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Sublinear Explicit Incremental Planar Voronoi Diagrams

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Abstract: A data structure is presented that explicitly maintains the graph of a Voronoi diagram of N point sites in the plane or the dual graph of a convex hull of points in three dimensions while allowing insertions of new sites/points. Our structure supports insertions in $\tilde{O}(N^{3/4})$ expected amortized time, where $\tilde{O}$ suppresses polylogarithmic terms. This is the first result to achieve sublinear time insertions; previously it was shown by Allen et al. that $\Theta(\sqrt{N})$ amortized combinatorial changes per insertion could occur in the Voronoi diagram but a sublinear-time algorithm was only presented for the special case of points in convex position.

Keywords: data structures, voronoi diagrams, computational geometry, combinatorial change, big cell

1. Introduction

Voronoi diagrams and convex hulls are two keystone geometric structures of central importance to computational geometry. We focus the description here on planar Voronoi diagrams of points; the results can be extended to the dual graph of 3D-convex hulls: we describe the way to do so later in the paper, see Section 9. Several algorithms, based on various different techniques, have been developed over the years that compute the Voronoi diagram of a set of N points in optimal time $O(N \log N)$ [4]. Surprisingly however, the problem of maintaining dynamically a Voronoi diagram subject to insertions/deletions of points is not well understood.

1.0.1 Incremental Voronoi Diagrams

In this paper we focus on the problem of maintaining the Voronoi diagram under insertion of new sites. In Allen et al. [1] it was observed that while there could be a linear number of changes to the embedded Voronoi diagram with each site insertion, this is not equivalent to the number of combinatorial changes (i.e., edge insertions and deletions) to the graph of the Voronoi diagram. What is more, it was proved that the maximum number of combinatorial changes per site insertion is $\Theta(\sqrt{N})$ amortized. This opened the possibility of maintaining the Voronoi diagram graph under insertions with sublinear update time. Allen et al. [1] achieved this for the restricted case where the sites are in convex position, for which they designed a data structure with $O(\sqrt{N}\log^7 N)$ amortized insertion time. This result relies crucially on the fact that the Voronoi diagram of a set of points in convex position is a tree. Other more restricted special cases have been studied where the number of combinatorial changes is $\Theta(\log N)$ [2], [10].

1.0.2 Our Result

In this work, we provide a data structure that explicitly maintains the graph of a Voronoi diagram of arbitrary point sites in $\mathbb{R}^2$ while allowing insertions of new sites in $\tilde{O}(N^{3/4})$ amortized time, where $\tilde{O}$ suppresses polylogarithmic terms. This is the first data structure supporting insertions in sublinear time for this problem.

We crucially note that we are interested in maintaining explicitly the Voronoi diagram. In particular, we store the diagram as a graph in adjacency list format on which primitive operations, including links and cuts, are performed. This is different than just maintaining a data structure that answers nearest neighbor queries. This case can be solved dynamically in polylogarithmic time by Dynamic Nearest Neighbor (DNN) structures [5], [6], [9]; this relies heavily on the fact that nearest neighbor is a decomposable search problem, whereas maintaining the Voronoi diagram is clearly not. In fact, here we use those DNN structures as subroutines for solving our problem.

We remark that maintaining the Voronoi diagram in sublinear time in the fully dynamic setting (i.e., with both insertions and deletions) is hopeless as the $\Theta(\sqrt{N})$ amortized bound of combinatorial changes for insertions becomes $\Theta(N)$.

1.1 Brief Description of Our Approach

We now give a high-level overview of our approach and the organization of the rest of the paper.

We store the Voronoi diagram as a combinatorial graph, which allows the quick retrieval of any geometric information if needed.

Suppose we wish to insert a new site s_N into the diagram, and let f be the cell of the diagram that contains s_N. This cell can be found in polylogarithmic time using DNN structures. It is known [1] that the affected cells that need to be updated, i.e., that undergo *combinatorial changes*, form a connected region including cell f. To update the diagram, we discover all the affected cells by a variation of the breadth-first search starting from f.

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1.1.1 Small and Big Cells

The main high-level idea is to divide the cells of the Voronoi diagram into *small* and *big*: small are the ones that have at most $N^{1/4}$ vertices and big are the ones that have more. For a Voronoi cell f, by the paws of f we denote the Voronoi vertices connected to the boundary of f by one edge. What we do to process small and big cells is different, but is based on the following fact. Given an affected cell f the neighboring cells of f that change can be identified by finding all the paws of f whose Voronoi circles contain the newly inserted site s_N.

A small cell is small enough to be processed by simply checking all $O(N^{1/4})$ Voronoi circles of its paws in a brute force way. This takes $O(N^{1/4}\,\mathrm{polylog}(N)) = \tilde{O}(N^{1/4})$ time per small cell. Since the amortized number of cells changing is $O(\sqrt{N})$, it takes $\tilde{O}(\sqrt{N} \cdot N^{1/4}) = \tilde{O}(N^{3/4})$ amortized time to perform all updates involving small cells.

For each big cell with b_i neighbors, we store a circular linked list of $\Theta(b_i/N^{1/4})$ data structures, each associated with the consecutive range of $O(N^{1/4})$ of its paws. Each structure stores the Voronoi circles for those paws that are relevant. These Dynamic Circle-Reporting structures (DCRs) are known variants of the DNN structure that support insertion and deletion of circles in polylogarithmic time, and given a query point, report all k circles containing the point in time $\tilde{O}(k)$. Overall, operations involving a big cell require polylogarithmic time in the number of affected neighbors. Since there are at most $O(N^{3/4})$ big cells, the total time to process them is $O(N^{3/4}\,\mathrm{polylog}(N)) = \tilde{O}(N^{3/4})$.

Overall, we need $\tilde{O}(N^{3/4})$ amortized time for updating both small and big cells, thus the main result follows.

1.1.2 Note on Previous Work

In a preliminary version of this paper [3] we presented the same result using a randomized data structure and providing bounds on the expected running time; the randomization came solely from the shallow cuttings used in the DNN structure [5]. Since the initial development of this work, Chan presented DNN structures that use shallow cuttings deterministically [6] (implicit in [7]); using this structure all DNN and DCR structures used in this paper can be implemented deterministically. As a result, our overall structure is deterministic.

1.1.3 Organization

The rest of the paper is organized as follows. In Section 2.1 we characterize affected cells that undergo combinatorial changes. In Section 3 we give a detailed description of our data structure. In Section 4 we present the sequence of actions performed with each insertion of a site. In Section 5 we present the procedure to find all the affected cells, and in Sections 6 and 7 we describe the procedure to implement the combinatorial changes and update the data structure accordingly.

2. Preliminaries and Definitions

We begin with standard definitions related to Voronoi diagrams and their basic properties.

Let $S := \{s_1, s_2, \ldots, s_N\}$ be a set of N distinct points in $\mathbb{R}^2$; these points are called *sites*. Let $\mathrm{dist}(\cdot,\cdot)$ denote the Euclidean distance between two points in $\mathbb{R}^2$. We assume that the sites in S are in *general position*, that is, no four sites lie on a common circle.

Definition 1. The *Voronoi diagram* of S is the subdivision of $\mathbb{R}^2$ into N cells, called *Voronoi cells*, one cell for each site in S, such that a point q lies in the Voronoi cell of a site s_i if and only if

$$\mathrm{dist}(q, s_i) < \mathrm{dist}(q, s_j)$$

for each $s_j \in S$ with $j \neq i$.

Let f_i denote the Voronoi cell of a site s_i. Edges of the Voronoi diagram, called *Voronoi edges,* are portions of bisectors between two sites which are the common boundary of the corresponding Voronoi cells. *Voronoi vertices* are points where at least three Voronoi cells meet. The *Voronoi circle* of a Voronoi vertex v is the circle passing through the sites whose cells are incident to v, see **Fig. 1** (a). Vertex v is the center of its Voronoi circle.

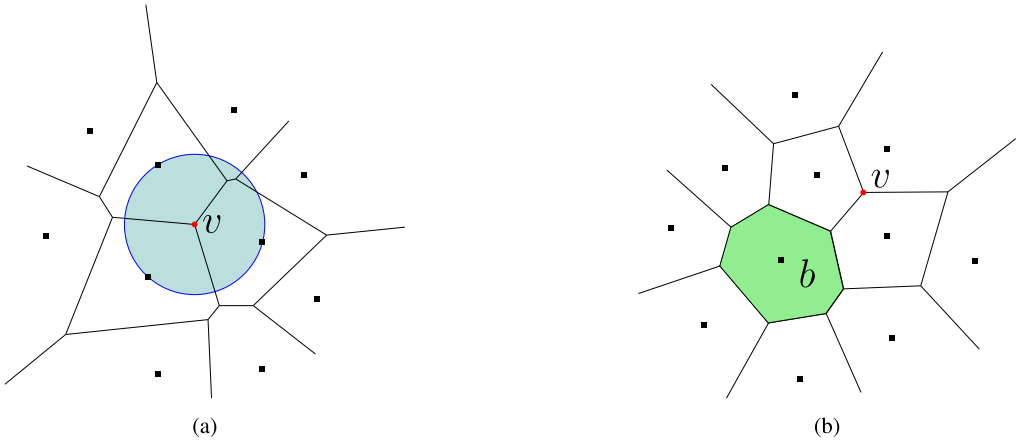
Since the sites are in the general position, each Voronoi vertex has degree three. Each Voronoi edge is either a segment or a ray and the graph of the Voronoi diagram formed by its edges and vertices is planar and connected.

The next three definitions are specific to our data structure.

Definition 2. The *size* of a Voronoi cell is the number of Voronoi edges constituting its boundary. We denote the size of cell f by $|f|$.

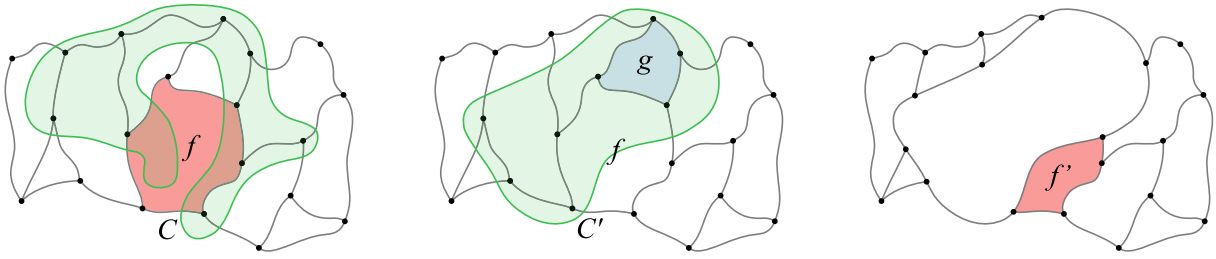
Definition 3. A Voronoi cell of a Voronoi diagram with N sites is called a *big cell* if it has size more than $N^{\frac{1}{4}}$. Otherwise it is called *small*.

Definition 4. The *paws* of Voronoi cell f are the Voronoi vertices



(a) (b)

Fig. 1 A Voronoi diagram and (a) a Voronoi circle of vertex v and (b) paw v of Voronoi cell f.



(a) (b) (c)

Fig. 2 Examples of (a) not flarbable curve (b) flarbable curve; (c) result of applying the flarb operation for curve C'.

that are connected to the boundary of f by an edge and are not themselves on the boundary of f, see Fig. 1 (b). A paw is called *relevant* if it is not incident to a big cell.

2.1 Combinatorial Changes to the Voronoi Diagram and the Flarb Operation

We now overview the definitions and results from Allen et al. [1] that we need to present our approach. In order to prove the $\Theta(N^{\frac{1}{2}})$ bound on the number of combinatorial changes caused by insertion of a site, a graph operation called *flarb* is introduced.

Let G be a planar 3-regular graph embedded in $\mathbb{R}^2$ without edge crossings (edges are not necessarily straight-line). Let C be a simple closed Jordan curve in $\mathbb{R}^2$.

Definition 5. Curve C is called *flarbable* for G if:

- the graph induced by vertices inside the interior of C is connected,
- C intersects each edge of G either at a single point or not at all,
- C passes through no vertex of G, and
- the intersection of C with each face of G is path-connected.

For example, curve C in **Fig. 2** (a) is not flarbable since the intersection between its interior (shaded green) and the highlighted face (red) consists of two disconnected parts. In Fig. 2 (b) curve C' is flarbable.

Given a graph G and a curve C flarbable for G, the *flarb* operation is, informally, removing part of G that is inside C and replacing it with C. Formally, the flarb operation for G and C is defined as follows (see Fig. 2 (b), Fig. 2 (c)):

- For each edge $e_i \in G$ that intersects C let u_i be its vertex lying inside C and v_i its vertex outside C. Create a new vertex $w_i = C \cap e_i$ and connect it to v_i along e_i.
- Connect consecutive vertices w_i along C.
- Delete all the vertices and edges inside C.

Let $\mathcal{G}(G,C)$ denote the graph obtained by applying the flarb operation to graph G and curve C.

Proposition 1. *The following holds for graph $\mathcal{G}(G,C)$: (a) $\mathcal{G}(G,C)$ has at most two more vertices than G does; (b) $\mathcal{G}(G,C)$ is a 3-regular planar graph; (c) $\mathcal{G}(G,C)$ has at most one more face than G does.*

Proof. Items (a) and (b) are proved in Ref. [1], Lemma 2.2. To prove (c) note that there is one new face bounded by the cycle added along C while performing the flarb. All the other faces of G are either deleted, left intact, or cropped by C; these operations obviously do not increase the number of faces. □

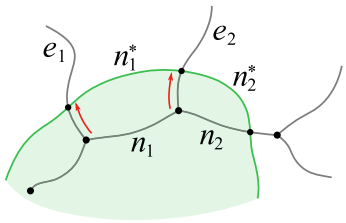


Fig. 3 Edges n_1^* and n_2^* can be obtained without any links or cuts.

Theorem 2 (Ref. [1])**.** *Let G be a graph of the Voronoi diagram of a set of $N-1$ sites $s_1 \dots s_{N-1}$. For any new site s_N there exists a flarbable curve C such that the graph of the Voronoi diagram of sites $s_1 \dots s_N$ is $\mathcal{G}(G,C)$.*

2.1.1 Cost of the Flarb

We want to analyze the number of structural changes that a graph undergoes when we apply the flarb operation to it. There are two basic combinatorial operations on graphs:

- *Link* is the addition of an edge between two non-adjacent vertices.
- *Cut* is the removal of an existing edge.

Other combinatorial operations, for example insertion of vertex of degree 2, are assumed to have no cost.

Definition 6. $\mathrm{cost}(G,C)$ is the minimum number of links and cuts needed to transform G into $\mathcal{G}(G,C)$.

Note that sometimes there are less combinatorial changes needed than the number of edges intersected by C. Consider edges e_1, e_2 of G crossed consecutively by C and edge n adjacent to them that reappears in $\mathcal{G}(G,C)$ as a part n^* of C. Then n^* can be obtained without any links or cuts by lifting n along e_1 and e_2 until it coincides with n^* or (which is the same) shrinking e_1 and e_2 until their endpoints coincide with their intersections with C (see **Fig. 3**). We will call it *preserving operation*.

Theorem 3 (Ref. [1])**.** *For a flarbable curve C, it holds that*

$$\mathrm{cost}(G,C) \le 12|\mathcal{S}(G,C)| + 3|\mathcal{B}(G,C)| + O(1).$$

Where

- $|\mathcal{B}(G,C)|$ *is the number of faces of G wholly contained inside C (g is such a face on* Fig. 2 (b)*).*
- $|\mathcal{S}(G,C)|$ *is the number of* shrinking *faces — i.e., the faces whose number of edges decreases when flarb operation is applied (face f is shrinking on* Fig. 2 (b)–2 (c)*).*

The following upper bound can be used to evaluate the number of combinatorial changes needed to update the graph of a Voronoi diagram when a new site is inserted.

Theorem 4 (Ref. [1]). *Consider one insertion of a new site to a Voronoi diagram V.*

- *The number of cells of Vundergoing combinatorial changes is $O(N^{\frac{1}{2}})$ amortized in a sequence of insertions;*
- *There are a constant number of combinatorial changes per cell;*
- *The cells of Vwith combinatorial changes form a connected region.*

Further in this paper by *a change in cell* we always mean a combinatorial change, that is a *link* or a *cut*.

3. Description of Data Structure

Our data structure consists of the following parts.

- The graph G_N of the Voronoi diagram represented by its adjacency list: for each Voronoi vertex v we store the list of all Voronoi vertices connected to v. Since the sites are in the general position, each list has length 3, therefore we can find and replace its elements in constant time. Thus any link or cut can be performed in constant time as well as insertion or deletion of a vertex of degree 2.
- For each vertex v its *data* D_v is stored. It is a list of the three sites that define the Voronoi circle of v, that is, the sites whose cells are incident to v.
- A dynamic nearest neighbor structure (DNN) [6] for the sites which supports insertion and deletion of sites and nearest neighbor queries in $\tilde{O}(1)$ amortized time.
- The *graph* Γ_N *of big cells* which is simply the dual graph to the subgraph of G_N formed by big cells. Vertices of Γ_N are big cells themselves and edges connect vertices corresponding to pairs of big cells that are adjacent. Graph Γ_N has $O(N^{\frac{3}{4}})$ edges, since it is a planar graph of at most $N^{\frac{3}{4}}$ vertices. For each pair of adjacent big cells b_1, b_2 we also store two Voronoi vertices they share. We store graph Γ_N as an adjacency list, where for each vertex, its edges are stored in a binary search tree ordered counterclockwise around the corresponding big cell. The vertices of Γ_N are stored in a binary search tree. This allows us to access any edge of Γ_N in $\tilde{O}(1)$ time.
- For each big cell b_i store a circular linked list of $\Theta(|b_i| / N^{\frac{1}{4}})$ data structures each associated with a consecutive range of $O(N^{\frac{1}{4}})$ paws of B_i, see **Fig. 4**. Each structure stores the Voronoi circles of the relevant paws of b_i (recall that a paw is relevant if it is not incident to a big cell, see Definition 4). The collections of circles are stored using *dynamic circle-reporting structures (DCRs)* that are variants of the DNN structure constructed in Ref. [1]. DCRs support insertion and deletion of circles in time $\tilde{O}(1)$, and given a query point, report all k circles containing the point in time $\tilde{O}(k)$.
- For each big cell b_i a *yard tree* T_{b_i} supporting the following operations in $\tilde{O}(1)$ time:
  - for a specified continuous range $v_1 \dots v_m$ of vertices of b_i updating $D_{v_1} \dots D_{v_m}$, changing s_i to a given site s_j.
  - removing a continuous range of vertices from b_i and create a new cell with these vertices preserving their order *(split)*,
  - merging the trees that correspond to big cells b_i and b_j (when two cells are merged their common edge is deleted), so that the same operations can apply to the resulting tree.

  One can use link-cut trees [11] or a collection of red-black trees with two-way pointers for this purpose, see Cormen et al. [8] for details.
- For each cell f_i we need to store its size $|f_i|$.

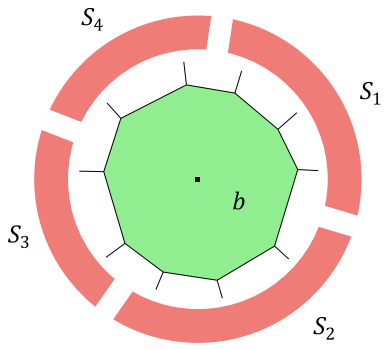


Fig. 4 b is a big cell; each of data structures $S_1 \dots S_4$ is associated with a consecutive range of its paws and stores Voronoi circles of the relevant ones.

4. Insertion of a Site

We aim to implement the update of graph G_{N-1} to become G_N when a *new site* s_N is added to the Voronoi diagram. Our goal is to quickly locate the affected cells that need combinatorial changes, and to avoid processing the other cells. When the cells that need changes are located, we implement these changes using the techniques of Ref. [1].

Let the cell of the new site s_N be called f_N. We denote the boundary of f_N by C_N. According to Theorem 2, what we are about to perform is the *flarb* operation on graph G_{N-1} of the current Voronoi diagram and curve C_N.

We first use the DNN structure to locate one Voronoi cell, call it f_{dnn}, that must change — the one whose site is the closest to newly added s_N. We then add s_N to the DNN. Finally we create the queue with all big cells of G_{N-1} and cell f_{dnn}. This whole procedure takes $\tilde{O}(1)$ time as the list of all big cells is already stored.

We then remove each cell f from the queue, process it, and add into the queue the small cells neighboring f with unprocessed changes. We do not have to add big cells neighboring f as all of them were already in the initial queue and thus will be processed. **Figure 6** shows a pseudocode of this procedure.

5. Recognizing Cells with Changes

Let f be a cell with combinatorial changes. We can identify the neighboring cells of f that change using the following theorem:

Theorem 5 (Ref. [1]). *Let g be a cell adjacent to f. Let v_1, v_2 be the vertices of g that are paws of f. Cell g needs to undergo combinatorial changes if and only if the Voronoi circle of v_1 or v_2 encloses s_N.*

See **Fig. 5** (a) for an example. Cell f is a cell with changes, n_1 and n_2 are its paws. The Voronoi circle of n_1 encloses the new site s_N and the one of n_2 does not. Therefore cells f_1 and f_2 need combinatorial changes as they are incident to vertex n_1, and cell f_3 does not need any changes.

We now consider separately the case when cell f is a big cell

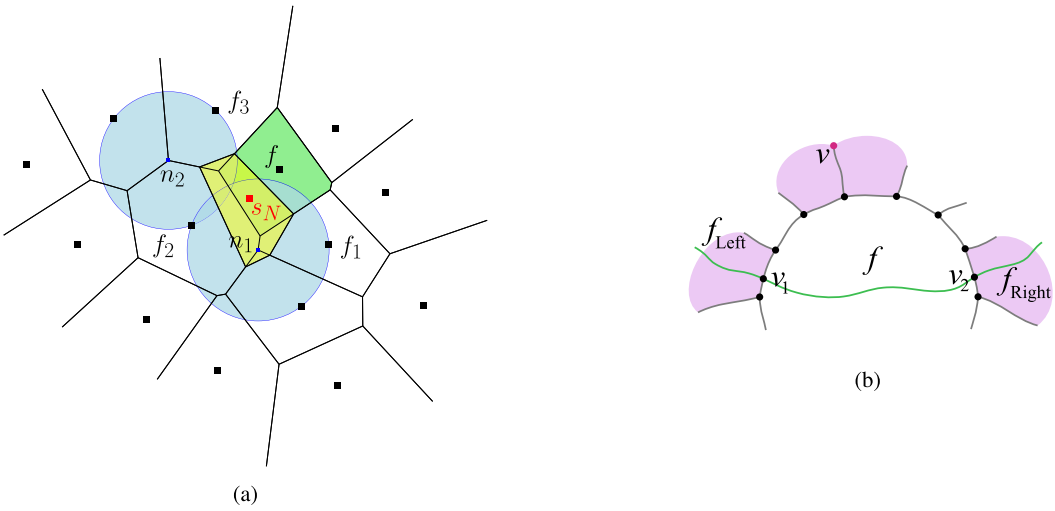


Fig. 5 Identifying Voronoi cells that need changes. (a) Voronoi circle of vertex n_1 encloses s_N, and the circle of n_2 does not. (b) v is a paw of big cell f returned by a DCR. Highlighted are the cells that are to be added to the queue.

```
1: Q := queue of all big cells
2: Changed := empty array of cells
3: Q.append (f_dnn)
4: DNN.insert(s_N)

5: while not Q.empty do
6:     f := Q.dequeue
7:     Changed.append(f)
8:     if f is big then
9:         add f's small neighbors that need changes
               to Q using Section 5.1
10:    else                                   * f is small
11:        add f's neighbors that need changes
               to Q using Section 5.2
12:    end if
13: end while

14: implement changes in cells in Changed
       as described in Section 6
15: Some small cells have become big and some big have become small,
       fix corresponding data structures as described in Section 7
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Fig. 6 Pseudocode describing insertion of new site s_N

(Section 5.1) and the case when it is a small cell (Section 5.2).

5.1 Cell f is Big

We use DCRs of cell f: they return all the relevant paws of f whose Voronoi circles enclose s_N. Small cells that are incident to these paws and are adjacent to f need combinatorial changes and thus have to be added to queue Q.

Two cells are to be considered separately: those that are neighboring f through an edge that is crossed by C_N. Denote them by f_{Left} and f_{Right}, see Fig. 5 (b). If they are small, we check whether the Voronoi circles of at most four paws of f incident to them (call these paws $p_1 \dots p_4$) enclose s_N, and, if yes, add the corresponding cell to the queue. To find Voronoi circles of these paws we get the data $D_{p_1} \dots D_{p_4}$ from the structures T_{b_i} associated with big cells adjacent to f_{Left} and f_{Right}, which requires $\tilde{O}(1)$ time.

5.2 Cell f is Small

We can look at every paw n_i of f and identify those, whose Voronoi circle encloses s_N. This requires $\tilde{O}(N^{\frac{1}{4}})$ time in total. We add to Q small cells adjacent to f that are incident to these paws as they need changes according to Theorem 5.

6. Implementing Combinatorial Changes

In this section we describe how to implement combinatorial changes in a cell f which lies in Changed. We again consider separately the case when f is big (Section 6.1) and the case when f is small (Section 6.2).

6.1 Processing Big Cells

Processing a big cell f consists of the following four steps:

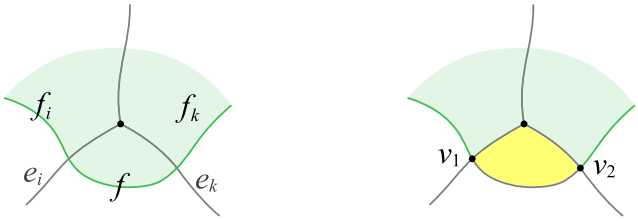
6.1.1 Updating the Vertices

We update a continuous range of f's vertices $v_1 \dots v_k$ — we need to change their data $D_{v_1} \dots D_{v_k}$ to indicate that these vertices are now incident to the cell of s_N and not the cell of s. This can be done in $\tilde{O}(1)$ time using the yard tree T_f.

6.1.2 Updating the Graph of the Voronoi Diagram

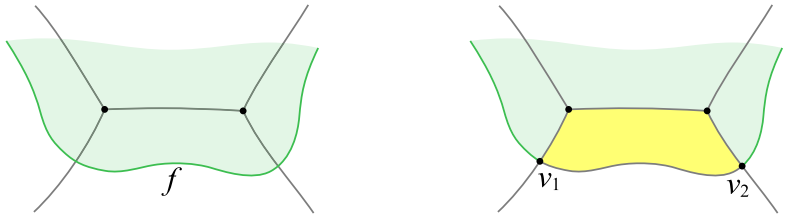
The first thing to do is a *link* along C_N creating two vertices: vertex v_1 incident to f, f_N, f_{Left}, and vertex v_2 incident to f, f_N, f_{Right} (see Fig. 5 (b)). After this *link* the part of f inside C_N becomes a part of the new cell f_N — luckily, all vertices of this part are already updated during the previous step.

There may be some big cells adjacent to f that are already processed, creating other parts of the new cell. We have to join these parts together by cutting edges of f that have portions inside C_N and are incident to already processed big cells. Finding these edges in a straightforward way could be slow as f can have a really large number of edges inside C_N and we do not have enough time to look at each of them individually. Luckily, graph Γ_{N-1} contains the information about edges shared by big cells. Thus we can in $\tilde{O}(1)$ time find and delete edges incident to both f and already processed big cells inside C_N. The edges shared by f and other big cells inside C_N will be deleted when these other big cells will be processed.



(a) (b)

Fig. 7 Processing a cell with one vertex inside the new cell.



(a) (b)

Fig. 8 Processing a cell with two vertices inside the new cell when no surrounding cells are yet processed.

6.1.3 Updating the Graph of Big Cells

The two operations we just carried out — split of f by the new edge v_1v_2 and joining of some cells that are parts of f_N — can change the set of big cells and add or cut some connections between them. However, Γ_{N-1} can be updated accordingly in $\tilde{O}(1)$ time when such an operation is executed. It can be done as follows:

While undergoing a split, vertex f falls apart into two vertices: f' and $f_N^{(f)}$ (the latter represents a part of the new cell). They share newly created edge v_1v_2 of the Voronoi diagram. Note that the cells adjacent to $f_N^{(f)}$ form a continuous range of cells that were adjacent to f.

Thus we need $\tilde{O}(1)$ time to cut a continuous range from the binary search tree of cells adjacent to f, $\tilde{O}(1)$ time to add a new edge between f' and $f_N^{(f)}$ to their binary search trees and $\tilde{O}(1)$ time to re-balance the binary search tree of all big cells.

Joining can be also done in $\tilde{O}(1)$ time. When two cells f_1, f_2 are joined we remove a node corresponding to f_1 from binary search tree of f_2 and vice versa, this takes $\tilde{O}(1)$ time. We then join the trees of f_1 and f_2 also in $\tilde{O}(1)$ time since cells that were adjacent to f_1 form a continuous range of cells adjacent to the new one.

6.1.4 Fixing Data Structures

There are two data structures associated with f that have to be considered:

- T_f can be updated in $\tilde{O}(1)$ time the same way we did with the graph of big cells.
- DCRs of relevant paws: when big cells are joined or split, most of DCR-s stay intact. The only DCRs that need to be rebuilt are those whose range contains the endpoints of the edge that is either cut or added. Rebuilding of a DCR takes $\tilde{O}(N^{\frac{1}{4}})$ time since at most $\emptyset(N^{\frac{1}{4}})$ circles are stored there.

6.2 Processing Small Cells

A small cell is different from a big cell in that we can consider every edge of it, and it will take us $\tilde{O}(N^{\frac{1}{4}})$ time. We will implement the combinatorial changes in f, and after this we update the DCRs of neighboring big cells.

We can in time $\tilde{O}(N^{\frac{1}{4}})$ distinguish whether f has one, two, or more vertices inside the new cell (if they exist). Below we describe these three cases separately.

6.2.1 One Vertex Inside the New Cell

See **Fig. 7** (a). Let f_i, f_k be the cells adjacent to f that share an edge with f inside C_N. Let those edges be called e_i, e_k respectively.

It is certain that neither f_i nor f_k have been processed yet: if f_i is processed then there would be a vertex $v = C_N \cap e_i$. Then we have to create the face in the graph that is separated from f, f_i, f_k, bounded by C_N, and is a part of the new cell f_N.

To do so, we perform a *link* operation inside f along C_N: we create new vertices v_1 on e_i, v_2 on e_k and add an edge v_1v_2 to G_{N-1}, see Fig. 7 (b). v_1 is incident to the cells of sites s, s_i and s_N; v_2 is incident to the cells of s, s_j and s_N.

6.2.2 Two Vertices Inside the New Cell

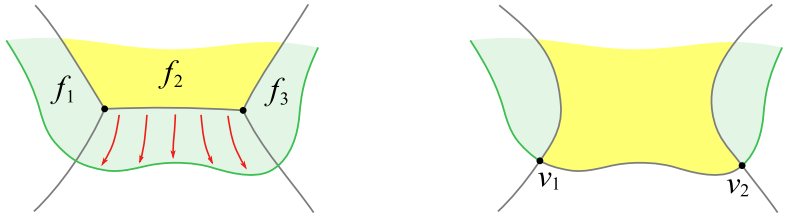
We check whether cells adjacent to f that share an edge with it inside C_N have been already processed. If not (see **Fig. 8** (a)), we perform a *link* operation inside f similarly to the previous paragraph, see Fig. 8 (b).

Otherwise let us denote three faces sharing an edge with cell f inside curve C_N by f_1, f_2, f_3, see **Fig. 9** (a).

Lemma 6. *It is only f_2 that can have been already processed.*

Proof. Suppose f_1 is processed. It must then have an edge along C_N. It implies that there is a vertex where C_N meets the common edge of f_1 and f. This vertex becomes the third vertex of f inside C_N. However, f has only two such vertices, which is a contradiction. □

If f_2 is processed and is part of f_N then the data D_{v_1}, D_{v_2} of its vertices was updated when we were processing it. Therefore f does not need to undergo any combinatorial changes, the common edge of f and f_N can be obtained by preserving operation which was described in Section 2.1, see Fig. 3. Thus no links and cuts are required, see Fig. 9 (b). This completes implementing changes in f.



(a) (b)

Fig. 9 Processing a cell with two vertices inside the new cell when there are processed neighboring cells — no structural changes are needed.

(a) (b)

Fig. 10 Processing a cell with three or more vertices inside the new cell. (a) No adjacent faces have been processed yet, (b) Adjacent face f' has been processed.

6.2.3 More Vertices Inside the New Cell

Again we check whether any of the cells adjacent to f have been processed already. If not, it is enough to perform one *link* creating two new vertices v_1, v_2 and to update the data D_{v_j} of all the vertices of cell f between v_1 and v_2: now they are incident to the cell of s_N, see **Fig. 10** (a).

If some cells sharing an edge with f inside C_N are already processed and represent a part of new cell, then for each processed cell f' adjacent to f we also perform a *cut* removing their common edge e and then remove vertices incident to this edge that now have degree 2, see Fig. 10 (b).

6.2.4 Updating the DCRs of Big Neighbors of f

The last step is that for every vertex v of f whose list of adjacent cells has changed during update of G_{N-1} we find all big cells for which v is a paw (there are at most three such cells, since v has degree 3), recalculate the Voronoi circle of v, and update Voronoi circle of v in DCRs of those cells which takes $\tilde{O}(1)$ time.

7. When Small Cells Become Big

When the size of a cell crosses the threshold of $N^{\frac{1}{4}}$, it can be easily identified since we store all the sizes. If a big cell b is split into a number of cells and one of them is small, or if N becomes greater than $|b|^4$, we delete all the structures associated with it, including DCRs and the structure T_f. We also remove from Γ_{N-1} the vertex corresponding to b.

The other way around, a new big cell can appear in the diagram when:

- the new cell f_N intersects many of the old cells and has more than $N^{\frac{1}{4}}$ vertices, or
- a cell f_k with $N^{\frac{1}{4}} - 1$ vertices has one vertex inside f_N and gets one additional vertex while being processed, see Fig. 7.

New cell f_N inherits the portion of its DCRs from its parts that previously were parts of big cells. The number of circles of paws of previously small cells that need to be added to DCRs can be bounded from above by the size of a small cell times the number of cells that undergo changes — that is,

$$N^{\frac{1}{2}} \cdot N^{\frac{1}{4}} = N^{\frac{3}{4}}.$$

The structure T_{f_N} is inherited in part by f_N from big cells that intersect curve C_N. The number of vertices that have to be added to T_{f_N} after that is bounded from above by the number of combinatorial changes in current insertion.

The cell f_k still has size $|f_k| \leq 2N^{\frac{1}{4}}$, so yard tree T_{f_k} can be built in $\tilde{O}(N^{\frac{1}{4}})$: it only takes time polylogarithmic in the size of the cell to add each vertex.

8. Correctness and Time Complexity

Theorem 7. *Inserting a new site s_N to our data structure and updating it requires $\tilde{O}(N^{\frac{3}{4}})$ amortized time.*

Proof. Let s be the number of small cells that change, and $b_1, b_2, \ldots, b_{|B|}$ be the big cells. Let ℓ_i be the number of circles returned by the DCR structures of b_i.

All the operations on a small cell take $\tilde{O}(N^{\frac{1}{4}})$ time. For all the big cells together the operations on updating the graph structure and the graph of big cells require $\tilde{O}(N^{\frac{3}{4}})$ total time. The number of DCR-s that have to be rebuilt is bounded from above by the number of changes in the graph.

Finally, the amortized time complexity is

$$\tilde{O}\left(sN^{\frac{1}{4}} + \sum_{i=1}^{|B|}\left(\left\lceil \frac{|b_i|}{N^{\frac{1}{4}}} \right\rceil + \ell_i\right) + N^{\frac{3}{4}} + sN^{\frac{1}{4}}\right).$$

Since s is $O(N^{\frac{1}{2}})$ amortized [1], $\sum_{i=1}^{|B|} |b_i| \leq N$, $|B| \leq N^{\frac{3}{4}}$, and $\sum_{i=1}^{|B|} \ell_i \leq sN^{\frac{1}{4}}$, this is simply $\tilde{O}(N^{\frac{3}{4}})$ amortized. □

9. Discussion

The problem of maintaining the convex hull of a set of points in $\mathbb{R}^3$ subject to point insertion can also be solved using our data structure. Namely, consider the dual problem of maintaining the

intersection of a set of halfspaces. The two blocks of our data structure that are specific for Voronoi diagrams, translate in this setting as follows. To find the first face affected by the insertion (or report that it does not exist) we need to find the vertex extreme in the direction normal to the plane being inserted; if it is affected, then all the three incident faces are affected. We check whether a vertex is affected by determinimg the above/below relation between this vertex and the plane baing inserted. Thus Chan's structure [6] is again enough for our needs.

There remains a gap between the $\tilde{O}(N^{3/4})$ expected amortized runtime of our structure and the $\Theta(\sqrt{N})$ amortized number of combinatorial changes to the Voronoi diagram. Also, it would be interesting to get output-sensitive bounds, where the update time depends on the number of combinatorial changes. This was achieved by Allen et al. [1] for points in convex position, where their update time is $O(K \log^7 N)$, where K the number of combinatorial changes. We are unable to show this using our technique, due to the fact that we need to process all $\Theta(N^{3/4})$ big cells, no matter how many of them undergo combinatorial changes.

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