Invited Paper

An Algorithm for Folding a Conway Tile into an Isotetrahedron or a Rectangle Dihedron

Jin Akiyama[1,a)] Kiyoko Matsunaga[1,b)]

Received: January 16, 2020, Accepted: September 28, 2020

Abstract: Every net of an isotetrahedron (I) or a rectangle dihedron (RD) is a Conway tile. Reversely, it is shown by using Alexandrov's theorem that every Conway tile can be folded into either I or RD. However, it was not known how to fold a given Conway tile into I or RD. The purpose of this paper is to give an algorithm for folding a Conway tile into I or RD. Moreover, for a given Conway tile we present a method to identify the exact shape of I or RD into which it can be folded.

Keywords: foldability, Conway tile, isotetrahedron, rectangle dihedron, reversibility

1. Conway Criterion and Conway Tiles

We first state the Conway criterion which is used throughout the paper.

Conway criterion [13]

A given region (figure) can tile the plane using only translations and 180° rotations if its perimeter can be divided into six parts by six consecutive points A, B, C, D, E and F, all located on its perimeter, such that:

(a) The perimeter part AB is congruent to the perimeter part ED by translation τ, in which

$$\tau(A) = E, \tau(B) = D.$$

(b) Each of the perimeter parts BC, CD, EF and FA is centrosymmetric, that is, each part coincides with itself when the region (figure) is rotated by 180° around its midpoint.

(c) Some of the six points may coincide but at least three of them must be distinct.

A region satisfying the Conway criterion is called a **Conway tile**. **A cutting tree**, denoted by CT, of a polyhedron P or dihedron D is a tree drawn on the surface of P or D which spans all vertices of P or D. An **unfolding** (or a **net**) of P or D is a planar region obtained by cutting along all edges of a CT of P or D (**Fig. 1**).

Theorem 1-1 [2], [7]

Every unfolding of an isotetrahedron or a rectangle dihedron is a Conway tile.

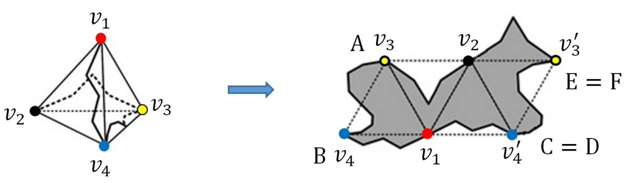


Fig. 1 A cutting tree and a net of P.

For a Conway tile N, a **4-base** of N is defined as a set of four midpoints of centrosymmetric parts of N under the assumption that the midpoint of a centrosymmetric part XY is X (= Y) if X coincides with Y. Thus, there exists a 4-base for any Conway tile N. Notice that a Conway tile may have many different 4-bases (**Fig. 2**).

Theorem 1-2 [9]

Let N be a Conway tile with its 4-base v_1, v_2, v_3 and v_4. Then these four points form a parallelogram.

The four points in the 4-base of a Conway tile N play an important role when N is folded into an isotetrahedron or a rectangle dihedron.

Theorem 1-3

Every Conway tile is foldable into either an isotetrahedron or a rectangle dihedron whose vertices are four points of its 4-base.

Sketch of Proof

Let N be an arbitrary Conway tile. Let A, B, C, D, E and F be the six consecutive points on the perimeter of N, which satisfy the conditions of Conway criterion.

A perimeter of N consists of at most 4 centrosymmetric pairs Bv_1 and Cv_1, Cv_2 and Dv_2, Ev_3 and Fv_3 and Fv_4 and Av_4 of the perimeter parts, having their midpoints v_1, v_2, v_3 *and* v_4 and at most one pair of congruent perimeter parts AB and ED.

Glue centrosymmetric pairs of the perimeter parts together and also glue one pair of congruent perimeter parts. The gluing result is a topological sphere. Besides, the gluing result has just 4 points, where the sum of the angles is 180°, which is proved as follows. Let P be a point on the perimeter of N other than v_1,

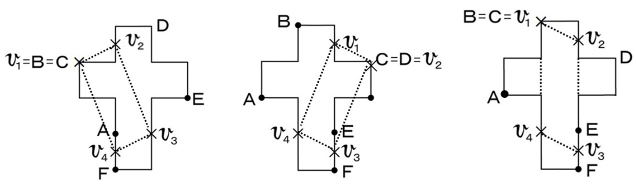


Fig. 2 Three different 4-bases $\{v_1, v_2, v_3, v_4\}$ of a Conway tile.

1 Tokyo University of Science, Shinjuku, Tokyo 162–8601, Japan
a) ja@jin-akiyama.com
b) matsunaga@mathlab-jp.com

v_2, v_3 *and* v_4, and the inner angle of P be α. Since P is on the parallel congruent part or on the centrosymmetric part, P has a unique point P′ with the inner angle α' ($= 360° - \alpha$), where P and P′ are glued. Therefore, the sum of the angles of the other points than v_1, v_2, v_3 and v_4 on the topological sphere is 360°. By Alexandrov's theorem (see details [12]), N is folded into either a polyhedron or a dihedron whose vertices are v_1, v_2, v_3 and v_4. Therefore the resultant polyhedron or dihedron is either an isotetrahedron or a rectangle dihedron whose four vertices are four points of the 4-base of N. ■

Let the gluing of the perimeter of a Conway tile N be called "**Conway gluing**" when centrosymmetric pairs of parts of perimeter of N are glued and a pair of congruent perimeter parts of perimeter of N is glued.

In Refs. [5], [11], [12], all convex polyhedra and dihedra are determined, into which a square is folded. **Figure 3** illustrates four cases of them where a square is folded into an isotetrahedron or a rectangle dihedron. In all those cases, the perimeter of a square is glued by Conway gluing.

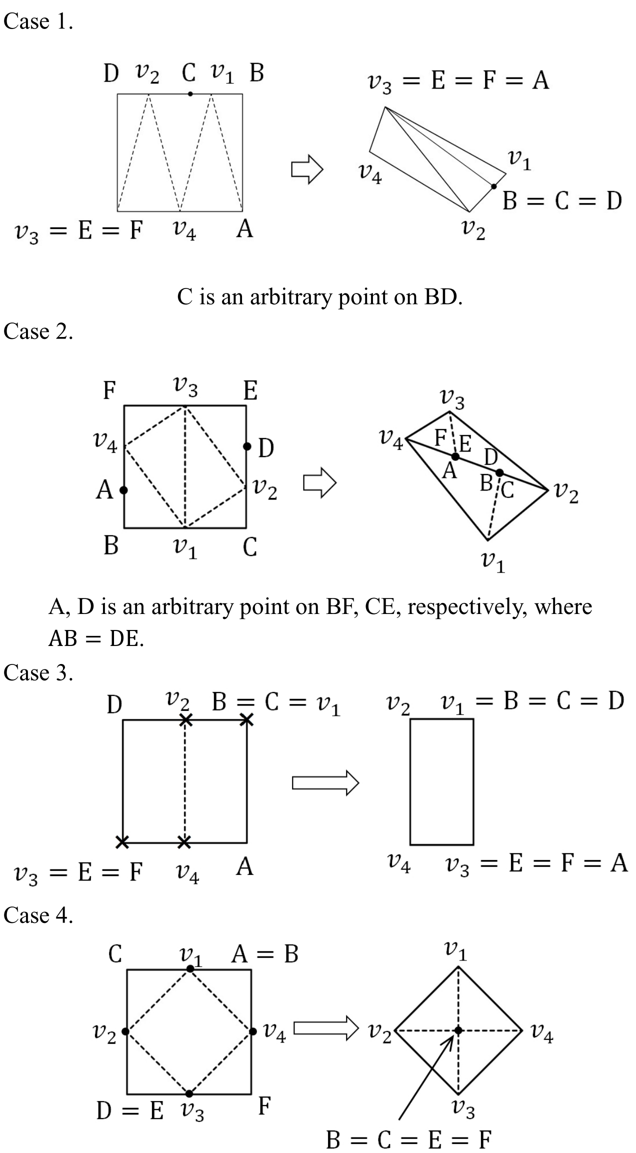


Fig. 3 A, B, C, D, E and F on the perimeter of a square satisfying the Conway criterion.

2. Foldability and Reversible Transformation

Since foldability of figures has a strong connection with their reversibility, we explain the known results on reversible transformation in this section.

Figure 4 illustrates the famous Dudeney's Harberdasher's puzzle. Haberdasher's puzzle asks one to dissect an equilateral triangle (referred to as P) into several pieces and rearrange them to make a square (referred to as Q) after hinging the pieces like a chain (Fig. 4). By scrutinizing the essence of the Haberdasher's puzzle, a reversible transformation between a pair of figures P and Q is defined in Refs. [7], [8] as follows:

A pair of figures P and Q is said to be **reversible** if P and Q satisfy the following conditions:

1. There exists a cutting tree CT_Q along which P is dissected into *n* pieces.
2. Hinge *n* pieces at $n-1$ endvertices of CT_Q to make a chain of *n* pieces.
3. Fix an endpiece of the chain of pieces and rotate the remaining pieces monotonously clockwise, counter-clockwise to obtain P, Q respectively.
4. In this reversible transformation, all dissection lines of P (i.e., edge of CT_Q inside P) are located on the perimeter of Q and all perimeter parts of P are located in the interior of Q (which form CT_P inside Q), and vice-versa (**reversible condition**).

Notice that both an equilateral triangle and a square are Conway tiles. By Conway gluing an equilateral triangle, a square, respectively, an identical isotetrahedron is obtained (**Fig. 5**). In other word, this istotetrahedron generates a reversible pair of Haberdasher's Puzzle.

In general, the following theorem holds for every reversible pair of plane figures.

Theorem 2-1 [4]

A pair of figures P and Q is reversible if and only if there exists a polyhedron or a dihedron W such that the cutting trees CT_P and CT_Q do not intersect on the surface of W.

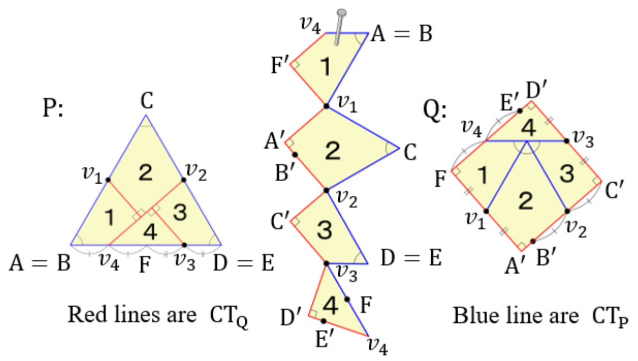


Fig. 4 Haberdasher's Puzzle.

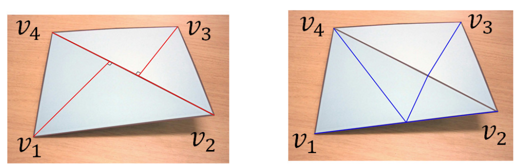


Fig. 5 An identical isotetrahedron generates a reversible pair of Haberdasher's Puzzle.

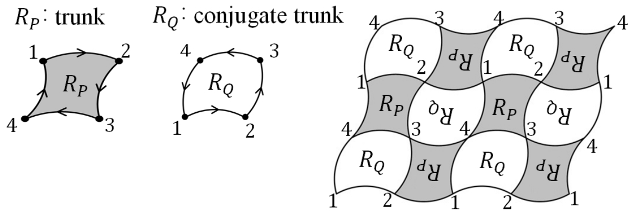
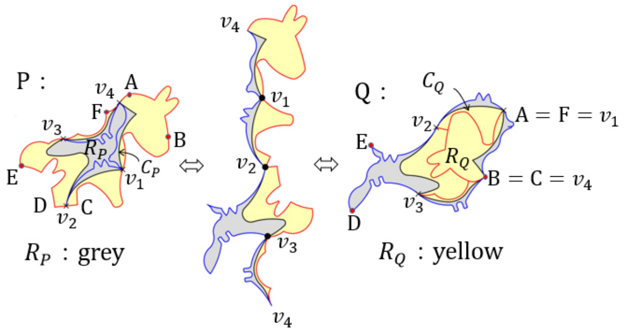
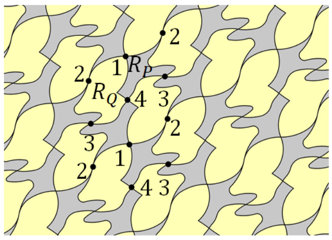


Fig. 6 R_P and R_Q tile the plane like a checkerboard.



(a)



(b) checkerboard tiling by R_P and R_Q

Fig. 7 (a) illustrates a pair of Conway tiles P, Q and their trunk and conjugate trunk R_P, R_Q. (b) shows a checkerboard tiling by R_P and R_Q. @Jin Akiyama

The polyhedron or a dihedron W in Theorem 2-1 is called a **mother** (or **generating**) **polyhedron** for P and Q.

Theorem 2-1 suggests the following fact: *Suppose A and B are a reversible pair and W is their mother polyhedron or dihedron, then try to fold B into W if it is hard to fold A into W.*

A pair of Conway tiles P and Q is called **non-intersecting** if P and Q satisfy the following two conditions (1) and (2):

(1) A 4-base of P is identical to a 4-base of Q.

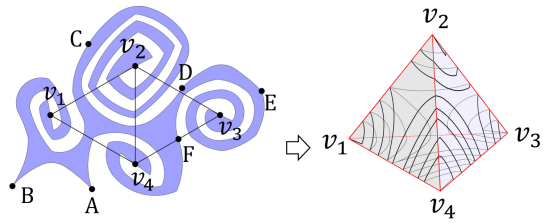
(2) Two regions R_P inside P and R_Q inside Q (they are called **trunk** and **conjugate trunk**, respectively in Ref. [7]) can be drawn such that each perimeter C_P of R_P and C_Q of R_Q passes through all vertices of a 4-base of P, Q, respectively, and R_P and R_Q tile the plane alternately like a checkerboard as shown in **Fig. 6**.

Theorem 2-2 [6], [9]

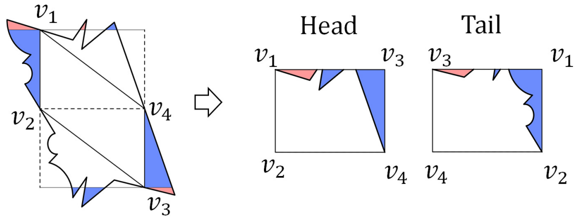
A pair of Conway tiles P and Q is reversible if and only if P and Q are non-intersecting.

Example 2-1: A reversible pair of Conway tiles P and Q (**Fig. 7**).

It is easy to fold a Conway tile into an isotetrahedron I or a rectangle dihedron RD if its 4-base parallelogram can be divided into two identical acute or right triangles by its diagonal as illustrated in **Fig. 8** (a) and (b). However, it becomes difficult to fold a Conway tile P into I or RD whose 4-base parallelogram is thin and long like one in **Fig. 9**. Notice that the 4-base parallelogram of P can not be divided by its diagonal into two identical acute or right triangles. Folding a such Conway tile P comes down to thinking about how to fold a stripe Q, which is reversible to P around the 4-base of P. Therefore, we discuss about folding a stripe into I or RD in the next section.



(a)



(b)

Fig. 8 Easy cases in folding.

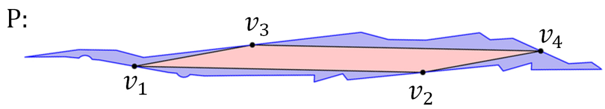


Fig. 9 One example of difficult cases in folding.

3. Fold a Long Stripe into an Isotetrahedron or a Rectangle Dihedron

Two ways of folding a stripe into a rectangle dihedron are known in Ref. [3]. We call these methods of folding, FF1, FF2, respectively. Generalizing these methods, a stripe can be folded into isotetrahedra.

In order to obtain our main results (Section 4), we use the results obtained in Ref. [1] which are generalization of FF1 and FF2. Therefore, we summarize the results briefly below.

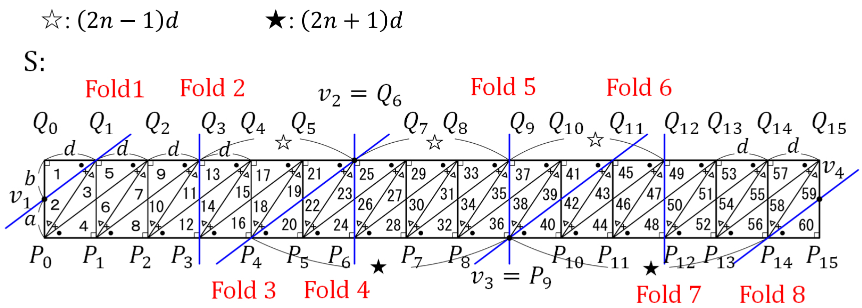
Generalization of FF1 and FF2 to fold stripes into isotetrahedra.

—FF1—

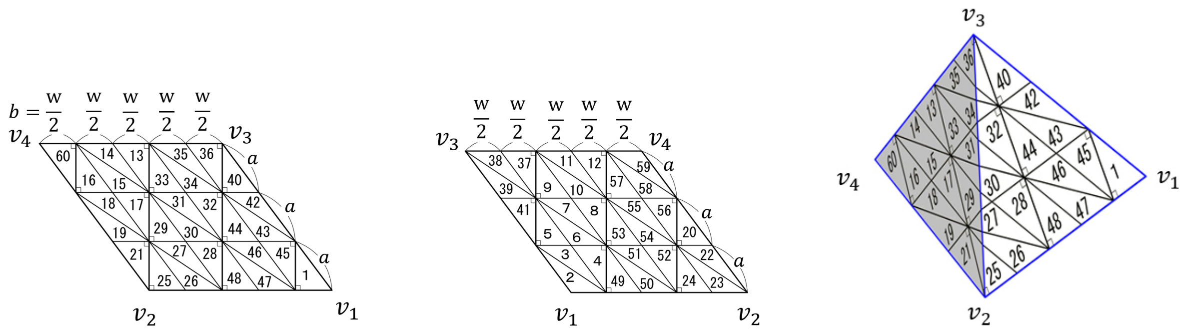
We first divide a parallelogram (or a rectangle) stripe S into $4n(n + 1)$ acute or obtuse triangles as shown in **Fig. 10** (a). This example is the case $n = 2$. By folding S according to FF1 and gluing pairs of adjacent perimeter parts folded by FF1, a topological sphere TS whose head and tail faces are shown in Fig. 10 (b) and (c) is obtained. Creasing each face of TS along the diagonals v_1v_2 and v_3v_4, we obtain an istotetrahedron (Fig. 10 (d)). In general, S is folded into isotetrahedra whose vertices v_1, v_2, v_3 and v_4 are the points of triangles with label 1, $2(n+1)$, $2n(n+1)+1$ and $2(n+1)^2$ of S by FF1. Notice that these four points compose of one of 4-bases of S.

—FF2—

We divide a rectangle stripe S into $4(2n-1)(2n+1)$ triangles as shown in **Fig. 11** (a). This example is the case $n = 2$. The way of dividing is the following: Each of long sides of S is divided into $(2n - 1)(2n + 1)$ segments with the same length d, P_0P_1, P_1P_2,

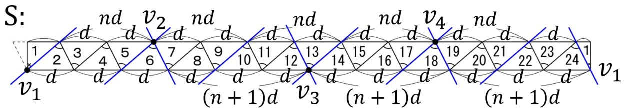


(a) A trianglulated stripe S (Blue lines are fold lines of FF2). Let the line $v_2P_{n(2n-1)}$, $v_2P_{n(2n+1)}$ be ℓ, m, respectively. Each fold line is parallel to either ℓ or m.

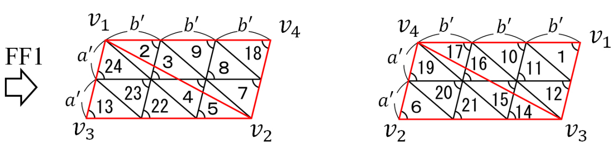


(b) Head face of a TS (c) Tail face of a TS (d) An isotetrahedron is folded

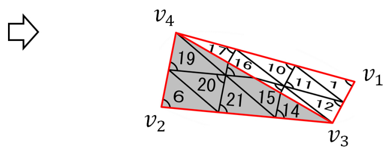
Fig. 11 Generalization of FF2. (b), (c) A topological sphere composed of two parallelograms. In this figure, $n = 2$, v_1 = the midpoint of P_0Q_0, $v_2 = Q_6$, $v_3 = P_9$, v_4 = the midpoint of $P_{15}Q_{15}$.



(a) A triangulated stripe S (Blue lines are fold lines of FF1. Two fold lines ℓ and m pass through v_2. Each fold line is parallel to either ℓ or m).



(b) Head face of TS (topological sphere) (c) Tail face of TS (topological sphere)



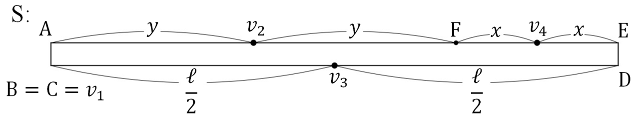
(d) An istotetrahedron is folded.

Fig. 10 Generalization of FF1.

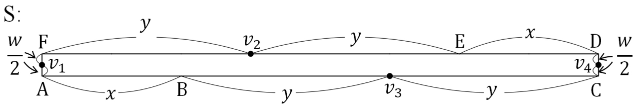
$\cdots$, Q_0Q_1, Q_1Q_2, $\cdots$.

Let the midpoint of P_0Q_0, $Q_{n(2n-1)}$, $P_{n(2n+1)-1}$, the midpoint of $P_{(2n-1)(2n+1)}Q_{(2n-1)(2n+1)}$ be v_1, v_2, v_3, v_4, respectively. Notice that these four points compose of one of the 4-bases of S. Last, draw v_1Q_1, $P_{(2n-1)(2n+1)-1}v_4$, P_0Q_1, P_1Q_1, $P_{i-1}Q_i$, P_iQ_i, and $P_{i-2}Q_i$, $(i = 2, 3, 4, \cdots, (2n-1)(2n+1))$. By folding S accoding to FF2 and gluing pairs of adjacent perimeter parts folded by FF2, a topological sphere TS whose head and tail faces are shown in (Fig. 11 (b), (c)) is obtained. Creasing each face of TS along v_2v_3, v_4v_1, we have an isotetrahedron in FFig. 11 (d).

Let a pair of head and tail faces of a topological sphere TS from a stripe by FF1 be called a **FF1-arrangement**. And let a unit triangle T into which a rectangle stripe is triangulated be called a **cell of FF1-arrangement**.



(a)



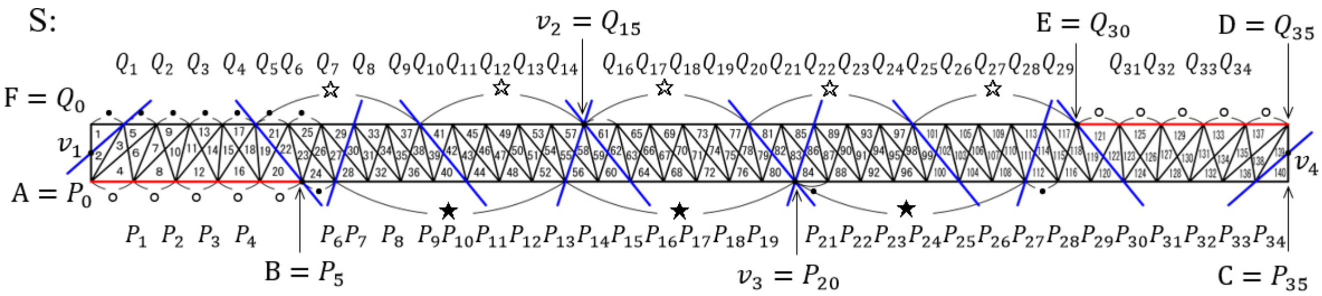
(b)

Fig. 12 (a) AB and ED locate on the vertical sides of S. (b) AB and ED is located on the holizontal sides of S.

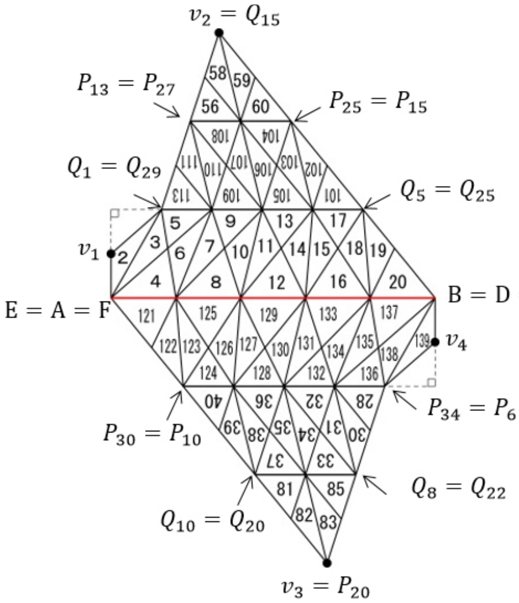
Similarly, a pair of head and tail faces of topological sphere TS from a stripe by FF2 is called a **FF2-arrangement**. For a given stripe S, there are two important Conway-gluings. We call them **Conway-gluing Type 1** (simply **CGT1**) and **Conway-gluing Type 2** (simply **CGT2**). CGT1, CGT2 is a Conway gluing of S when the perimeter of S is divided into six parts as illustrated in **Fig. 12** (a), (b), respectively. In Fig. 12 (a) and (b), the six points A, B, C, D, E and F follow the Conway criterion. The difference between CGT1 and CGT2 results from the locations of their parallel-congruent parts (i.e., AB and ED) of S. Notice that a pair of parallel-congruent parts AB and ED in Fig. 12 (a) is located on the vertical sides of S.

On the other hand, a pair of parallel-congruent parts AB and ED in Fig. 12 (b) is located on the holizontal sides of S. In both Fig. 12 (a) and (b), the four points v_1, v_2, v_3 and v_4 compose of a 4-base of S.

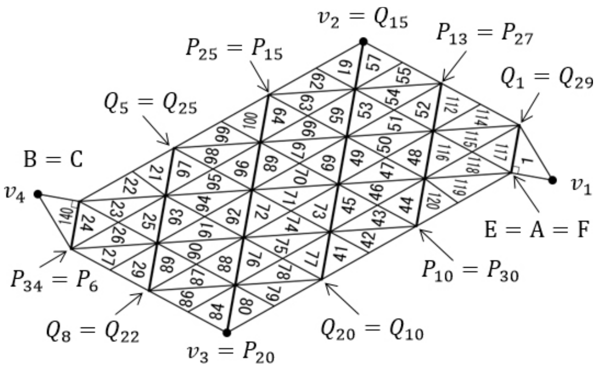
For S in Fig. 12 (a), the FF1-arrangement of S is called a **proper FF1-arrangement for *n*** of S when S is triangulated into $4n(n+1)$ triangles as shown in **Fig. 13** (a) (i.e., divide BD into $2n(n+1)$ segments P_iP_{i+1}. Divide $v_2\mathrm{E} \cup \mathrm{A}v_2$ into $2n(n+1)$



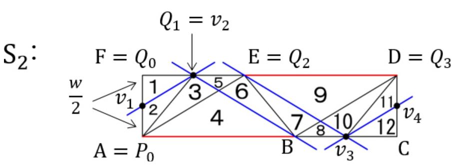
(a) Blue lines are fold lines. ☆: $(2n-1)d$, ★: $(2n+1)d$, • = d_2, ∘ = d_1. Triangulated stripe S which induces proper FF2-arrangement for n. In this figure, $n = 3$.



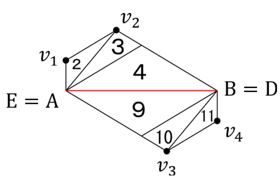
(b) Head face of TS



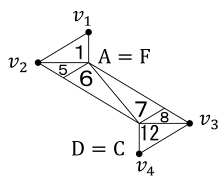
(c) Tail face of TS



(d)

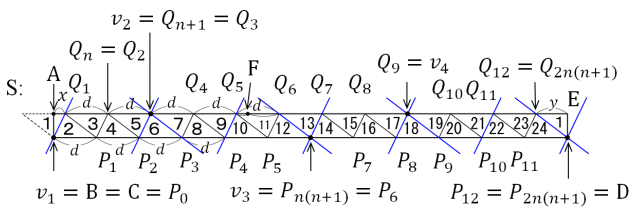


(e) Head face of TS

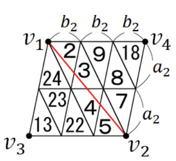


(f) Tail face of TS

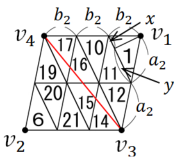
Fig. 14 (b), (c) a proper FF2-arrangement for 3. (d) Triangulated stripe which induces FF2-arrangement for $n = 1$. (e), (f) proper FF2-arrangement for 1.



(a) Blue lines are fold lines.



(b) Head face of TS



(c) Tail of TS

Fig. 13 (a) Triangulated stripe S which induces proper FF1-arrangement for n of S. In this figure $n = 2$. (b), (c) a proper FF1-arrangement for 2.

segments $Q_{n+1}Q_{n+2}$, $Q_{n+2}Q_{n+3}$, $\cdots$, $Q_{2n(n+1)}Q_1$, Q_1Q_2, $\cdots$, Q_nQ_{n+1}), and is folded and glued according to FF1 in Fig. 13 (b) and (c).

For S in Fig. 12 (b), the FF2-arrangement of S is called a **proper FF2-arrangement for n** of S when S is triangulated into $4(2n-1)(2n+1)$ triangles as shown in **Fig. 14** (a) $n = 3$, (d) $n = 1$ and is folded and glued according to FF2 in Fig. 14 (b), (c) and (e), (f). The way of dividing S in Fig. 14 (a), (d) is the following:

First, AB, ED is divided into $2n-1$ segments with the same length d_1, respectively.

Second, BC, EF is divided into $2n(2n-1)$ segments with the same length d_2, respectively. Third, draw segments v_1Q_1, $P_{i-1}Q_i$, P_iQ_i, $P_{i-1}Q_{i+1}$, $P_{(2n-1)(2n+1)-1}Q_{(2n-1)(2n+1)}$ and $P_{(2n-1)(2n+1)-1}v_4$ ($i = 1, 2, \cdots, (2n-1)(2n+1)-1$).

Theorem 3-1 [1]

*S (Fig. 13 (a)) is folded into an isotetrahedron or a rectangle dihedron which is made by creasing proper FF1-arrangement for n of S (**Fig. 15**), when S is glued by CGT1 and its 4-base parallelogram of S can not be divided into two acute or right triangles by its diagonal.*

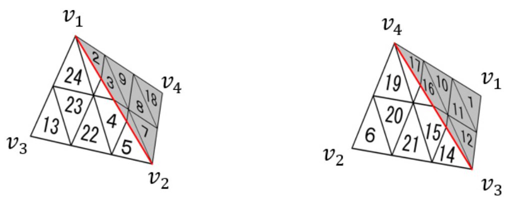
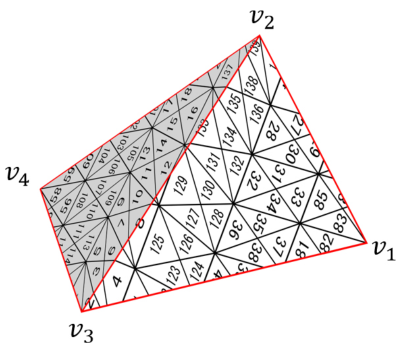


Fig. 15 An isotetrahedron obtained by CGT1 for S in Fig. 13 (seen from the two different views).



(a)

(b)

Fig. 16 (a) An isotetrahedron obtained by CGT2 for S in Fig. 14 (a). (b) An isotetrahedron obtained by CGT2 for S in Fig. 14 (d).

Theorem 3-2 [1]

*S (Fig. 14 (d)) is folded into an isotetrahedron which is made by creasing proper FF2-arrangement for n of S (**Fig. 16** (b)), when S is glued by CGT2 and its 4-base parallelogram of S can not be divided into two acute or right triangles by its diagonal. In particular, an isotetrahedron is made by creasing proper FF2-arrrangement for 1 of S when the length of parallel-congruent part (AB or ED) of S is longer than $\ell/3$, where ℓ is the length of S.*

4. Fold a Long Conway Tile into an Istotetrahedron or a Rectangle Dihedron

We have already mentioned that it is easy to fold a Conway tile N into an isotetrahedron or a rectangle dihedron if a 4-base parallelogram of N is divided into two identical acute or right triangles in Section 2. We now consider how to fold a long and thin Conway tile whose 4-base parallelogram is not divided into two acute or right triangles. That is, we consider the Conway tile N like in **Fig. 17** (Type 1) and **Fig. 18** (Type 2).

The difference between Type 1 and Type 2 is exposed when their parallelo-congruent parts AB and ED are glued. A 4-base parallelogram of Type 1 is located inside a ring when AB and ED of N is glued (**Fig. 19** (a)). On the other hand, a 4-base parallelogram of Type 2 is located inside a one-round spiral when AB and ED of N is glued (Fig. 19 (b)).

If the 4-base parallelogram of N is located inside a ring after gluing AB and ED of N, we call N **ring type**. If the 4-base parallelogram of N is located inside a one-round spiral after gluing AB and ED of N, we call N **spiral type**.

Each of Conway tiles N of ring type and spiral type is reversible to the parallelogram M_1, M_2, respectively as to its 4-base (or sometimes M_1, M_2 might be a deformed parallelogram, but if so, the following argument is valid) which inscribes its 4-base parallelogram (blue parallelogram in **Fig. 20** (a), (b)).

From Theorem 2-1, an isotetrahedron I or a rectangle dihedron RD obtained from a Conway tile of ring type is nothing but I or RD obtained by folding M_1 such that $v_1'v_2'$ & v_1v_2, v_4v_2 & v_4v_2', and v_3v_1 & v_3v_1' of M_1 are adjacent, respectively.

Similarly, I or RD obtgained from a Conway tile of spiral type is nothing but I or RD obtained by folding M_2 such that G′v_4 & Gv_4, Hv_2 & H′v_2, G′H″ & G″H′, Gv_1 & G″v_1, and H″v_3 & Hv_3 of M_2 are adjacent, respectively. Therefore, folding a Conway tile of ring type, spiral type into an isotetrahedron I or a rectangle di-

Type 1:

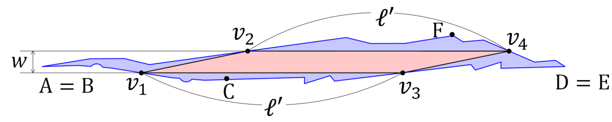


Fig. 17 A long Conway tile N of Type 1; ring type.

Type 2:

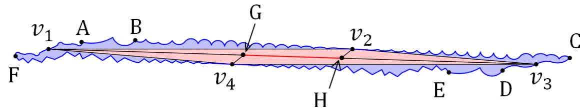
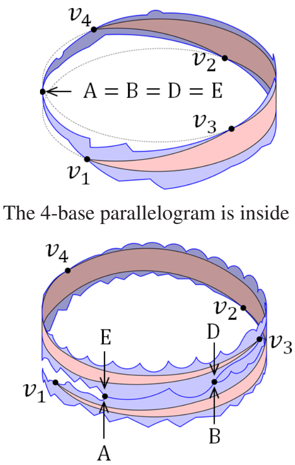


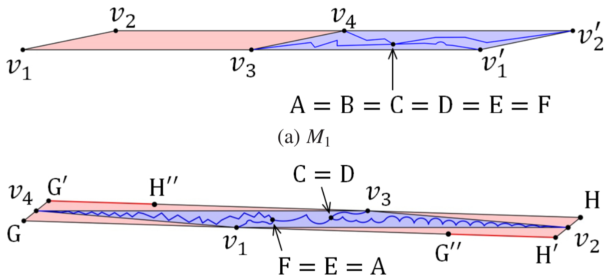
Fig. 18 A long Conway tile N of Type 2; Spiral type.



(a) The 4-base parallelogram is inside a ring

(b) The 4-base parallelogram is inside a one-round spiral

Fig. 19 The difference between two types of long and thin Conway tiles.



(a) M_1

(b) M_2

Fig. 20 N is reversible to the parallelogram. Ring type, spiral type is reversible to M_1 type, M_2 type, respectively.

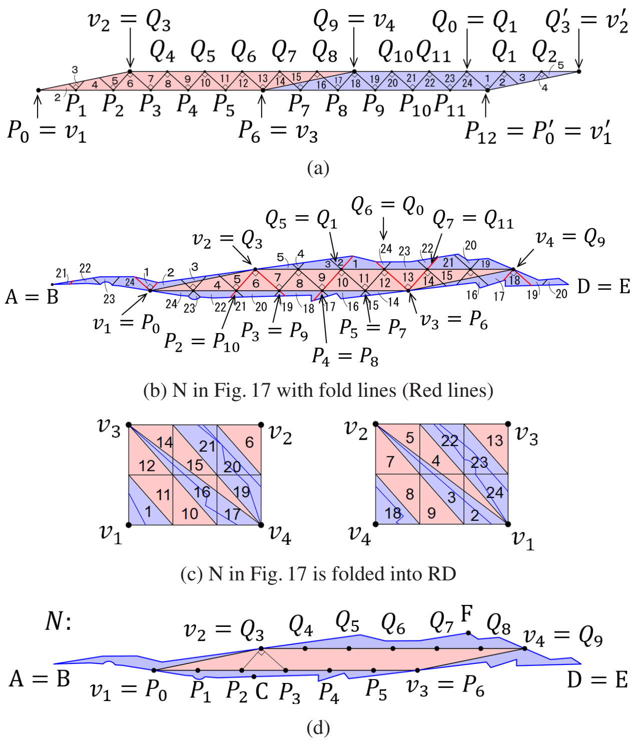


Fig. 21 How to fold N of ring type.

hedron RD comes down to folding a parallelogram stripe M_1, M_2 into I or RD according to FF1, FF2, from Theorem 3-1, Theorem 3-2, respectively.

Example 4-1 (Fold a Conway tile N of ring type in Fig. 17)

(1) The perimeter of N is divided into four parts BC, CD, EF and FA (B = A, E = D) such that BC, CD, EF, FA is centrosymmetric around its midpoint v_1, v_3, v_4 v_2, respectively, as shown in Fig. 17. The 4-base parallelogram $v_1v_3v_4v_2$ can not be divided into two acute or right triangles by its diagonal.

(2) Determine the suitable value of n. Reference [1] suggests that the suitable value of n ($n \geq 2$) is roughly determined such that $\sqrt{2}n(n+1)w/\sqrt{3n+1} \leq \ell' \leq 2n(n+1)w$ for the length ℓ' and the width w of the 4-base parallelogram $v_1v_3v_4v_2$. In this case, $n = 2$.

(3) Dissect the Conway tile N along v_1v_2, v_2v_4 and v_3v_1 and reverse N into a parallelogram $v_2v_1v_1'v_2'$ as shown in **Fig. 21** (a). Divide each of the segments v_1v_1', v_2v_2' into $2n(n+1)$ (= 12 in this case) segments P_0P_1, P_1P_2, $\cdots$, $P_{n(n+1)}P_{n(n+1)+1}$, $P_{n(n+1)+1}P_{n(n+1)+2}$, $\cdots$, $P_{2n(n+1)-1}P_0$; $Q_{n+1}Q_{n+2}$, $Q_{n+2}Q_{n+3}$, $\cdots$, $Q_{2n(n+1)-1}Q_0$, Q_0Q_1, $\cdots$, Q_nQ_{n+1}, respectively, as shown in Fig. 21 (a).

(4) Make a ring from the parallelogram $v_2v_1v_1'v_2'$ in Fig. 21 (a) by gluing v_2v_1 and $v_2'v_1'$. Then decompose the ring into $4n(n+1)$ cells. In this case, the cell is a right triangle. And fold the ring into a rectangle dihedron along the method of FF1 (i.e., this rectangle dihedron RD is the proper FF1-arrangement for 2 in this case). Reverse the parallelogram $v_2v_1v_1'v_2'$ with fold-lines to a Conway tile N again as shown in Fig. 21 (b). Fold N along fold-lines (Red lines in Fig. 21 (b)), and then N is folded into a rectangle dihedron RD as shown in Fig. 21 (c).

(5) In fact, for a Conway tile N in Fig. 17 and its 4-base parallelogram $v_2v_1v_3v_4$, the triangle $P_nv_2P_{n+1}$ (i.e., $P_2Q_3P_3$ in this case) symbolizes the cell of M_1 which plays an important role in folding M_1 into I or RD by FF1, after dividing v_1v_3 into $n(n+1)$ segments with the same length. Since a triangle $P_2Q_3P_3$ in this case is a right triangle as shown in Fig. 21 (d), it suggests that this Conway tile N is folded into a rectangle dihedron with size $2 \cdot P_3Q_3 \times 3 \cdot P_2Q_3$ by FF1.

Example 4-2 (Fold a Conway tile N of spiral type in Fig. 18)

(1) The perimeter of N is divided into six parts AB, BC, CD, DE and FA such that AB is parallel-congruent to ED and BC, CD, EF, FA is centrosymmetric around its midpoint v_2, v_3, v_4, v_1, respectively, as shown in Fig. 18. A 4-base parallelogram $v_1v_2v_3v_4$ can not be divided into two acute or right triangles by its diagonal.

(2) Draw the line v_1v_3, ℓ_1, ℓ_2 such that each ℓ_1, ℓ_2 passes through v_4, v_2, respectively, where ℓ_1 is parallel to ℓ_2 (**Fig. 22** (a)). Denote by G, H, the intersection of ℓ_1 and v_1v_3, ℓ_2 and v_1v_3, respectively. Dissect N along the segments v_1v_3, v_4G and v_2H.

(3) Reverse N to a parallelogram GG′HH′ as to its 4-base as shown in Fig. 22 (b). Notice that the parallelogram $v_4v_3v_2v_1$ in GG′HH′ is congruent to the parallelogram $v_1v_2v_3v_4$ inside N.

(4) Since GH of v_1v_3 in N (Fig. 22 (a)) is shorter than Hv_3 (= Gv_1) for N (i.e., G′H″ = G″H′ < Hv_3 = Gv_1 in Fig. 22 (b)), then $n \geq 2$ from Theorem 3-2. In this case, for $n = 2$, the parallelogram GG′HH′ is divided into sixty (= $4\times(2\times2-1)\times(2\times2+1)$) triangles as shown in Fig. 22 (c). Fold it along the fold-lines according to FF2 for 2 and glue adjacent pairs of perimeter parts of a stripe. Then, the topological sphere composed of proper FF2-arrangement for 2 is obtained (Fig. 22 (d)). Crease each face of the topological sphere along v_2v_4 and v_1v_3 (red lines in Fig. 22 (d)). And then, an isotetrahedron I is obtained (Fig. 22 (e). Therefore, the parallelogram GG′HH′ can be folded into an isotetrahedron I as shown in Fig. 22 (e)).

(5) Reverse a parallelogram GG′HH′ with fold-lines to a Conway tile N again. Fold N along these fold-lines (Fig. 22 (f)), and then N is folded into an isotetrahedron (Fig. 22 (g)) which is congruent to I in Fig. 22 (e).

We have our main results generalizing the argument in the two examples mentioned above.

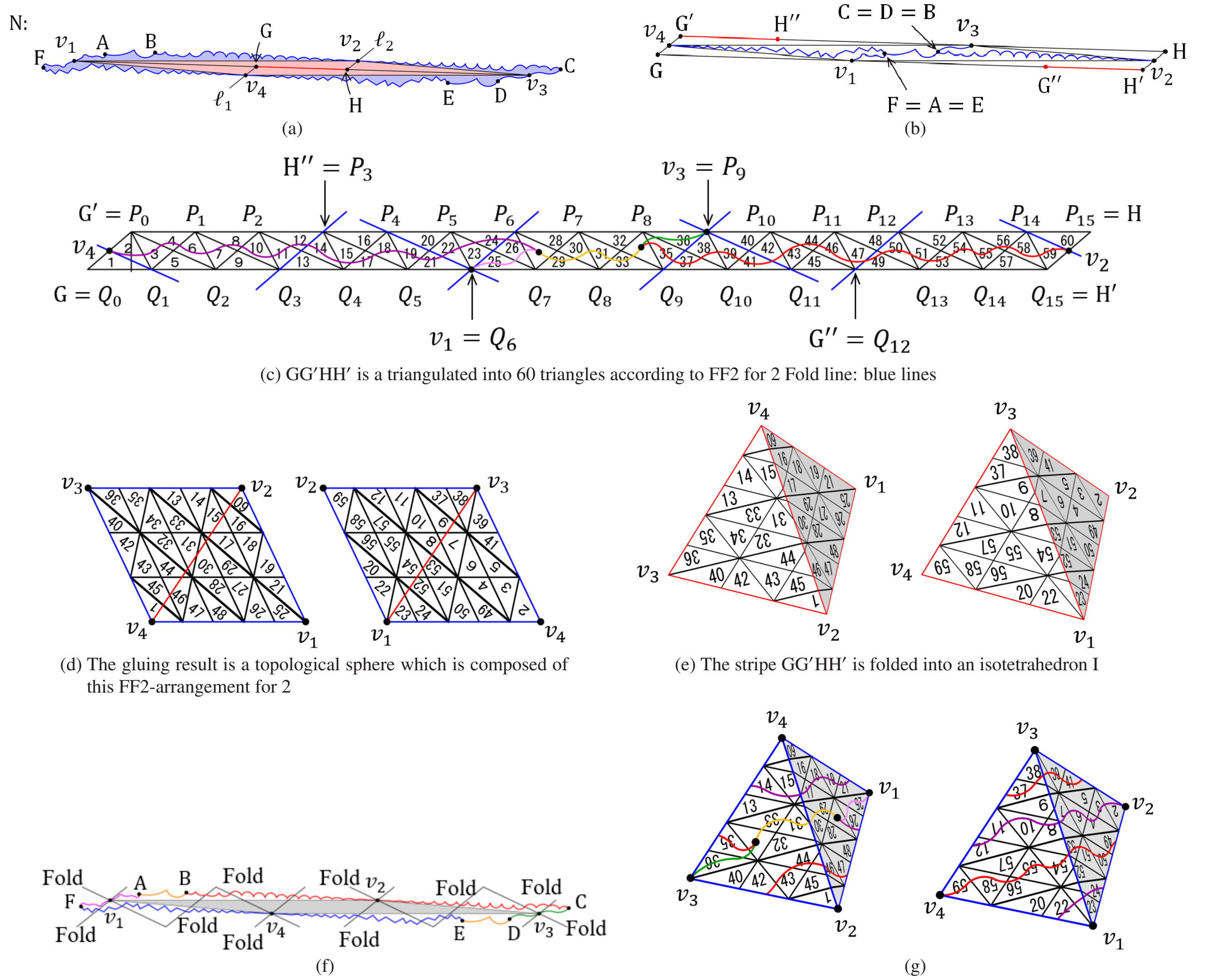
Theorem 4-1

Any Conway tile N with width w is folded into either an isotetrahedron or a rectangle dihedron by the following procedure applying FF1 (FF2), if N is a ring type (a spiral type).

Procedure

The first two steps are common for all Conway tiles:

1. Find the 4-base v_1, v_2, v_3 and v_4 of a given Conway tile N.
2. If a 4-base parallelogram is divided into two identical acute or right triangles T_1 and T_2, the Conway tile N can be folded into an isotetrahedron with four identical faces to T_1 (= T_2) or a rectangle dihedron whose face is identical to a rectangle composed of T_1 and T_2. If the 4-base parallelogram of N cannot be divided into right or acute triangles, glue the parallel-congruent



(a) (b)

(c) GG′HH′ is a triangulated into 60 triangles according to FF2 for 2 Fold line: blue lines

(d) The gluing result is a topological sphere which is composed of this FF2-arrangement for 2

(e) The stripe GG′HH′ is folded into an isotetrahedron I

(f) (g)

Fig. 22 How to fold N of spiral type.

parts (i.e., AB and ED) of N. In the case that N is ring type, go to the FF1-procedure mentioned below. If not, go to the FF2-procedure.

FF1-procedure

Without loss of generality, v_1v_3 (= v_2v_4) is longer than v_1v_2 (= v_3v_4) and the angle $v_2v_1v_3$ is acute angle of the 4-base parallelogram $v_1v_2v_3v_4$ of N. If the angle $v_2v_1v_3$ is obtuse, turn over the Conway tile N and rename the points v_1, v_2, v_3, v_4 into “v_2, v_1, v_3, v_4”, respectively, and then the angle $v_2v_1v_3$ is acute.

3. Determine n (≥ 2) such that $\sqrt{2}n(n+1)w/\sqrt{3n+1} \leq \ell' \leq 2n(n+1)w$, where ℓ' is the length of a 4-base parallelogram.
4. Divide the base side v_1v_3 of the 4-base parallelogram into $n(n+1)$ segments $P_0P_1, P_1P_2, \cdots, P_{n(n+1)-1}P_{n(n+1)}$ with the same length $d = \ell'/n(n+1)$, where $P_0 = v_1$ and $P_{n(n+1)} = v_3$.
5. Draw the segments P_nv_2 and v_2P_{n+1}. The triangle $P_nv_2P_{n+1}$ symbolizes the cell of the parallelogram stripe M_1 below.
6. Dissect the Conway tile N along v_1v_2, v_2v_4, v_4v_3 and v_3v_1 and reverse the Conway tile N to the parallelogram M_1: $v_2v_1v_1'v_2'$ with the length $2\ell'$ and the width w, where v_2', v_1' is a centrosymmetric point to v_2, v_1 (= P_0) around v_4, v_3 (= $P_{n(n+1)}$), respectively.
7. Divide the segment v_3v_1' into $n(n+1)$ segments with the length d; $v_3P_{n(n+1)+1}$, $P_{n(n+1)+1}P_{n(n+1)+2}$, $\cdots$, $P_{2n(n+1)-1}v_1$, where $v_1 = P_0 = P_{2n(n+1)}$. Divide also the segment v_2v_2' with the length d into $2n(n+1)$ segments $Q_{n+1}Q_{n+2}$, $Q_{n+2}Q_{n+3}$, $\cdots$, $Q_{2n(n+1)-1}Q_0$, Q_0Q_1, $\cdots$, where $v_4 = Q_{(n+1)^2}$, $v_2 = v_2' = Q_{n+1}$ and $Q_0 = Q_{2n}$.
8. Glue the side v_2v_1 and $v_2'v_1'$ of the parallelogram of M_1 to make a ring and decompose a parallelogram $P_0P_{2n(n+1)}Q_{2n(n+1)}Q_0$ into $4n(n+1)$ triangles (i.e., cells), $Q_0P_0Q_1$, $P_0Q_1P_1$, $\cdots$ each of which is identical to the triangle $P_nv_2P_{n+1}$ in Step 5 above.
9. Fold the parallelogram M_1: $P_0P_{2n(n+1)}Q_{2n(n+1)}Q_0$ into an isotetrahedron I or a rectangle dihedron RD according to FF1 for n. From Theorem 3-1, the FF1-arrangement for n is the topological sphere which induces I or RD by creasing it. I or RD can be exactly described by FF1-arrangement of the parallelogram M_1 (Ref. [1]).
10. Reverse the parallelogram M_1: $P_0P_{2n(n+1)}Q_{2n(n+1)}Q_0$ with fold-lines to a Conway tile N again. Fold a Conway tile N along the fold-lines, and then a Conway tile is folded into an isotetrahedron or a rectangle dihedron which is congruent to I or RD in Step 9.

FF2-procedure

3. For the 4-base parallelogram $v_1v_2v_3v_4$ of N, let v_1v_3 be the longer diagonal and $v_1v_4 < v_1v_2$ (i.e., $v_3v_2 < v_3v_4$).
4. Draw the line v_1v_3. Draw the line ℓ_1, ℓ_2 through v_4, v_2, respectively such that ℓ_1 is parallel to ℓ_2. Denote by G, H the intersection of v_1v_3 and ℓ_1, v_1v_3 and ℓ_2, respectively.
5. Dissect N along v_1v_3, Gv_4 and Hv_2 into four pieces, and then reverse the Conway tile N to the parallelogram M_2: GG′HH′, which inscribes the parallelogram $v_4v_3v_2v_1$ which is identical to the parallelogram $v_1v_2v_3v_4$ in N.
6. If GH of v_1v_3 in N is longer than Gv_1 (= Hv_3) in N, triangulate the parallelogram M_2: GG′HH′ and fold it into an isotetrahedron I or a rectangle dihedron RD in the manner of FF2 for 1. If not, triangulate the parallelogram M_2: GG′HH′ into $4(2n-1)(2n+1)$ triangles and fold it into an isotetrahedron I or a rectangle dihedron RD in the manner of FF2 for n. From Theorem 3-2, I or RD can be exactly described by FF2-arrangement of the parallelogram M_2 (Ref. [1]).
7. Reverse the parallelogram M_2: GG′HH′ with fold-lines to a Conway tile N again. Fold a Conway tile N along the fold-lines, and then the Conway tile N is folded into an isotetrahedron or a rectangle dihedron which is congruent to I or RD in Step 6.

Acknowledgments The authors would like to thank Professor David Eppstein and Professor Stefan Langerman for their invaluable comments.

References

[1] Matsunaga, K.: *The mechanism of folding stripes into isotetrahedra or rectangle dihedra*, in preparation.
[2] Akiyama, J.: Tile-Makers and Semi-Tile-Makers, *American Math. Monthly*, Vol.114, pp.602–609 (2007).
[3] Abel, Z., Demaine, E.D., Demaine, M., Matsui, H., Rote, G. and Uehara, R.: Common Developments of Several Different Orthogonal Boxes, *23rd CCCG 2011* (2011).
[4] Akiyama, J., Demaine, E.D. and Langerman, S.: Polyhedral Characterization of Reversible Hinged Dissection, *Graphs Combin.*, Vol.32, No.2, pp.25–33 (2020).
[5] Alexander, R., Dyson, H. and O'Rourke, J.: The Foldings of a Square to Convex Polyhedra, LNCS 2866, pp.38–50 (2003).
[6] Akiyama, J., Langerman S. and Matsunaga, K.: Reversible Nets of Polyhedra, LNCS 9943, pp.13–23 (2016).
[7] Akiyama, J. and Matsunaga, K.: *Treks in Intuitive Geometry*, Springer (2015).
[8] Akiyama, J. and Matsunaga, K.: Generalization of Haberdasher's puzzle, *Discrete Comput. Geom.*, Vol.58, No.1, pp.30–50 (2017).
[9] Akiyama, J. and Matsunaga, K.: Reversibility and Foldability of Conway tiles, *Comput. Geom.*, Vol.64, pp.30–45 (2017).
[10] Akiyama, J. and Matsunaga, K.: Unfoldings of an envelope, *European Journal of Combinatorics*, Vol.80, pp.3–16 (2019).
[11] Demaine, E.D. and O'Rourke, J.: *Geometric Folding Algorithms*, Cambridge University Press (2007).
[12] O'Rourke, J.: *How to fold it*, Cambridge University Press (2011).
[13] Schattschneider, D.: Will It Tile? Try the Conway Criterion, *Mathematics Magazine*, Vol.53, No.4, pp.224–233 (1980).

Jin Akiyama is a mathematician at heart. Currently, he is a fellow of the European Academy of Science, the founding editor of the journal Graphs and Combinatorics, and a specially appointed vice-president at Tokyo University of Science. He is particularly interested in graph theory and discrete and solid geometry, and has published many papers in these fields. Aside from his research, he is best known for popularizing mathematics, first in Japan and then in other parts of the globe: his lecture series was broadcast on NHK (Japan's only public broadcaster) television and radio from 1991 to 2013. He was a founding member of the Organizing Committee of the UNESCO-sponsored traveling exhibition "Experiencing Mathematics", which debuted in Denmark in 2004. In 2013, he built a hands-on mathematics museum called "Akiyama's Math Experience Museum" in Tokyo. He has authored and co-authored more than 200 books, including Factors and Factorizations of Graphs (jointly with Mikio Kano, Lect. Notes Math., Springer, 2011) and, A Day's Adventure in Math Wonderland (jointly with Mari-Jo Ruiz, World Scientific, 2008), which has been translated into ten languages.

Kiyoko Matsunaga received her B.Sc. in Mathematics from Waseda University. Since 1995, she works at Math Lab, Japan as a science writer for math and published many books including Math Time Travel for senior high school students, and The Joy of Math and Math Wonderland for the broader public. Her expository and research papers in mathematics have appeared in Mathematics Magazine, European Journal of Combinatorics, Computational Geometry, Discrete Computational Geometry etc. Her main interest in mathematics is Discrete Geometry and she co-authored the book "Treks into Intuitive Geometry (jointly with Jin Akiyama, Springer, 2015)."