit will lie in the plane.

Definition 4 A sequence of points $\{p_i\}$ is said to be periodic if for some constant k it satisfies $p_i = p_{i+kn}$, for $n = 1, 2, \ldots$.

Definition 5 An orbit $O_p(\mathbf{a}, \mathbf{q})$ is called bounded if it is a subset of some closed disk in the plane and unbounded, otherwise. Further, the $(\mathbf{a}, \mathbf{q})$-orbits are unbounded (resp. bounded) if all non-periodic $O_p(\mathbf{a}, \mathbf{q})$ are unbounded (resp. bounded) for all points p.

Definition 6 An orbit $O_p(\mathbf{a}, \mathbf{q})$ is called closed if $p_0 = p_{m+1}$, for some $m \geq 1$.

Closed orbits are clearlt periodic orbits. A closed orbit is also of course bounded.

Now we can formulate one of the main questions which will be studied in the remainder of this paper. (Note that the definitions and discussion above are valid for any dimension $d = 1, 2, 3$ in Euclidean space.)

Problem 1 (Orbits from Anchor Points) Suppose that n anchor points $\mathbf{q} = (q_0, q_1, \ldots, q_{n-1})$ are located in the plane. Under what conditions on the real numbers $a_0, a_1, \ldots, a_{n-1}$ is the $(\mathbf{a}, \mathbf{q})$-orbit bounded or unbounded?

More specifically, we are interested in whether or not there is a phase transition concerning the boundedness of the orbits which depends on the product $a_0 a_1 \cdots a_{n-1}$.

1.2 Orbits from Vertices of a Convex Polygon

The orbit problem on point sets proposed in this paper is somewhat related to outer billiards (see Ref. [6]), that is a dynamical system defined in the Euclidean plane. This involves a discrete sequence of moves taking place outside a given bounded convex set K. The boundary of K is a closed curve which may be smooth or polygonal. In addition, let a be an arbitrary positive real number.

To form an outer billiard (orbit), we start with an arbitrary point $p := p_0$ which lies outside the convex set K. We draw the straight line tangent to K emanating from p_0 and intersecting K at a vertex of K, for example denoted by p_{01}, so that K is to the left of this line. Let p_1 be the point on this line antipodal to p_0 with respect to p_{01} so that $|p_0 p_{01}| = a|p_{01} p_1|$. Note that the point p_{01} will be a vertex of the convex polygon which depends on the point p_0 and the convex polygon K.

Now iterate the same operation starting with the point p_1 leading to the segment $p_1 p_{12} p_2$, and so on.

Definition 7 The sequence of points resulting when we iterate the previously described operation of drawing tangential line segments with expansion factor a and which always keeps the convex polygon to their left (resp. right) called forward (resp. backward) a-orbit.

Figure 3 depicts an example of a forward outer billiard orbit with respect to a convex polygon starting from a point p_0.

It makes no difference whether one uses forward or backward a-orbits. For example, any finite backward orbit arises by reversing the direction of a finite forward orbit. For this reason, all a-orbits are considered forward a-orbits. Notice that an arbitrary orbit (outer billiard) may not necessarily traverse all the vertices of the convex polygon when rotating around the convex polygon and it may well traverse different vertices in another round [*1]. Orbits from convex polygons will be visited again in Section 3.

The definitions above are inspired from and are a generalization of analogous definitions restricted to $a = 1$ and which can be found in the book [6] by Schwartz.

1.3 Related Work

Neumann [4] was the first to introduce outer billiards in the late 1950s. In the 1970s, Moser [3] popularized outer billiards as a toy model for celestial mechanics. More precisely, Moser [2][p.11] attributes the following question to Ref. [4]: Assume $a = 1$.

> "Is there an outer billiards system with an unbounded orbit?"

Moser [2] considers the above question as an idealized version of the problem of understanding the stability of the solar system.

For a book-length treatment of the topic as well as a chronological list of most known works related to the question of the boundedness of the resulting orbit when $a = 1$ the reader is referred to the book by Schwartz [6]. We note that only outer orbits on Penrose kites are known to be unbounded [5]. To find general related work on billiards the reader is referred to the books [7], [8] by Tabachnikov.

There are certain similarities and differences between the problem of boundedness of orbits from anchor points and the polygonal case of outer billiard problem. There are similarities because they both generate orbits of points in the plane. There are also differences because of the way outer billiard orbits are generated. Given a convex polygon K, its sequence of vertices $\mathbf{q} = (q_0, q_1, \ldots, q_{n-1})$ is also a sequence of anchor points. However an outer billiard $(\mathbf{a}, \mathbf{q})$-orbit on K is a trajectory that traverses these vertices cyclically but might skip some of them when it makes its next move since it requires that the next vertex is tangent to the polygon and this may not necessarily be the next vertex of the convex polygon in the natural cyclical order of the vertices on its perimeter. However, this is not the case in our problem on orbits for anchor points because we cannot skip any points in forming a trajectory.

To the best of our knowledge the phase transition phenomenon for the boundedness of orbits from anchor point sets considered in the present paper is new and has never been studied in the past.

1.4 Outline and Results of the Paper

For any set of anchors $\mathbf{q} = (q_0, q_1, \ldots, q_{n-1})$ in the plane and associated expansion factors (positive real numbers) $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$, we analyze and characterize the boundedness of the $(\mathbf{a}, \mathbf{q})$-orbit. Theorem 1 is the main result in Section 2 and proves the phase transition in d-dimensional Euclidean space, for $d = 1, 2, 3$, of the resulting orbit for any starting point p. Further, in Section 3 we look at similar questions for orbits on convex polygons and propose interesting open problems. Finally, Section 4 presents the conclusion.

2. Orbits from Anchor Point Sets

In this section we consider phase transitions for orbits arising from a set of anchor points. The main theorem to be proved in the sequel is the following.

Theorem 1 (Phase Transition on Boundedness of Orbits) Consider a sequence of n anchor points $q_0, q_1, \ldots, q_{n-1}$ and associated expansion factors $a_0, a_1, \ldots, a_{n-1}$, respectively.

(1) Assume $a_0 a_1 \cdots a_{n-1} \neq 1$. Any $(\mathbf{a}, \mathbf{q})$-orbit visiting the points cyclically in the order $q_0, q_1, \ldots, q_{n-1}$ is either closed or else if the product $a_0 \cdots a_{n-1}$ is greater than 1, it is unbounded, and if the product $a_0 \cdots a_{n-1}$ is less than 1 it is bounded.

(2) Assume $a_0 a_1 \cdots a_{n-1} = 1$.
   (a) If n is odd then any $(\mathbf{a}, \mathbf{q})$-orbit visiting the points cyclically in the order $q_0, q_1, \ldots, q_{n-1}$ is periodic.
   (b) If n is even then any $(\mathbf{a}, \mathbf{q})$-orbit visiting the points cyclically in the order $q_0, q_1, \ldots, q_{n-1}$ is either periodic or unbounded.

The statements above are equally valid for anchor points in 1D, 2D and 3D space.

Details of the proof of Theorem 1 will be deferred until some basic concepts are introduced and necessary lemmas are proven.

The main ideas of the proof are as follows. In Section 2.1 we present the well-known concept of the *pantograph* leading to a simple methodology for deriving the first phase transition result presented in Section 2.2 and which is valid for $a_0 a_1 \cdots a_{n-1} \neq 1$. The phase transition for the case $a_0 a_1 \cdots a_{n-1} = 1$ is analyzed in Section 2.3. We also prove the phase transition result in 1D, 2D and 3D space by applying the idea of projection. Finally, the proof of the main theorem will be completed in Section 2.4 by putting all these ideas together.

2.1 Pantographs

The approach of this section is inspired from the *pantograph*, a mechanical device used for copying and scaling in industrial design work. The pantograph is based on linkages of connected parallelograms so that the movement of one pen, in tracing an image, causes identical (up to scale) movements in another one pen or even more pens (see Ref. [9]). This simple principle of copying "up to scale" is founded on the simple Lemma 1 given below.

Lemma 1 (The Pantograph Lemma) Consider a sequence of n anchor points $q_0, q_1, \ldots, q_{n-1}$ and associated expansion factors $a_0, a_1, \ldots, a_{n-1}$, respectively. Consider two different starting points p_0, p'_0 and the resulting orbits $p_0, p_1, \ldots, p_{m-1}, \ldots$ and $p'_0, p'_1, \ldots, p'_{m-1}, \ldots$ generated by the respective antipodal operations on these anchor points. If the point p_0 moves to the new point p'_0 at distance x from p_0 then the distance of the point p_i from the point p'_i will be equal to $(a_0 \cdots a_{i-1})x$, for $1 \leq i \leq n$, respectively.

Proof. Since both transitions (see **Fig. 4**)

$$p_{j-1} \Rightarrow_{a_{j-1}, q_{j-1}} p_j \text{ and } p'_{j-1} \Rightarrow_{a_{j-1}, q_{j-1}} p'_j$$

hold true, the triangles

*1 In a way, this is the main difference between orbits generated by the convex polygon with vertices $q_0, q_1, \ldots, q_{n-1}$ and the orbits generated from the anchor points $q_0, q_1, \ldots, q_{n-1}$.

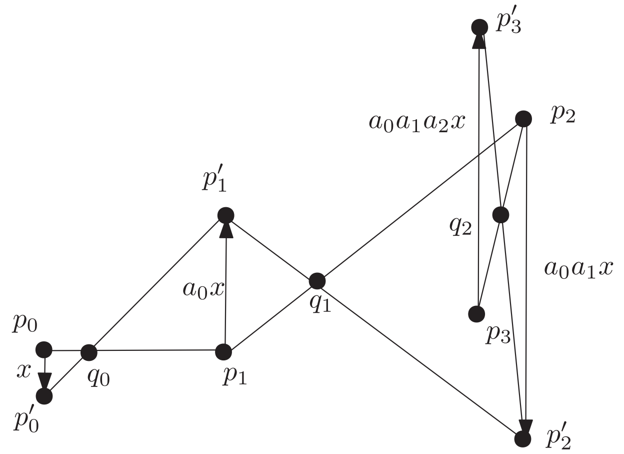

Fig. 4 Using similarity of triangles, it follows that if $|p_0p_1| = x$ then $|p_0'p_1'| = a \cdot x$.

$$\triangle(p_{j-1}p_{j-1}'q_{j-1}) \text{ and } \triangle(p_jp_j'q_{j-1})$$

must be similar, for all $j \geq 1$. Therefore the distance of the points p_i and p_i' are as required, which proves Lemma 1. ■

Notice that the drawing resulting from the original pen is proportional (similar) to the original drawing by a factor arising as a product of the expansion factors of the corresponding anchors.

As another useful corollary of Lemma 1 we note that if the point p_0 is moving towards the point p_0' tracing a smooth rectifiable curve γ_0 then the point p_j will be moving towards the point p_j' tracing a smooth rectifiable curve γ_j which can be obtained from γ_0 by uniformly scaling by a corresponding factor $a_0a_1 \cdots a_j$, possibly with additional translation, rotation and reflection. Therefore, as can be seen easily from Fig. 4, if p_0 is moving towards p_0' along a straight line then also p_i is moving towards p_i' (along a straight line). Moreover, the orientation of the curve is reflected, in the sense that if the expansion factor $\frac{|p_jp_j'|}{p_0p_0'}$ is equal to a, where $a > 0$, and the curve γ_0 represents an arrow $\overrightarrow{p_0p_0'}$ then the arrow $\overrightarrow{p_jp_j'}$ corresponding to the curve γ_j is reflected so as to satisfy

$$\overrightarrow{p_jp_j'} = \begin{cases} a(\overrightarrow{p_0p_0'}), & \text{if } j \text{ is even} \\ -a(\overrightarrow{p_0p_0'}), & \text{if } j \text{ is odd} \end{cases}$$

This property will turn out to be important in the proof of Lemma 3 which is used in the proof of the main Theorem.

2.2 Phase Transition for $a_0a_1 \cdots a_{n-1} \neq 1$

Now we state and prove the existence of a phase transition phenomenon when $a_0 \cdots a_{n-1} \neq 1$.

Lemma 2 (Phase transition when $a_0a_1 \cdots a_{n-1} \neq 1$**)** Consider a sequence of n anchor points $q_0, q_1, \ldots, q_{n-1}$ and associated expansion factors $a_0, a_1, \ldots, a_{n-1}$, respectively. Assume $a_0a_1 \cdots a_{n-1} \neq 1$. Any $(\mathbf{a}, \mathbf{q})$-orbit visiting the points cyclically in the order $q_0, q_1, \ldots, q_{n-1}$ is either closed or else if the product $a_0 \cdots a_{n-1}$ is greater than 1, it is unbounded, and if the product $a_0 \cdots a_{n-1}$ is less than 1 it is bounded.

Proof. Suppose that an orbit is started at a given point p_0, and after visiting the anchor points $q_0, q_1, \ldots, q_{n-1}$ cyclically in this order it ends up at a point p_n using the sequence of consecutive antipodal operations as given below

$$p_0 \Rightarrow_{a_0,q_0} p_1 \Rightarrow_{a_1,q_1} p_2 \cdots p_{n-2} \Rightarrow_{a_{n-2},q_{n-2}} p_{n-1} \Rightarrow_{a_{n-1},q_{n-1}} p_n$$

If $p_n = p_0$ then the orbit is closed (periodic).

Therefore without loss of generality we may suppose that p_0 is different from p_n and let $d_0 := |p_0p_n|$ be the Euclidean distance between p_0 and p_n. Now continue the orbit starting from the point p_n and by traversing the anchor points $q_0, q_1, \ldots, q_{n-1}$ one is ending at the new point p_{2n}. By repeated application of the pantograph Lemma 1 mentioned above, if one traces the movement of the point p_0 to the point p_n, the point p_n will move along the line joining p_0 to p_n by a distance $d_0 \cdot (a_0 \cdots a_{n-1})$. Moreover, the final position of p_n will be p_{2n} (the position at which p_0 will end up after visiting all of the q_i's twice). If one visits all of the q_i's k times, the distance between $p_{(k-1)n}$ and p_{nk} will be $d_0 \cdot (a_0 \cdots a_{n-1})^{k-1}$.

Now we can prove the phase transition claimed in the statement of the lemma. On the one hand, if the product $a_0 \cdots a_{n-1}$ is bigger than one, then $|p_{(k-1)n}p_{kn}| \to \infty$, as $n \to \infty$. On the other hand, if $a_0 \cdots a_{n-1}$ is less than one then using the triangle inequality we see that

$$\begin{aligned} |p_{kn}p_0| &\leq \sum_{i=1}^{k} |p_{(i-1)n}p_{in}| \\ &= d_0 \sum_{i=1}^{k} (a_0 \cdots a_{n-1})^{i-1} \\ &\leq d_0 \frac{1 - (a_0 \cdots a_{n-1})^k}{1 - a_0 \cdots a_{n-1}}, \end{aligned}$$

which is bounded because $a_0 \cdots a_{n-1} < 1$.

To prove that the entire orbit is bounded we must employ a similar argument for the remaining points of the orbit. To this end, it is enough to define $d_i := |p_0p_i|$, for $i < n$ and repeat the previous argument. This completes the proof of Lemma 2. ■

Lemma 2 shows that there is a phase transition for the orbits generated from a point set $q_0, q_1, \ldots, q_{n-1}$ of anchors with respective expansion factors $a_0, a_1, \ldots, a_{n-1}$, provided that $a_0a_1 \cdots a_{n-1} \neq 1$. Next we look at the case $a_0a_1 \cdots a_{n-1} = 1$.

2.3 Phase Transition for $a_0a_1 \cdots a_{n-1} = 1$

Now we state and prove the existence of a phase transition phenomenon when $a_0 \cdots a_{n-1} = 1$.

Lemma 3 (Phase transition when $a_0a_1 \cdots a_{n-1} = 1$**)** Consider a sequence of n anchor points $q_0, q_1, \ldots, q_{n-1}$ and associated expansion factors $a_0, a_1, \ldots, a_{n-1}$, respectively. Assume $a_0a_1 \cdots a_{n-1} = 1$.

(1) If n is odd then any $(\mathbf{a}, \mathbf{q})$-orbit visiting the points cyclically in the order $q_0, q_1, \ldots, q_{n-1}$ is periodic.

(2) If n is even then any $(\mathbf{a}, \mathbf{q})$-orbit visiting the points cyclically in the order $q_0, q_1, \ldots, q_{n-1}$ is either periodic or unbounded.

Proof. Suppose that an orbit is started at a given point p_0, and after visiting the anchor points $q_0, q_1, \ldots, q_{n-1}$ cyclically in this order it ends up at a point p_n using the sequence of consecutive antipodal operations as follows

$$p_0 \Rightarrow_{a_0,q_0} p_1 \Rightarrow_{a_1,q_1} p_2 \cdots p_{n-2} \Rightarrow_{a_{n-2},q_{n-2}} p_{n-1} \Rightarrow_{a_{n-1},q_{n-1}} p_n$$

If $p_n = p_0$ then the resulting orbit is closed (periodic). So without

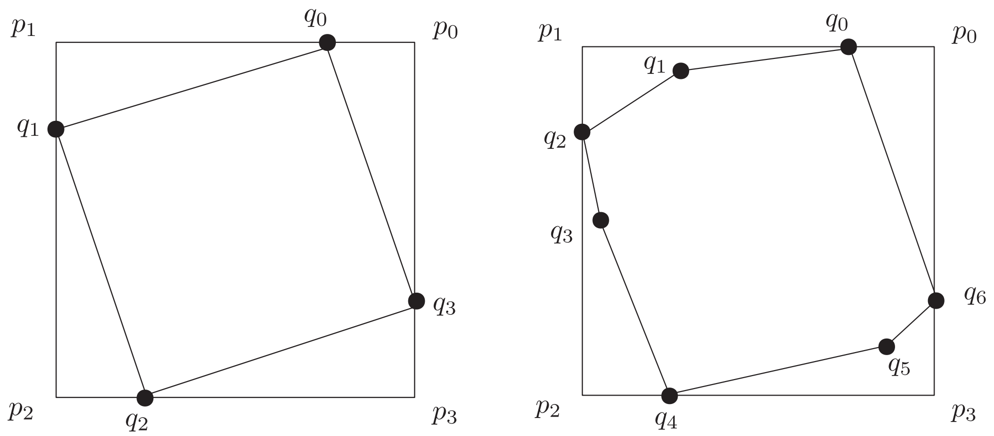

Fig. 6 Left picture: The four q_0, q_1, q_2, q_3 vertices of a square having the same expansion factor and a periodic orbit with four points p_0, p_1, p_2, p_3. Right picture: The seven vertices $q_0, q_1, \ldots, q_6$ of a convex polygon not having the same expansion factors and the resulting periodic orbit consisting of four points p_0, p_1, p_2, p_3.

loss of generality one can assume that $p_n \neq p_0$.

To complete the proof we argue as follows. As noted above, recall from Fig. 4 that every application of an antipodal operation reverses the direction of the arrow indicating the direction of movement from the point p_i to the point p'_i. Therefore the rest of the argument will depend on the parity of the index n.

Part 1. If n is odd. Consider the next n points $p_n, p_{n+1}, \ldots, p_{2n-1}$ in the orbit of p_0 and the line segments $p_0 p_n$ and $p_n p_{2n}$. As a consequence of the pantograph Lemma 1 these segments are parallel and further $|p_0 p_n| = |p_n p_{2n}|$. Moreover, it follows from the observation above that the movement between the points p_0 and p_n has opposite orientation from the movement between the points p_n and p_{2n}. As a result, $p_{2n} = p_0$, which proves that the orbit is periodic.

Part 2. If n is even. Consider the consecutive groups of n points $p_{kn}, p_{kn+1}, \ldots, p_{(k+1)n-1}$ as generated by the antipodal operation which follows the anchor points $q_0, q_1, \ldots, q_{n-1}$ cyclically, for $k = 0, 1, \ldots$. As a consequence of the pantograph Lemma 1, the line segments $p_{kn} p_{(k+1)n}$, for $k - 0, 1, \ldots$ are parallel and of equal length. Since $|p_{kn} p_{(k+1)n}| \neq 0$, as a consequence of the pantograph Lemma 1, these are parallel and further $|p_0 p_n| = |p_n p_{2n}|$. Moreover, it follows from the observation above that the movement between the points p_{kn} and $p_{(k+1)n}$, for all k, has the same orientation. As a consequence, the resulting orbit is unbounded.

This completes the proof of Lemma 3. ■

2.4 Proof of Theorem 1

Proof. Now we are in a position to complete the details of the proof of Theorem 1. The results proven so far are equally valid in 1D with the interpretation that the anchor points lie on a straight line and the starting points of the orbits are placed on this line.

We can extend the result to anchor points in space by projecting the anchor points to points of an infinite plane so that different points in 3D are projected to different points in 2D (see **Fig. 5**). The projection clearly maintains the same expansion factors for the projected points. Additional details required to complete the proof are left up to the reader.

By combining Lemmas 2 and 3 together with the above observation on projecting anchor points the proof of Theorem 1 is

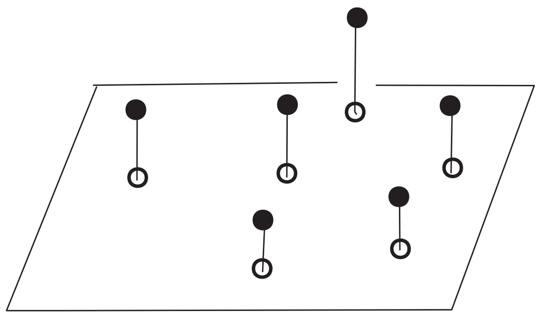

Fig. 5 A set $q_0, q_1, \ldots, q_{n-1}$ of n anchors in 3D space. Each anchor q_i is projected to the infinite plane.

easily completed. ■

3. Orbits from Convex Polygons

Consider a convex polygon with vertices $q_0, q_1, \ldots, q_{n-1}$ in the counterclockwise order (see Fig. 3). Each vertex q_i is associated with a positive real number a_i, called the *expansion factor*. To form an outer billiard orbit, one starts with an arbitrary point p_0 which lies outside the convex polygon K and draws a straight line tangent to K emanating from p_0 and intersecting K at a vertex so that K is to the left of this line. Let p_1 be the point on this line antipodal to p_0 with expansion factor a_0 [*2]. Now iterate the same operation starting with the point p_1 so that p_2 is the point antipodal to p_1 and expansion factor a_1, leading to the segment $p_1 p_2$, and so on. See Fig. 3 which depicts an $(a_0, a_1, \ldots, a_{n-1})$-orbit (outer billiard) generated from a starting point p_0.

Example 1 Consider the example depicted in **Fig. 6** below. The left side of Fig. 6 depicts a periodic orbit with four points p_0, p_1, p_2, p_3 having identical expansion factors arising from a square with four vertices q_0, q_1, q_2, q_3. The right side of Fig. 6 depicts a periodic orbit consisting of four points p_0, p_1, p_2, p_3 for a convex polygon with seven vertices $q_0, q_1, \ldots, q_6$ not having the same expansion factors.

The example just described indicates the main difficulties with characterizing phase transition phenomena for orbits arising from

[*2] If the tangent is parallel to an edge then by convention we select as antipodal point the first vertex of the convex polygon intersected by this line.

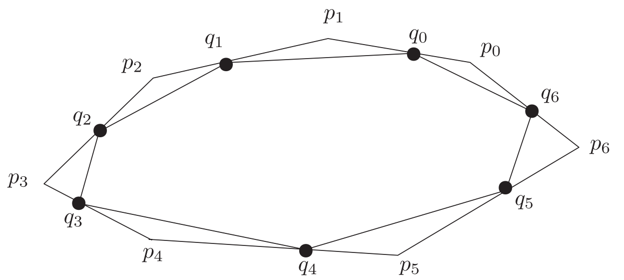

Fig. 7 A convex polygon with 7 vertices $q_0, q_1, \ldots, q_6$ and a periodic outer billiard $p_0, p_1, \ldots, p_6$ in an outer billiard (orbit) for a convex polygon with starting point p_0.

convex polygons. First of all we must restrict our attention to non-periodic orbits. Second, even for non-periodic orbits one must take into account the fact that (non-)periodic orbits may omit vertices of the convex polygon.

Therefore, unlike the case of orbits from anchor point sets already analyzed in Section 2 it is not a priori obvious that a condition on the size of the product $a_0 a_1 \cdots a_{n-1}$ can ensure a phase transition on the boundedness of the resulting orbits. Therefore the following question (which is similar to question posed in Problem 1) is left as an open problem.

Problem 2 (Phase transition on convex polygons) Suppose we are given a convex polygon with vertices $q_0, q_1, \ldots, q_{n-1}$ and associated expansion factors, a_i, for $i = 0, 1, 2, \ldots, n-1$ respectively. Is there a phase transition for the boundedness of non-periodic orbits which depends on the value of the product $a_0 a_1 \cdots a_{n-1}$, in other words, if $a_0 a_1 \cdots a_{n-1} < 1$ then the orbit generated is bounded and if $a_0 a_1 \cdots a_{n-1} \geq 1$ then the orbit generated is unbounded?

This problem subsumes the main question on the boundedness of outer billiards initially posed by Neumann [4] and popularized by Moser [3], already mentioned in the related work in Section 1.3. Therefore it is not expected to be easy.

In general we can prove the following theorem.

Theorem 2 For any sequence of n expansion factors $a_0, a_1, \ldots, a_{n-1}$ there esist a sequence of n points $q_0, q_1, \ldots, q_{n-1}$ forming a convex polygon and a periodic orbit $p_0, p_1, \ldots, p_{n-1}, p_n = p_0$ visiting all the vertices of the convex polygon in their given order such that

$$\frac{|p_i q_i|}{|q_i p_{i+1}|} = a_i, \text{ for all } i = 0, 1, \ldots, n-1.$$

Proof. The proof is depicted in **Fig. 7**. Start with a sequence of points $p_0, p_1, \ldots, p_{n-1}, p_n = p_0$ forming a convex polygon. On each line segment $p_i p_{i+1}$ select a point q_i so that

$$\frac{|p_i q_i|}{|q_i p_{i+1}|} = a_i,$$

for all $i = 0, 1, \ldots, n-1$. Clearly, the points points $q_0, q_1, \ldots, q_{n-1}$ form a convex polygon. Moreover, the orbit associated with the convex polygon $q_0, q_1, \ldots, q_{n-1}$ and emanating from the point p_0 is periodic, by construction. This completes the proof of Theorem 2. ■

4. Conclusion

In this paper we introduce a new dynamical system defined in the Euclidean plane or space and study its phase transition for point sets. For any sequence of n positive real numbers $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$, we analyze and characterize the boundedness of the resulting $(\mathbf{a}, \mathbf{q})$-orbits, where $\mathbf{q} = (q_0, q_1, \ldots, q_{n-1})$ is a corresponding arbitrary sequence of n anchors in the plane. In particular, we prove that there is a phase transition concerning the boundedness of an $(\mathbf{a}, \mathbf{q})$-orbit which depends on whether or not $a_0 a_1 \cdots a_{n-1} < 1$. We also propose several related problems for orbits arising from considering collections of anchor points forming a convex polygon.

References

[1] Kranakis, E.: A Phase Transition Concerning the Boundedness of Orbits on a Point Set (2 Page Abstract), *Japanese Conference on Discrete and Computational Geometry, Graphs and Games* (*JCDCGGG*) (2019).

[2] Moser, J.: *Stable and random motions in dynamical systems: With special emphasis on celestial mechanics* (*AM-77*), Princeton University Press (1973).

[3] Moser, J.: Is the solar system stable?, *The Mathematical Intelligencer*, Vol.1, No.2, pp.65–71 (1978).

[4] Neumann, B.H.: Sharing ham and eggs, *Iota, Manchester University*, pp.14–18 (1959).

[5] Schwartz, R.E.: Unbounded orbits for outer billiards I, *Journal of Modern Dynamics*, Vol.1, No.3, pp.371–424 (2007).

[6] Schwartz, R.E.: *Outer Billiards on Kites* (*AM-171*), Princeton University Press (2009).

[7] Tabachnikov, S.: *Billiards*, Panoramas et Synthèses, Vol.I, Société Mathématique de France, "Panoramas et Syntheses" (1995).

[8] Tabachnikov, S.: *Geometry and Billiards*, Student Mathematical Library, Vol.30, American Mathematical Society (2005).

[9] Wikipedia: Pantograph, available from ⟨https://en.wikipedia.org/wiki/Pantograph⟩ (accessed 2019-08-07).

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