Vol.2019-AL-175 No.10
2019/11/28

An Envy-free and Truthful Mechanism for the Cake-cutting Problem without Expansion Process with Unlocking

Takao Asano[1,a)] Hiroyuki Umeda[1,b)]

Abstract: We show an input example in the cake-cutting problem which cannot be correctly solved by the expansion process with unlocking proposed in the paper [1], [6], and give an alternative envy-free and truthful mechanism which is not based on the expansion process with unlocking.

Keywords: cake-cutting problem, envy-freeness, fairness, truthfulness, mechanism design

1. Introduction

The problem of dividing a cake among players in a fair manner has been widely studied since it was first defined by Steinhaus [7]. Prcocaccia has claimed in his survey paper [5] as follows: insight from the study of cake-cutting problem can be applied to the allocation of computational resources, and designing cake-cutting algorithms that are computationally efficient and immune to manipulation is a challenge for computer scientists. Recently, the cake-cutting problem has been studied by computer scientists, not only from the computational complexity point of view [3], but also from the game theoretical point of view [2].

Alijani, Farhadi, Ghodsi, Seddighin, and Tajik considered the following cake-cutting problem from the game theoretical point of view [1], [6]: Given a divisible heterogeneous cake $C = (0, 1] = \{x \mid 0 < x \leq 1\}$, n strategic players $N = \{1, 2, \dots, n\}$ with valuation interval $C_i = (\alpha_i, \beta_i] = \{x \mid 0 \leq \alpha_i < x \leq \beta_i \leq 1\} \subseteq C$ of each player $i \in N$, find a mechanism (that is, a polynomial time algorithm) for dividing the cake into pieces and allocating pieces of the cake to n players to meet the following conditions (Fig.1):

(i) the mechanism is envy-free, i.e., each player (weakly) prefers his/her allocated pieces to any other player's allocated pieces,

(ii) the mechanism is strategy-proof (truthful), i.e., each player's dominant strategy is to reveal his/her own true valuation interval over the cake (i.e., making a lie will not lead to a better result), and

(iii) the number of cuts made on the cake is small.

They proposed an expansion process with unlocking and using it they gave a mechanism for the above cake-cutting problem [1], [6]. They claimed that their mechanism satisfies the above three conditions, i.e., it is envy-free, truthful and the the number of cuts made on the cake is at most $2(n-1)$.

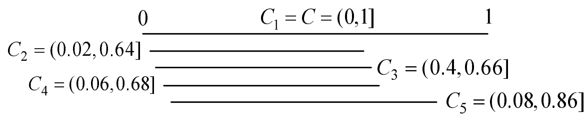

Fig. 1 An input example for the cake-cutting problem ($n = 5$). player 1 is allocated $(0, 0.06] \cup (0.86, 1]$, player 2 is allocated $(0.06, 0.26]$, player 3 is allocated $(0.26, 0.46]$, player 4 is allocated $(0.46, 0.66]$ and player 5 is allocated $(0.66, 0.86]$.

In this paper, we first note that, for the input example in Fig.1 of the above cake-cutting problem, the mechanism proposed based on the expansion process with unlocking in the paper [1], [6] is not envy-free, since, by their mechanism, first player 2 is allocated $(0.08, 0.28]$, player 3 is allocated $(0.28, 0.48]$, player 4 is allocated $(0.48, 0.68]$, then player 5 is allocated $(0.73, 0.86]$, and finally player 1 is allocated $(0, 0.08] \cup (0.68, 0.73] \cup (0.86, 1]$ (thus, player 5 will envy players 2,3 and 4). We give an alternative envy-free and truthful mechanism for the above cake-cutting problem which is not based on the expansion process with unlocking.

2. Preliminaries

In this section, we give notations which will be used in this paper. They are a little different from those used in the paper [6].

We denote by $\mathcal{C}_N$ the set of valuation intervals of all the players N, i.e., $\mathcal{C}_N = \{C_i \mid i \in N\}$. We assume that $\bigcup_{C_i \in \mathcal{C}_N} C_i = C$.

Definition 2.1 A *piece* of the cake C is a separated interval of C, and a *set of pieces* of C is a set of disjoint pieces of C. Thus, $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \dots, A_{i_{k_i}}\}$ is a set of pieces of C if and only if each A_{i_j} ($j = 1, 2, \dots, k_i$) is a piece of C and any two distinct A_{i_j} and $A_{i_{j'}}$ ($1 \leq j < j' \leq k_i$) are disjoint (i.e., $A_{i_j} \cap A_{i_{j'}} = \emptyset$).

Definition 2.2 Let $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \dots, A_{i_{k_i}}\}$ be a set of pieces of the cake C for each $i \in N$, and let $A_i = A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_{k_i}}$. A union A_i of mutual disjoint sets $A_{i_1}, A_{i_2}, \dots, A_{i_{k_i}}$ is called a *direct sum* of $A_{i_1}, A_{i_2}, \dots, A_{i_{k_i}}$ and is denoted by $A_i = A_{i_1} + A_{i_2} + \dots + A_{i_{k_i}}$.

1 Chuo University
a) asano@ise.chuo-u.ac.jp
b) wordmindblack@gmail.com

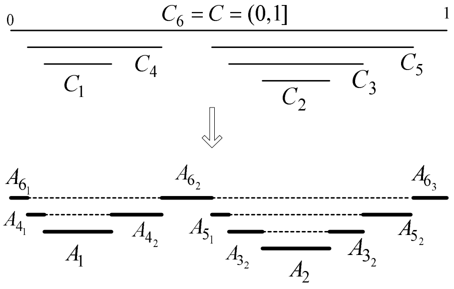

Fig. 2 An allocated set of pieces to player 5 is $\{A_{5_1}, A_{5_2}\}$ ($n = 6$). size(A_6) = size(A_{6_1}) + size(A_{6_2}) + size(A_{6_3}) for $A_6 = A_{6_1} + A_{6_2} + A_{6_3}$.

Then $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ is called an *allocation* of the cake C to n players N if any two distinct A_i and A_j ($1 \leq i < j \leq n$) are disjoint and $\sum_{i \in N} A_i = A_i + A_2 + \cdots + A_n = C$. We consider $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ (also $A_i = \sum_{j=1}^{k_i} A_{i_j} = A_{i_1} + A_{i_2} + \cdots + A_{i_{k_i}}$) is an allocated set of pieces of the cake C to player i in $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ (Fig.2).

Definition 2.3 For an interval $X = (x', x'']$ of C, the *size* of X, denoted by size(X), is defined by $x'' - x'$.

For a direct sum $X = X_1 + X_2 + \cdots + X_k$ (a union $X = X_1 \cup X_2 \cup \cdots \cup X_k$ of mutual disjoint intervals X_j of C), the *size* of X, denoted by size(X), is defined by the total sum of size(X_j) (Fig.2), i.e., size(X) = size(X_1) + size(X_2) + $\cdots$ + size(X_k).

Definition 2.4 Let $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ be a set of pieces of C and let $X = X_1 + X_2 + \cdots + X_k$. For each $i \in N$ and valuation interval C_i of player i, the *utility* of $\mathcal{X}$ for player i, denoted by $\mathrm{ut}_i(\mathcal{X})$, is the total sum of size($X_j \cap C_i$) for all pieces $X_j \in \mathcal{X}$, i.e.,

$$\mathrm{ut}_i(\mathcal{X}) = \mathrm{size}(X_1 \cap C_i) + \cdots + \mathrm{size}(X_k \cap C_i). \tag{1}$$

We also write $\mathrm{ut}_i(\mathcal{X})$ as $\mathrm{ut}_i(X) = \mathrm{size}(X_1 \cap C_i) + \mathrm{size}(X_2 \cap C_i) + \cdots + \mathrm{size}(X_k \cap C_i)$.

Definition 2.5 Let $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ be an allocation of the cake C to n players N and let $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ be a set of pieces of C allocated to player $i \in N$. If

$$\mathrm{ut}_i(\mathcal{A}_i) \geq \mathrm{ut}_i(\mathcal{A}_j) \quad \text{for all } j \in N - \{i\}, \tag{2}$$

then the allocated set of pieces $\mathcal{A}_i$ is called *envy-free* for player i. If, for every player $i \in N$, the allocated set of pieces $\mathcal{A}_i$ is envy-free for player i, then the allocation $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ of the cake C to n players N is called *envy-free*.

Definition 2.6 Let $\mathcal{M}$ be a mechanism for the cake-cutting problem. Let $\mathcal{C}_N = \{C_1, C_2, \ldots, C_n\}$ be an arbitrary input to the mechanism $\mathcal{M}$ and $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ be an allocation of the cake C to n players N obtained by $\mathcal{M}$ with $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ for each $i \in N$. If the allocation $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ is envy-free then the mechanism $\mathcal{M}$ is called *envy-free*.

Now, for each player $i \in N$, assume that only player i makes a lie and gives a false valuation interval C'_i. Thus, let

$$\mathcal{C}'_N(i) = \{C_1, C_2, \ldots, C_{i-1}, C'_i, C_{i+1}, \ldots, C_n\} \tag{3}$$

be an input to the mechanism $\mathcal{M}$ and let an allocation of the cake C to n players N obtained by $\mathcal{M}$ be

$$\mathcal{A}'_N(i) = \{\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_{i-1}, \mathcal{A}'_i, \mathcal{A}'_{i+1}, \ldots, \mathcal{A}'_n\} \tag{4}$$

with $\mathcal{A}'_j = \{A'_{j_1}, A'_{j_2}, \ldots, A'_{j_{k'_j}}\}$ for each $j \in N$. The utilities of $\mathcal{A}_i$ and $\mathcal{A}'_i$ for player i are

$$\mathrm{ut}_i(\mathcal{A}_i) = \sum_{j=1}^{k_i} \mathrm{size}(A_{i_j} \cap C_i), \quad \mathrm{ut}_i(\mathcal{A}'_i) = \sum_{j=1}^{k'_i} \mathrm{size}(A'_{i_j} \cap C_i). \tag{5}$$

If $\mathrm{ut}_i(\mathcal{A}_i) \geq \mathrm{ut}_i(\mathcal{A}'_i)$, i.e., if the utility of player i ($\mathrm{ut}_i(\mathcal{A}'_i)$), even if player i (and only player i) makes a lie on the valuation interval and gives any C'_i, will not become better than the utility of player i ($\mathrm{ut}_i(\mathcal{A}_i)$) when player i gives true valuation interval C_i, then player i does not want to tell a lie and player i reports the true valuation interval C_i to the mechanism $\mathcal{M}$.

For each player $i \in N$, if this holds, then no player wants to tell a lie and all players want to report true valuation intervals to the mechanism $\mathcal{M}$. In this case, the mechanism $\mathcal{M}$ is called *truthful* and an allocation $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ of the cake C to n players N obtained by the mechanism $\mathcal{M}$ is also called *truthful*.

Definition 2.7 For an interval $X = (x', x'']$ of C, let $N(X)$ be the set of players in N whose valuation intervals are entirely contained in X and let $\mathcal{C}_{N(X)}$ be the set of valuation intervals in $\mathcal{C}_N$ which are entirely contained in X. Let n_X be the cardinality of $N(X)$ ($\mathcal{C}_{N(X)}$), i.e., n_X is the number of players in $N(X)$ (the number of valuation intervals of $\mathcal{C}_N$ in $\mathcal{C}_{N(X)}$). Thus,

$$N(X) = \{i \in N \mid C_i \subseteq X, C_i \in \mathcal{C}_N\}, \tag{6}$$

$$\mathcal{C}_{N(X)} = \{C_i \in \mathcal{C}_N \mid i \in N(X)\}, \tag{7}$$

$$n_X = |N(X)| = |\mathcal{C}_{N(X)}|. \tag{8}$$

Then the *density* of the interval X, denoted by $\rho(X)$, is defined by

$$\rho(X) = \frac{\mathrm{size}(X)}{|\mathcal{C}_{N(X)}|} = \frac{x'' - x'}{n_X}. \tag{9}$$

Definition 2.8 For an interval $X = (x', x'']$ of C, if there are valuation intervals $C_i = (\alpha_i, \beta_i]$ and $C_j = (\alpha_j, \beta_j]$ in $\mathcal{C}_{N(X)} = \{C_k \in \mathcal{C}_N \mid k \in N(X)\}$ with $x' = \alpha_i$ and $x'' = \beta_j$, then $X = (x', x'']$ is called a *minimal interval with respect to density* (there are at most n^2 minimal intervals with respect to density).

Definition 2.9 For a subset $S \subseteq N$ of players, let $\mathcal{C}_S$ be the set of valuation intervals of players in S, i.e.,

$$\mathcal{C}_S = \{C_i = (\alpha_i, \beta_i] \in \mathcal{C}_N \mid i \in S\}. \tag{10}$$

Let $C(S)$ be the interval of C defined by

$$C(S) = (\min_{C_i=(\alpha_i,\beta_i]\in\mathcal{C}_S} \alpha_i, \max_{C_i=(\alpha_i,\beta_i]\in\mathcal{C}_S} \beta_i] \tag{11}$$

Then the *density* of $\mathcal{C}_S$, denoted by $\sigma(\mathcal{C}_S)$, is defined by

$$\sigma(\mathcal{C}_S) = \frac{\mathrm{size}(C(S))}{|S|}. \tag{12}$$

Note that, for an interval $X = C(S)$ of C, we have

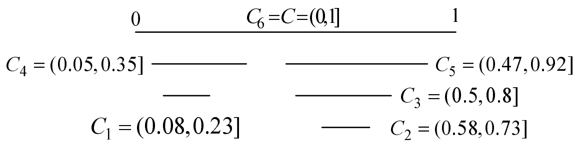

Fig. 3 Example of the valuation intervals $C_1, C_2, \ldots, C_6$. The minimum density is $\rho_{\min} = 0.15$ and the intervals of minimum density are C_1, C_2, C_3, C_4, C_5. Among them, C_1 and C_2 are the minimal intervals of minimum density and C_4 and C_5 are the maximal intervals of minimum density and interval C_6 is of density $\rho(C_6) = \frac{1}{6} = 0.1666...$

$$|\mathcal{C}_{N(X)}| = |N(X)| \geq |S| = |\mathcal{C}_S|, \quad \text{size}(C(S)) = \text{size}(X), \quad (13)$$

$$\rho(X) = \rho(C(S)) = \frac{\text{size}(C(S))}{|N(X)|} \leq \sigma(\mathcal{C}_S) = \frac{\text{size}(C(S))}{|S|}. \quad (14)$$

since $S \subseteq N(X) = \{i \in N \mid C_i \subseteq X\}$ and $\mathcal{C}_S \subseteq \mathcal{C}_{N(X)} = \{C_i \in \mathcal{C}_N \mid i \in N(X)\}$. In this paper, we are mainly interested in intervals of minimum density among all intervals in C.

Definition 2.10 Let $\mathcal{X}$ be the set of all nonempty intervals in C. Let $\rho_{\min}$ be the minimum density among the densities of all intervals in C (Fig.3). Thus,

$$\rho_{\min} = \min_{X \in \mathcal{X}} \rho(X). \quad (15)$$

Let $\mathcal{X}_{\min}$ be the set of all intervals in C of minimum density, i.e.,

$$\mathcal{X}_{\min} = \{X \in \mathcal{X} \mid \rho(X) = \rho_{\min}\}. \quad (16)$$

An interval $X \in \mathcal{X}_{\min}$ is called an *interval of minimum density*. An interval X of minimum density is called a *minimal interval of minimum density* if X contains no interval in $\mathcal{X}_{\min}$ properly. Simlarly, an interval X of minimum density is called a *maximal interval of minimum density* if no interval in $\mathcal{X}_{\min}$ contains X properly.

(Note that, an interval of minimum density is always a minimal interval with respect to density.)

3. Structures of Intervals of Minimum Density

In this section, we discuss structures of intervals of minimum density which play a central role in our mechanism.

Lemma 3.1 Let $X_i = (x'_i, x''_i]$ and $X_j = (x'_j, x''_j]$ be two distinct minimal intervals in C with respect to density. Let

$$X_i \cap X_j \neq \emptyset, \quad X_i - X_j \neq \emptyset, \quad X_j - X_i \neq \emptyset,$$
$$\rho(X_i) \geq \rho(X_j), \quad \rho(X_i \cap X_j) \geq \rho(X_j).$$

Then $\rho(X_i \cup X_j) \leq \rho(X_i)$.

Proof: By symmetry we can assume $x'_i < x'_j < x''_i < x''_j$ since $X_i \cap X_j \neq \emptyset$, $X_i - X_j \neq \emptyset$, and $X_j - X_i \neq \emptyset$ (Fig.4). Let

$$Y = X_i \cap X_j = (y', y''], \quad Z = X_i \cup X_j = (z', z''].$$

Thus, $y' = x'_j$, $y'' = x''_i$, $z' = x'_i$, $z'' = x''_j$. By Definition 2.7,

$$N(X_i) = \{k \in N \mid C_k \subseteq X_i\}, \quad \mathcal{C}_{N(X_i)} = \{C_k \in \mathcal{C}_N \mid k \in N(X_i)\},$$

$$N(X_j) = \{k \in N \mid C_k \subseteq X_j\}, \quad \mathcal{C}_{N(X_j)} = \{C_k \in \mathcal{C}_N \mid k \in N(X_j)\},$$

$$N(Y) = \{k \in N \mid C_k \subseteq Y\}, \quad \mathcal{C}_{N(Y)} = \{C_k \in \mathcal{C}_N \mid k \in N(Y)\},$$

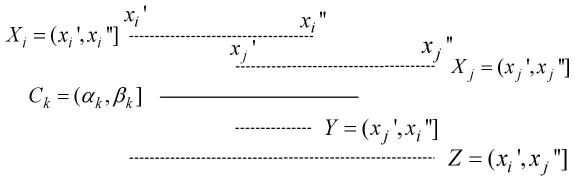

Fig. 4 Two intervals $X_i = (x'_i, x''_i]$ and $X_j = (x'_j, x''_j]$ in Proof of 3.1. A valuation interval $C_k = (\alpha_k, \beta_k]$ is not in X_i nor X_j, but in $Z = X_i \cup X_j$.

$$N(Z) = \{k \in N \mid C_k \subseteq Z\}, \quad \mathcal{C}_{N(Z)} = \{C_k \in \mathcal{C}_N \mid k \in N(Z)\},$$
$$\mathcal{C}_{N(Y)} = \mathcal{C}_{N(X_i)} \cap \mathcal{C}_{N(X_j)}.$$

Let

$$n_{X_i} = |N(X_i)|, \; n_{X_j} = |N(X_j)|, \; n_Y = |N(Y)|, \; n_Z = |N(Z)|,$$

and let

$$\mathcal{C}_W = \mathcal{C}_{N(Z)} - (\mathcal{C}_{N(X_i)} \cup \mathcal{C}_{N(X_j)}), \quad n_W = |\mathcal{C}_W|.$$

Note that a valuation interval $C_k = (\alpha_k, \beta_k] \in \mathcal{C}_N$ with $x'_i < \alpha_k < x'_j$ and $x''_i < \beta_k < x''_j$ is in $\mathcal{C}_W = \mathcal{C}_{N(Z)} - (\mathcal{C}_{N(X_i)} \cup \mathcal{C}_{N(X_j)})$ (Fig.4). Then, by the inclusion-exclusion principle, we have

$$n_Z = n_{X_i} + n_{X_i} - n_Y + n_W.$$

Thus, the density $\rho(Z)$ of interval $Z = X_i \cup X_j$ in Definition 2.7 is

$$\begin{aligned}\rho(Z) &= \frac{\text{size(Z)}}{n_Z} = \frac{x''_j - x'_i}{n_{X_i} + n_{X_j} - n_Y + n_W} \\ &= \frac{x''_j - x'_j + x''_i - x'_i - (x''_i - x'_j)}{n_{X_i} + n_{X_j} - n_Y + n_W}.\end{aligned}$$

We first discuss the case of $n_Y > 0$. Since $n_W \geq 0$ and by the definition of density of an interval in Definition 2.7, we have

$$\begin{aligned}n_{X_i}\,\rho(X_i) &= x''_i - x'_i, \quad n_{X_j}\,\rho(X_j) = x''_j - x'_j, \quad n_Y\,\rho(Y) = x''_i - x'_j, \\ \rho(Z) &= \frac{x''_j - x'_j + x''_i - x'_i - (x''_i - x'_j)}{n_{X_i} + n_{X_j} - n_Y + n_W} \\ &\leq \frac{x''_j - x'_j + x''_i - x'_i - (x''_i - x'_j)}{n_{X_i} + n_{X_j} - n_Y} \\ &= \frac{n_{X_i}\,\rho(X_i) + n_{X_j}\,\rho(X_j) - n_Y\,\rho(Y)}{n_{X_i} + n_{X_j} - n_Y}.\end{aligned}$$

Note that,

$$\rho(Z) = \frac{n_{X_i}\,\rho(X_i) + n_{X_j}\,\rho(X_j) - n_Y\,\rho(Y)}{n_{X_i} + n_{X_j} - n_Y}$$

if and only if $n_W = 0$. Since $\rho(Y) = \rho(X_i \cap X_j) \geq \rho(X_j)$, we have

$$\begin{aligned}\rho(Z) &\leq \frac{n_{X_i}\,\rho(X_i) + n_{X_j}\,\rho(X_j) - n_Y\,\rho(Y)}{n_{X_i} + n_{X_j} - n_Y} \\ &\leq \frac{n_{X_i}\,\rho(X_i) + n_{X_j}\,\rho(X_j) - n_Y\,\rho(X_j)}{n_{X_i} + n_{X_j} - n_Y}.\end{aligned}$$

Furthermore, since $\rho(X_i) \geq \rho(X_j)$ and $n_{X_j} \geq n_Y$, we have

$$\begin{aligned}\rho(Z) &\leq \frac{n_{X_i}\,\rho(X_i) + n_{X_j}\,\rho(X_j) - n_Y\,\rho(X_j)}{n_{X_i} + n_{X_j} - n_Y} \\ &= \frac{n_{X_i}\,\rho(X_i) + (n_{X_j} - n_Y)\,\rho(X_j)}{n_{X_i} + n_{X_j} - n_Y} \\ &\leq \frac{(n_{X_i} + n_{X_j} - n_Y)\,\rho(X_i)}{n_{X_i} + n_{X_j} - n_Y} = \rho(X_i).\end{aligned}$$

The case of $n_Y = 0$ can be discussed similarly.

By the argument above, we have $\rho(Z) = \rho(X_i)$ if and only if $n_W = 0$, $n_Y > 0$, and $\rho(Y) = \rho(X_j) = \rho(X_i)$. □

By Lemma 3.1, we have the following corollaries.

Collorary 3.1 Let $X_i = (x'_i, x''_i]$ and $X_j = (x'_j, x''_j]$ be two distinct intervals in C of minimum density. If $X_i \cap X_j \neq \emptyset$ then both $Y = X_i \cap X_j$ and $Z = X_i \cup X_j$ are of minimum density.

Proof: If $X_i - X_j = \emptyset$ (i.e., $X_i \subseteq X_j$), then clearly $Y = X_i$ and $Z = X_j$ are both of minimum density. Similarly, if $X_j - X_i = \emptyset$, (i.e., $X_j \subseteq X_i$), then clearly both $Y = X_j$ and $Z = X_i$ are of minimum density. Thus, we can assume $X_i - X_j \neq \emptyset$ and $X_j - X_i \neq \emptyset$. Since $\rho(X_i) = \rho(X_j) = \rho_{\min}$, we have $\rho(X_i) \geq \rho(X_j)$, $\rho(X_i \cap X_j) \geq \rho(X_j)$ and $\rho(Z) \leq \rho_{\min}$ by Lemma 3.1. It is clear that $\rho(Z) \geq \rho_{\min}$ and we have $\rho(Z) = \rho_{\min}$.

Furthermore, by the argument in the proof of Lemma 3.1, $\rho(Z) = \rho(X_i) = \rho_{\min}$ if and only if $n_W = 0$, $n_Y > 0$, and $\rho(Y) = \rho(X_j) = \rho(X_i)$. Thus, we have $\rho(Y) = \rho_{\min}$. □

Collorary 3.2 If $X_i = (x'_i, x''_i]$ and $X_j = (x'_j, x''_j]$ are two distinct minimal intervals of minimum density, then $X_i \cap X_j = \emptyset$.

Similarly, if $X_i = (x'_i, x''_i]$ and $X_j = (x'_j, x''_j]$ are two distinct maximal intervals of minimum density, then $X_i \cap X_j = \emptyset$.

4. Our Mechanism

We give a brief outline of our mechanism. We first cut the cake $C = (0, 1]$ at both endpoints of each maximal interval of minimum density. By Corollary 3.2, two distinct maximal intervals of minimum density are disjoint and we can cut the cake at both endpoints of each maximal interval of minimum density, independently. By these cuts, we can reduce the original cake-cutting problem into two types of cake-cutting subproblems (Fig.5):

(i) the cake-cutting problem within each maximal interval $X_i = (x'_i, x''_i]$ of minimum density (which consists of the players whose valuation intervals in X_i), and

(ii) the cake-cutting problem with all valuation intervals obtained by deleting all the valuation intervals within all the maximal interval of minimum density.

Note that the cake-cutting problem of type (i) is almost the same as the original cake-cutting problem. On the other hand, the cake-cutting problem of type (ii) is different from the original cake-cutting problem, because the resulting cake becomes a set of two or more disjoint intervals and a resulting valuation interval may also become a set of two or more disjoint intervals. However, the cake-cutting problem of type (ii) has a nice property.

Lemma 4.1 For two distinct intervals $X_i = (x'_i, x''_i]$ and $X_j = (x'_j, x''_j]$, let $X_i \cap X_j \neq \emptyset$ and $X_i - X_j \neq \emptyset$. Furthermore, let

$$Z = X_i \cup X_j, \quad \mathcal{C}_N(Z - X_j) = \{C_k - X_j \mid C_k \subseteq Z, C_k \in \mathcal{C}_N\}.$$

Suppose that $\rho(X_i) \geq \rho(X_j)$ and $\rho(Z) \geq \rho(X_j)$. Then, by cutting the cake at both endpoints of X_j and deleting X_j, Z becomes $Z - X_j = X_i - X_i \cap X_j$ and $\mathcal{C}_{N(Z)} = \{C_k \in \mathcal{C}_N \mid C_k \subseteq Z\}$ becomes

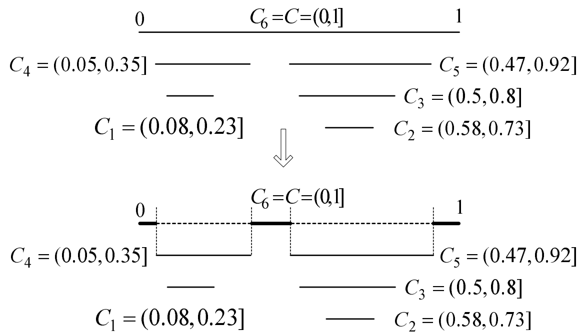

Fig. 5 The cake-cutting problem can be reduced into two types of cake-cutting subproblems by cutting the cake $C = (0, 1]$ at both endpoints of each maximal interval of minimum density: (i) one within each maximal interval of minimum density (players $S_1 = \{1, 4\}$ and players $S_2 = \{2, 3, 5\}$]), and (ii) one with all valuation intervals obtained by deleting all the valuation intervals within all maximal intervals of minimum density (players $R = \{6\}$).

$$\mathcal{C}_N(Z - X_j) = \{C_k - X_j \mid C_k \in \mathcal{C}_N, C_k \subseteq Z, C_k - X_j \neq \emptyset\} \quad (17)$$

and the density $\rho(Z)$ in the original cake-cutting problem becomes from

$$\rho(Z) = \rho_C(Z) = \frac{\text{size}(Z)}{|\mathcal{C}_N(Z)|}$$

to the density $\rho_{C-X_j}(Z - X_j)$ in the cake-cutting problem with the cake $C - X_j$, players $N - N(X_j)$ and valuation intervals

$$\mathcal{C}_N(C - X_j) = \{C_k - X_j \mid C_k \in \mathcal{C}_N, C_k - X_j \neq \emptyset\} \quad (18)$$

such that

$$\rho_{C-X_j}(Z - X_j) = \frac{\text{size}(Z - X_j)}{|\mathcal{C}_N(Z - X_j)|} \geq \rho(X_j). \quad (19)$$

Furthermore, if $\rho(Z) > \rho(X_j)$ then $\rho_{C-X_j}(Z - X_j) > \rho(X_j)$.

Proof: We divide the case into two subcases: (i) when $X_j - X_i \neq \emptyset$ and (ii) when $X_j - X_i = \emptyset$.

(i) when $X_j - X_i \neq \emptyset$: By symmetry we can assume

$$x'_i < x'_j < x''_i < x''_j$$

since $X_i \cap X_j \neq \emptyset$, $X_i - X_j \neq \emptyset$, and $X_j - X_i \neq \emptyset$ (Fig.4).

Let $n_{Z-X_j} = |\mathcal{C}_N(Z - X_j)|$, $n_Z = |\mathcal{C}_{N(Z)}|$ and $n_{X_j} = |\mathcal{C}_{N(X_j)}|$ by Eqs. (6), (7) and (8). Then

$$n_{Z-X_j} = n_Z - n_{X_j} \geq 0,$$

$$\text{size}(Z - X_j) = \text{size}(X_i) - \text{size}(X_i \cap X_j) = x'_j - x'_i$$

and, by $\rho(Z) \geq \rho(X_j)$, we have

$$\begin{aligned} \rho_{C-X_j}(Z - X_j) &= \frac{\text{size}(Z - X_j)}{n_{Z-X_j}} = \frac{x'_j - x'_i}{n_{Z-X_j}} \\ &= \frac{x''_j - x'_i - (x''_j - x'_j)}{n_Z - n_{X_j}} = \frac{n_Z \rho(Z) - n_{X_j} \rho(X_j)}{n_Z - n_{X_j}} \\ &\geq \frac{(n_Z - n_{X_j})\, \rho(X_j)}{n_Z - n_{X_j}} = \rho(X_j). \end{aligned}$$

Note that, $\rho_{C-X_j}(Z - X_j) = \rho(X_j)$ if and only if $\rho(Z) = \rho(X_j)$. Thus, if $\rho(Z) > \rho(X_j)$ then $\rho_{C-X_j}(Z - X_j) > \rho(X_j)$.

(ii) when $X_j - X_i = \emptyset$ (i.e., $X_j \subset X_i$ and $Z = X_i$): we can assume

$$x'_i \leq x'_j < x''_j < x''_i \quad \text{or} \quad x'_i < x'_j < x''_j \leq x''_i$$

by symmetry since $X_i \cap X_j \neq \emptyset$, $X_i - X_j \neq \emptyset$, $X_j - X_i = \emptyset$. Thus,

$Z - X_j = X_i - X_j$, $\mathrm{size}(Z - X_j) = x_i'' - x_i' - (x_j'' - x_j')$, $n_{Z-X_j} = n_{X_i} - n_{X_j}$,

and, by $\rho(X_i) = \rho(Z) \geq \rho(X_j)$, we have

$$\begin{aligned}\rho_{C-X_j}(Z - X_j) &= \frac{x_i'' - x_i' - (x_j'' - x_j')}{n_{Z-X_j}} \\ &= \frac{x_i'' - x_i' - (x_j'' - x_j')}{n_{X_i} - n_{X_j}} = \frac{n_{X_i}\rho(X_i) - n_{X_j}\rho(X_j)}{n_{X_i} - n_{X_j}} \\ &\geq \frac{(n_{X_i} - n_{X_j})\rho(X_j)}{n_{X_i} - n_{X_j}} = \rho(X_j).\end{aligned}$$

Note, $\rho_{C-X_j}(Z - X_j) = \rho(X_j)$ if and only if $\rho(X_i) = \rho(Z) = \rho(X_j)$.

By the argument above, we have the following: If $\rho(Z) > \rho(X_j)$ then $\rho_{C-X_j}(Z - X_j) > \rho(X_j)$. □

Using Lemma 4.1 repeatedly we have the following corollary.

Collorary 4.1 For the cake $C = (0, 1]$, n players $N = \{1, 2, \ldots, n\}$, $\mathcal{C}_N$ with valuation interval $C_i = (\alpha_i, \beta_i]$ of each player $i \in N$ and $\bigcup_{i=1}^{n} C_i = C$, let $H_1 = (h_1', h_1'']$, $H_2 = (h_2', h_2'']$, $\ldots$, $H_L = (h_L', h_L'']$ be all the maximal intervals of minimum density. Then by cutting the cake at both endpoints of each $H_\ell = (h_\ell', h_\ell'']$ we can reduce the original cake-cutting problem into two types of cake-cutting subproblems:
(i) the cake-cutting problem within each maximal interval $H_\ell = (h_\ell', h_\ell'')$ of minimum density (which consists of the players $N(H_\ell)$ and valuation intervals $\mathcal{C}_{N(H_\ell)} = \{C_k \in \mathcal{C}_N \mid C_k \subseteq H_\ell\}$,
(ii) the cake-cutting problem with cake $D = C - \sum_{\ell=1}^{L} H_\ell$, players $P = N - \sum_{\ell=1}^{L} N(H_\ell)$ and valuation intervals

$$\mathcal{D}_P = \{D_i = C_i - \sum_{\ell=1}^{L} H_\ell \mid C_i \in \mathcal{C}_N - \sum_{\ell=1}^{L} \mathcal{C}_{N(H_\ell)}\}.$$

Furthermore, the minimum density of intervals in each cake-cutting problem of type (i) is equal to $\rho_{\min}$.

On the other hand, the minimum density of intervals in the cake-cutting problem of type (ii) is greater than $\rho_{\min}$.

We denote, by Procedure CutCake($P, D, \mathcal{D}$), a method for solving the cake-cutting problem with the cake D which is a single interval, players P and valuation intervals $\mathcal{D}_P$ (where each valuation interval is a single interval in D). Thus, the original the cake-cutting problem with the cake C, players N and valuation intervals $\mathcal{C}_N$ can be solved by setting $P = N$, $D = C$ and $\mathcal{D}_P = \mathcal{C}_N$, and calling Procedure CutCake($N, C, \mathcal{C}$).

The cake-cutting problem of type (i) can be also solved by this procedure. However, we use a slightly different method for solving the cake-cutting problem of type (i) with the cake $H = H_\ell$, players $R = N(H_\ell) = \{i \in N \mid C_i \subseteq H_\ell\}$ and valuation intervals $\mathcal{D}_R = \mathcal{C}_{N(H_\ell)} = \{C_i \in \mathcal{C}_N \mid i \in N(H_\ell)\}$, since H is a maximal interval of minimum density. We call it Procedure CutMaxInterval($R, H, \mathcal{D}_R$).

Similarly, we denote, by Procedure CutMinInterval($S, X, \mathcal{D}_S$), a method for solving the cake-cutting problem of type (i) where the cake is a minimal interval X of minimum density in $H = H_\ell$, players $S = N(X) = \{i \in N(H) \mid C_i \subseteq X\}$ and valuation intervals $\mathcal{D}_S = \mathcal{C}_{N(X)} = \{C_i \in \mathcal{C}_{N(H)} \mid i \in S\}$.

On the other hand, as mentioned before, the cake-cutting problem of type (ii) with the cake $D = C - \sum_{\ell=1}^{L} H_\ell$, players $P = N - \sum_{\ell=1}^{L} N(H_\ell)$ and valuation intervals $\mathcal{D}_P = \{D_i = C_i - \sum_{\ell=1}^{L} H_\ell \mid C_i \in \mathcal{C}_N - \sum_{\ell=1}^{L} \mathcal{C}_{N(H_\ell)}\}$ may be different from the original cake-cutting problem with the cake C, players N and valuation intervals $\mathcal{C}_N$, because the cake $D = C - \sum_{\ell=1}^{L} H_\ell$ may be a set of two or more disjoint intervals and valuation interval $D_i \in \mathcal{D}_P$ may be so. However, we can consider the cake-cutting problem of type (ii) to be of almost the same type as the original cake-cutting problem by contraction of $H_1, H_2, \ldots, H_L$.

By *contraction* of $H_\ell = (h_\ell', h_\ell'']$ which is deleted from the cake C (H_ℓ becomes an empty piece in the remaining cake), we virtually consider two endpoints h_ℓ', h_ℓ'' of H_ℓ to be the same point $h_\ell = h_\ell' = h_\ell''$. Thus, we can consider the cake $D = C - \sum_{\ell=1}^{L} H_\ell$ as a single interval, and also valuation interval $D_i \in \mathcal{D}_P$ as a single interval in D. Note that, before the contraction and the after the contraction, the size of an interval in the cake $D = C - \sum_{\ell=1}^{L} H_\ell$ and valuation intervals $D_i \in \mathcal{D}_P$ remains the same, since the contracted intervals can be considered of size 0.

Thus, by contraction, we can solve the cake-cutting problem of type (ii) with the cake $D = C - \sum_{\ell=1}^{L} H_\ell$, players $P = N - \sum_{\ell=1}^{L} N(H_\ell)$ and valuation intervals $\mathcal{D}_P = \{D_i = C_i - \sum_{\ell=1}^{L} H_\ell \mid C_i \in \mathcal{C}_N - \sum_{\ell=1}^{L} \mathcal{C}_{N(H_\ell)}\}$ in the same way as the original cake-cutting problem. Of course, in a final output, all contracted point h_ℓ should be replaced by an empty cake $H_\ell = (h_\ell', h_\ell'']$, because $H_\ell = (h_\ell', h_\ell'']$ will be allocated to the players in $N(H_\ell)$. We call this operation as *inverse contraction* of H_ℓ.

Thus, we can describe a method for solving the cake-cutting problem of type (ii) with the cake $D = C - \sum_{\ell=1}^{L} H_\ell$, players $P = N - \sum_{\ell=1}^{L} N(H_\ell)$ and valuation intervals $\mathcal{D}_P = \{D_i = C_i - \sum_{\ell=1}^{L} H_\ell \mid C_i \in \mathcal{C}_N - \sum_{\ell=1}^{L} \mathcal{C}_{N(H_\ell)}\}$ as follows:
(a) First perform contraction of all $H_\ell = (h_\ell', h_\ell'')$ ($\ell = 1, 2, \ldots, L$). Let D', $\mathcal{D}_P'$ be obtained from D, $\mathcal{D}_P$ by performing contraction of all $H_\ell = (h_\ell', h_\ell'')$.
(b) Then recursively call CutCake($P, D', \mathcal{D}_P'$).
(c) Finally perform inverse contraction of all $H_\ell = (h_\ell', h_\ell'')$.

We will give detailed descriptions of Procedures CutCake($P, D, \mathcal{D}_P$) (for the cake D, players P and valuation intervals $\mathcal{D}_P$), CutMaxInterval($R, H, \mathcal{D}_R$) (for the cake H of maximal interval of minimum density, players R and valuation intervals $\mathcal{D}_R$), and CutMinInterval($S, X, \mathcal{D}_S$) (for the cake X of minimal interval of minimum density, players S and valuation intervals $\mathcal{D}_S$) later. By using them, we can write our mechanism.

Mechanism 4.1 Our cake-cutting mechanism.

Input: A cake $C = (0, 1]$, n players $N = \{1, 2, \ldots, n\}$, and valuation intervals $\mathcal{C}_N$ with valuation inteval $C_i = (\alpha_i, \beta_i]$ of each player $i \in N$ and $\bigcup_{i=1}^{n} C_i = C$.

Output: Allocation $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ to players N.

Algorithm {
  for each $i \in N$ **do** $a_i = \alpha_i$; $b_i = \beta_i$; $D_i = (a_i, b_i]$;
  $P = N$; $D = C$; $\mathcal{D}_P = \mathcal{C}_N$;
  CutCake($P, D, \mathcal{D}_P$);
}

Procedure 4.1 CutCake($P, D, \mathcal{D}_P$) {
Find all the maximal intervals of minimum density
in the cake-cutting problem with cake D, players P
and valuation intervals $\mathcal{D}_P$;
Let $H_1 = (h'_1, h''_1],\ H_2 = (h'_2, h''_2], \ldots,\ H_L = (h'_L, h''_L]$
be all the maximal intervals of minimum density;
// $H_1, H_2, \ldots, H_L$ are mutually disjoint by Corollary 3.2
for $\ell = 1$ **to** L **do**
cut the cake at both endpoints h'_ℓ, h''_ℓ of H_ℓ;
$R_\ell = \{i \in P \mid D_i \subseteq H_\ell\}$; $\mathcal{D}_{R_\ell} = \{D_i \in \mathcal{D}_P \mid i \in R_\ell\}$;
CutMaxInterval($R_\ell, H_\ell, \mathcal{D}_{R_\ell}$);
for $\ell = 1$ **to** L **do**
$P = P - R_\ell$; $D = D - H_\ell$;
if $P \neq \emptyset$ **then**
$\mathcal{D} = \emptyset$;
for each $D_i \in \mathcal{D}_P$ with $i \in P$ **do**
$D_i = D_i - (H_1 + H_2 + \cdots + H_L)$; $\mathcal{D} = \mathcal{D} + \{D_i\}$;
Perform contraction of all $H_1, H_2, \ldots, H_L$;
Let D, $\mathcal{D}$ become D', $\mathcal{D}'_P$ after contraction;
CutCake($P, D', \mathcal{D}'_P$);
Perform inverse contraction of all $H_1, H_2, \ldots, H_L$;
}

Procedure 4.2 CutMaxInterval($R, H, \mathcal{D}_R$) {
Let $X_1 = (x'_1, x''_1],\ X_2 = (x'_2, x''_2], \ldots,\ X_K = (x'_K, x''_K]$
be all the minimal intervals of density $\rho_{\min}$ in H;
// $X_1, X_2, \ldots, X_K$ are mutually disjoint by Corollary 3.2
for $k = 1$ **to** K **do**
cut the cake at both endpoints x'_k, x''_k of X_k;
$S_k = \{i \in R \mid D_i \subseteq X_k\}$; $\mathcal{D}_{S_k} = \{D_i \in \mathcal{D}_R \mid i \in S_k\}$;
CutMinInterval($S_k, X_k, \mathcal{D}_{S_k}$);
for $k = 1$ **to** K **do**
$R = R - S_k$; $H = H - X_k$;
if $R \neq \emptyset$ **then**
$\mathcal{D} = \emptyset$;
for each $D_i \in \mathcal{D}_R$ with $i \in R$ **do**
$D_i = D_i - (X_1 + X_2 + \cdots + X_K)$; $\mathcal{D} = \mathcal{D} + \{D_i\}$;
Perform contraction of all $X_1, X_2, \ldots, X_K$;
Let H, $\mathcal{D}$ become H', $\mathcal{D}'_R$ after contraction;
CutMaxInterval($R, H', \mathcal{D}'_R$);
Perform inverse contraction of all $X_1, X_2, \ldots, X_K$;
}

To describe Procedure CutMinInterval($S, F, \mathcal{D}_S$), we need some definitions.

Definition 4.1 Let $X = (x', x'']$ be a minimal interval of minimum density $\rho_{\min}$. A minimal interval $Y = (y', y'']$ with respect to density which is properly contained in X (i.e., $Y \subset X$) is called a *separable interval* of X, if size(Y) is less than $(n_Y + 1)\rho_{\min}$, where n_Y is the number of players whose valuation intervals are entirely contained in Y (Fig.6).

If there is no separable interval of $X = (x', x'']$, then X is called *nonseparable*.

Note that, since X is a minimal interval of minimum density, size(Y) for each $Y \subset X$ is always larger than $n_Y \rho_{\min}$ by the definition of a minimal interval of minimum density.

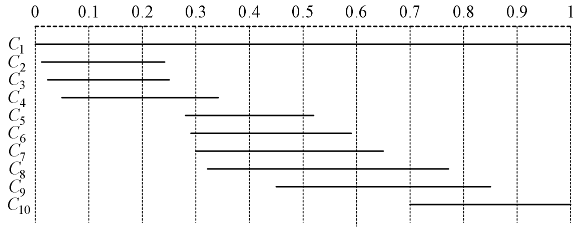

Fig. 6 Players $N = \{1, 2, \ldots, 10\}$ and their valuation intervals $C_1 = (0, 1]$, $C_2 = (0.01, 0.24]$, $C_3 = (0.02, 0.25]$, $C_4 = (0.05, 0.34]$, $C_5 = (0.28, 0.52]$, $C_6 = (0.29, 0.59]$, $C_7 = (0.3, 0.65]$, $C_8 = (0.32, 0.77]$, $C_9 = (0.45, 0.85]$, $C_{10} = (0.7, 1]$. In this case, $X = (0, 1]$ is a minimal interval of minimum density $\rho_{\min} = 0.1$, and there are several separable intervals of $X = (0, 1]$ such as $(0.01, 0.27]$, $(0.01, 0.25]$, $(0.01, 0.59]$, $(0.01, 1]$, $(0.28, 0.65]$, $(0.28, 0.77]$, $(0.28, 0.85]$. The largest left endpoint y^* of the separable intervals in X is 0.28 and the set of separable intervals with the largest left endpoint $y^* = 0.28$ is $\{(0.28, 0.65], (0.28, 0.77], (0.28, 0.85]\}$.

We first consider the case when a minimal interval X of minimum density is nonseparable. This has a nice property.

Lemma 4.2 Let $X = (x', x'']$ be a nonseparable minimal interval of minimum density $\rho_{\min}$. For simplicity, we assume $X = (0, 1]$, $N(X) = N$, $\mathcal{C}_{N(X)} = \mathcal{C}_N$. Let $I_j = ((j-1)\rho_{\min}, j\rho_{\min}]$ for each $j \in N$, and let $\mathcal{I}_N = \{I_1, I_2, \ldots, I_n\}$. Let $G = (\mathcal{C}_N, \mathcal{I}_N, E)$ be a bipartite graph with vertex set $\mathcal{C}_N + \mathcal{I}_N$ and edge set E where $(C_i, I_j) \in E$ if and only if $I_j \subseteq C_i$. Then G has a perfect matching $M = \{(C_i, I_{\pi(i)}) \mid i \in N\} \subseteq E$ (π is a permutation on N).

Lemma 4.2 can be proved by Hall's Theorem [4]: if $C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_k}$ contains ℓ intervals $I_{j_1}, I_{j_2}, \ldots, I_{j_\ell}$ and $\ell \geq i_k$ for all subsets $\{C_{i_1}, C_{i_2}, \ldots, C_{i_k}\} \subseteq \mathcal{C}_N$, then the bipartite graph $G = (\mathcal{C}_N, \mathcal{I}_N, E)$ has a perfect matching.

Lemma 4.2 can be also proved by using the expansion process with unlocking proposed in the paper [1], [6].

Let $M = \{(C_i, I_{\pi(i)}) \mid i \in N\}$ be a perfect matching of the bipartite graph $G = (\mathcal{C}_N, \mathcal{I}_N, E)$ defined in Lemma 4.2. Then we can allocate $A_i = I_{\pi(i)} \subseteq C_i$ of the cake $X = (0, 1]$ to player $i \in N$. Since a perfect matching can be obtained in polynomial time of n, we call this Procedure AllocateInterval($N(X), X, \mathcal{C}_{N(X)}$) and will use in Procedure CutMinInterval($N(X), X, \mathcal{C}_{N(X)}$).

Next we consider the case when the cake $X = (x', x'']$ is a minimal interval of minimum density and has a separable interval.

Let $X = (x', x'']$ be a minimal interval of minimum density $\rho_{\min}$. Let $\mathcal{Y}$ be the set of separable intervals in X and let

$$y^* = \max_{Y=(y',y'']\in\mathcal{Y}} y'. \tag{20}$$

That is, y^* is the largest left endpoint of the separable intervals in X. Let

$$\mathcal{Y}_{y^*} = \{Y = (y',\ y''] \in \mathcal{Y} \mid y' = y^*\}. \tag{21}$$

That is, $\mathcal{Y}_{y^*}$ is the set of separable intervals in X whose left endpoints are y^* (Fig.6).

For each interval $Y = (y',\ y'']$ of X, let

$$\gamma(Y) = \text{size}(Y) - n_Y\,\rho_{\min}. \tag{22}$$

If $Y = (y',\ y'']$ is a separable interval of the minimal interval X

of minimum density, then Y is a minimal interval with respect to density and

$$n_Y\,\rho_{\min} < \mathrm{size}(Y) < (n_Y+1)\,\rho_{\min} \tag{23}$$

and we have

$$0 < \gamma(Y) < \rho_{\min}. \tag{24}$$

Let γ^* be the minimum $\gamma(Y)$ among the separable intervals $Y = (y^*,\ y'']$ with the largest left endpoint y^*, i.e.,

$$\gamma^* = \min_{Y\in\mathcal{Y}_{y^*}} \gamma(Y). \tag{25}$$

Clearly, by Eqs. (24), (25),

$$0 < \gamma^* < \rho_{\min}. \tag{26}$$

Let Z_{y^*} be the set of right endpoints of the separable intervals whose left endpoints are y^*, i.e.,

$$Z_{y^*} = \{y'' \mid Y = (y^*,\ y''] \in \mathcal{Y}_{y^*}\}. \tag{27}$$

Let $\mathcal{Y}^{\gamma^*}_{y^*}$ be the set of separable intervals $Y = (y^*,\ y'']$ in $\mathcal{Y}_{y^*}$ with $\gamma(Y) = \gamma^*$, i.e.,

$$\mathcal{Y}^{\gamma^*}_{y^*} = \{Y = (y^*,\ y''] \in \mathcal{Y}_{y^*} \mid \gamma(Y) = \gamma^*\}. \tag{28}$$

Let $Z^{\gamma^*}_{y^*}$ be the set of right endpoints of the separable intervals in $\mathcal{Y}^{\gamma^*}_{y^*}$ and J be the cardinality of $Z^{\gamma^*}_{y^*}$, i.e.,

$$Z^{\gamma^*}_{y^*} = \{y'' \mid Y = (y^*,\ y''] \in \mathcal{Y}^{\gamma^*}_{y^*}\}, \quad J = |Z^{\gamma^*}_{y^*}|. \tag{29}$$

Let

$$Z^{\gamma^*}_{y^*} = \{z^*_1, z^*_2, \ldots, z^*_J\}, \quad z^*_1 < z^*_2 < \cdots < z^*_J. \tag{30}$$

For simplicity, we consider

$$z^*_0 = y^* + \gamma^*, \quad Y_0 = (y^*,\ z^*_0]. \tag{31}$$

Furthermore, if $z^*_J < x''$ then we consider $z^*_{J+1} = x''$ and $Y_{J+1} = (y^*,\ z^*_{J+1}] = (y^*,\ x'']$. Let, for each $k = 1, 2, \ldots, J$,

$$Y_k = (y^*, z^*_k]. \tag{32}$$

In the example of Fig.6, $Z_{y^*} = \{0.65, 0.77, 0.85\}$, $\mathcal{Y}^{\gamma^*}_{y^*} = \{(0.28, 0.65],\ (0.28, 0.85]\}$, $Z^{\gamma^*}_{y^*} = \{0.65, 0.85\}$, $J = 2$, $z^*_1 = 0.65 < z^*_2 = 0.85$.

Then we have the following lemma and corollary.

Lemma 4.3 Let $X = (x', x'']$ be a minimal interval of minimum density $\rho_{\min}$ in the cake C. Let $Y = (y^*,\ z]$ be an interval of X such that $z = \beta_i$ for some $C_i = (\alpha_i,\ \beta_i] \in \mathcal{C}_N$. Then $\gamma(Y) = \gamma^*$ for $z \in Z^{\gamma^*}_{y^*}$ and $\gamma(Y) > \gamma^*$ for $z \notin Z^{\gamma^*}_{y^*}$, i.e.,

$$\gamma(Y) \begin{cases} = & \gamma^* \quad (z \in Z^{\gamma^*}_{y^*}) \\ > & \gamma^* \quad (z \notin Z^{\gamma^*}_{y^*}). \end{cases} \tag{33}$$

Proof: By the definition of $\gamma(Y)$ of $Y = (y^*,\ z]$ in Eq.(22),

$$\gamma(Y) = \mathrm{size}(Y) - n_Y\rho_{\min} = z - y^* - n_Y\rho_{\min}.$$

It is clear that if $z \in Z^{\gamma^*}_{y^*}$ then $\gamma(Y) = \gamma^*$ by the definitions of $\mathcal{Y}^{\gamma^*}_{y^*}$ and $Z^{\gamma^*}_{y^*}$. Therefore, we can assume $z \notin Z^{\gamma^*}_{y^*}$ below.

We first consider the case when $Y = (y^*,\ z]$ is not a separable interval. Thus, $\mathrm{size}(Y) \geq (n_Y+1)\,\rho_{\min}$ and by Eq. (26),

$$\gamma(Y) = \mathrm{size}(Y) - n_Y\rho_{\min} \geq \rho_{\min} > \gamma^*.$$

We next consider the case when $Y = (y^*,\ z]$ is a separable interval. Thus, $n_Y\rho_{\min} < \mathrm{size}(Y) < (n_Y+1)\rho_{\min}$. By the definition of $Z^{\gamma^*}_{y^*} = \{y'' \mid Y = (y^*,\ y''] \in \mathcal{Y}^{\gamma^*}_{y^*}\}$ and Eq. (25), we have

$$\gamma(Y) = \mathrm{size}(Y) - n_Y\rho_{\min} > \gamma^*$$

since $z \notin Z^{\gamma^*}_{y^*}$. □

Collorary 4.2 Let $X = (x', x'']$ be a minimal interval of minimum density $\rho_{\min}$ in the cake C. Let $Y = (y^*,\ z]$ be an interval of X such that $z^*_k < z < z^*_{k+1}$ for some $Y_k = (y^*,\ z^*_k] \in \{Y_j = (y^*,\ z^*_j] \mid j = 0, 1, \ldots, J\}$ and that z is a right endpoint of some valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_N$. Then $z - z^*_k = \mathrm{size}(Y) - \mathrm{size}(Y_k) = (\gamma(Y) - \gamma(Y_k)) + \rho_{\min}(n_Y - n_{Y_k}) > \rho_{\min}(n_Y - n_{Y_k})$ by $\gamma(Y_k) = \gamma^*$.

Let $X = (x', x'']$ be a minimal interval of minimum density $\rho_{\min}$ in the cake C and let $S = N(X)$. For each $j = 1, 2, \ldots, J$, let

$$\begin{aligned} Z_j &= (z^*_{j-1},\ z^*_j], \\ \mathcal{D}_{S(Z_j)} &= \{D_i \in \mathcal{D}_S \mid D_i \subseteq (y^*,\ z^*_j], D_i \nsubseteq (y^*,\ z^*_{j-1}]\}, \\ S(Z_j) &= \{i \in S \mid D_i \in \mathcal{D}_{S(Z_j)}\}. \end{aligned} \tag{34}$$

Note that

$$\mathcal{D}_{S(Z_j)} = \{D_i \in \mathcal{D}_S \mid D_i \subseteq (y^*,\ z^*_j]\} - \{D_i \in \mathcal{D}_S \mid D_i \subseteq (y^*,\ z^*_{j-1}]\}.$$

Furthermore, for each $j = 1, 2, \ldots, J$, let

$$\mathcal{D}'_{S(Z_j)} = \{D'_i = D_i - (y^*,\ z^*_{j-1}] \mid D_i \in \mathcal{D}_{S(Z_j)}\}. \tag{35}$$

We also consider the remaining cake-cutting problem after deletion of the interval $(z^*_1,\ z^*_J]$. Note that $(z^*_1,\ z^*_J] = Z_1 + Z_2 + \cdots + Z_J$ and $S(Z_1) + S(Z_2) + \cdots + S(Z_J) = S((z^*_1,\ z^*_J])$. Let

$$S' = S - S((z^*_1,\ z^*_J]), \tag{36}$$

$$D' = D - (z^*_1,\ z^*_J], \tag{37}$$

$$\mathcal{D}_{S'} = \{D'_i \mid D'_i = D_i - (z^*_1,\ z^*_J], D_i \in \mathcal{D}_S, D_i \nsubseteq (y^*,\ z^*_J]\}. \tag{38}$$

Similarly, let D'' and $\mathcal{D}'_{S'}$ be obtained from D' and $\mathcal{D}_{S'}$ by performing contraction of all the $Z_1, Z_2, \ldots, Z_J$.

Thus, we reduce the remaining cake-cutting problem with the cake D', players S' and valuation intervals $\mathcal{D}_{S'}$ by performing contraction of all the $Z_1, Z_2, \ldots, Z_J$ to the cake-cutting problem with the cake D'', players S' and valuation intervals $\mathcal{D}'_{S'}$.

Then the following lemmas hold.

Lemma 4.4 Each interval $Z_j = (z^*_{j-1},\ z^*_j]$ $(j = 1, 2, \ldots, J)$ is a minimal interval with minimum density $\rho'_{\min} = \rho_{\min}$ for the cake-cutting problem with the cake Z_j, players $S(Z_j) = \{i \in S \mid D_i \in \mathcal{D}_{S(Z_j)}\}$, valuation intervals $\mathcal{D}'_{S(Z_j)}$ in Eq.(34) and the density ρ'.

Similarly, if we proform contraction of all the $Z_1, Z_2, \ldots, Z_J$, then the interval D'' obtained from $D' = D - (z^*_1, z^*_J]$ in Eq.(37) is also a minimal interval with minimum density $\rho'_{\min} = \rho_{\min}$ for the cake-cutting problem with the cake D'', players S' in Eq.(36), valuation intervals $\mathcal{D}'_{S'}$ obtained from $\mathcal{D}_{S'}$ in (38) and the density ρ'.

Proof: We first show that each $Z_j = (z^*_{j-1}, z^*_j]$ $(j = 1, 2, \ldots, J)$ is a minimal interval with minimum density $\rho_{\min}$.

It is clear that $\rho'(Z_j) = \rho_{\min}$, since

$$\begin{aligned}
\mathrm{size}((y^*, z^*_j]) &= \rho_{\min} n_{(y^*,\, z^*_j]} + \gamma^*,\\
\mathrm{size}((y^*, z^*_{j-1}]) &= \rho_{\min} n_{(y^*,\, z^*_{j-1}]} + \gamma^*,\\
n_{Z_j} &= n_{(y^*,\, z^*_j]} - n_{(y^*,\, z^*_{j-1}]},\\
\mathrm{size}(Z_j) &= \mathrm{size}((y^*, z^*_j]) - \mathrm{size}((y^*, z^*_{j-1}])\\
&= \rho_{\min}(n_{(y^*,\, z^*_j]} - n_{(y^*,\, z^*_{j-1}]}) = \rho_{\min} n_{Z_j}.
\end{aligned}$$

Let $Z = (z', z'']$ be a proper subinterval of Z_j (i.e., $Z \subset Z_j$) such that z' is z^*_{j-1} or a left endpoint of some valuation interval and that z'' is a right endpoint of some valuation interval in $\mathcal{D}'_{S(Z_j)}$. If $z' \neq z^*_{j-1}$ then $\rho'(Z) = \rho(Z) > \rho_{\min}$, since $Z \subset X$ $(Z \neq X)$ and X is a minimal interval with minimum density $\rho_{\min}$. Thus, we assume $z' = z^*_{j-1} < z'' < z^*_j$. Now consider the intervals $Y'_j = (y^*, z'']$ and $Y_{j-1} = (y^*, z^*_{j-1}]$. Then $n_Z = n_{Y'_j} - n_{Y_{j-1}}$. By Corollary 4.2, we have $\mathrm{size}(Z) = z'' - z^*_{j-1} > \rho_{\min}(n_{Y'_j} - n_{Y_{j-1}}) = \rho_{\min}\, n_Z$ and $\rho'(Z) = \frac{\mathrm{size}(Z)}{n_Z} > \rho_{\min}$. Thus, $Z_j = (z^*_{j-1}, z^*_j]$ is a minimal interval with minimum density $\rho'_{\min} = \rho_{\min}$.

Next we show that D'' is also a minimal interval with minimum density $\rho_{\min}$. $\rho'(D'') = \rho_{\min}$ can be obtained in an almost the same argument above.

Let $Z = (z', z'']$ be a proper subinterval in D'' (i.e., $Z \subset D''$). Thus, $z' < z^*_1$ or $z'' > z^*_J$. We will show that $\rho'(Z) > \rho_{\min}$ by dividing into two subcases: (i) the case of $z' < z^*_1$ and (ii) the case of $z^*_1 \leq z'$ and $z'' > z^*_J$.

We only discuss the case of $z' \leq y^* < z^*_1 < z'' \leq z^*_J$ in (i) (the other cases in (i) can be discussed similarly). Since the contraction of $Z_1 + Z_2 + \cdots + Z_J = (z^*_1, z^*_J]$ is performed, we can consider

$$z'' = z^*_J, \quad \mathrm{size}(Z) = z'' - z' = z^*_J - z', \quad \rho_{\min} n_{(y^*,\, z^*_J]} = z^*_J - z^*_1.$$

Thus, after the contraction is performed, Z becomes $Z' = Z - (z^*_1, z^*_J]$ and $n_{Z'} = n_Z - n_{(y^*,\, z^*_J]}$ and $\mathrm{size}(Z') = z^*_1 - z'$, we have

$$\rho'(Z') = \frac{\mathrm{size}(Z')}{n_{Z'}} = \frac{z^*_1 - z'}{n_{Z'}} > \rho_{\min}$$

by

$$\rho(Z) = \frac{\mathrm{size}(Z)}{n_Z} = \frac{z^*_J - z^*_1 + z^*_1 - z'}{n_{Z'} + n_{(y^*,\, z^*_J]}} = \frac{\rho_{\min} n_{(y^*,\, z^*_J]} + z^*_1 - z'}{n_{Z'} + n_{(y^*,\, z^*_J]}} > \rho_{\min}.$$

We only discuss the case of $z^*_1 \leq z' < z^*_J < z''$ in (ii) (the other cases in (ii) can be discussed similarly). By corollary 4.2 for $k = J$, $Y_J = (y^*, z^*_J]$ and $Y = (y^*, z'']$, we have $Z' = (z^*_J, z'']$,

$$\mathrm{size}(Z') = z'' - z^*_J = \mathrm{size}(Y) - \mathrm{size}(Y_J) > \rho_{\min}(n_Y - n_{Y_J})$$

and $n_{Z'} \leq n_Y - n_{Y_J}$. Thus, $\rho'(Z') = \frac{\mathrm{size}(Z')}{n_{Z'}} \geq \frac{\mathrm{size}(Z')}{n_Y - n_{Y_J}} > \rho_{\min}$. □

Based on Lemma 4.2 and Lemma 4.4, we can write Procedure CutMinInterval($S, X, \mathcal{D}_S$) as follows.

Procedure 4.3 CutMinInterval($S, X, \mathcal{D}_S$) {
if $X = (x', x'']$ is nonseparable **then**
  AllocateInterval($S, X, \mathcal{D}_S$);
  // this finds an allocation of X to players S by Lemma 4.2
else // there is a separable interval in X
  Find y^*, γ^*, $\mathcal{Y}^{\gamma^*}_{y^*}$, and $Z^{\gamma^*}_{y^*}$ defined by
   Eqs. (20), (25), (28), and (29), respectively;
  Let $Z^{\gamma^*}_{y^*} = \{z^*_1, z^*_2, \ldots, z^*_J\}$ and assume
   $z^*_0 = y^* + \gamma^* < z^*_1 < z^*_2 < \cdots < z^*_J \leq z_{J+1} = x''$;
  for $j = 1$ **to** J **do**
   $Z_j = (z^*_{j-1}, z^*_j]$;
   cut the cake at both endpoints z^*_{j-1}, z^*_j of $Z_j = (z^*_{j-1}, z^*_j]$;
   let $\mathcal{D}_{S(Z_j)}$ and $\mathcal{D}'_{S(Z_j)}$ be defined in Eqs. (34) and (35);
   $S(Z_j) = \{i \in S \mid D_i \in \mathcal{D}_{S(Z_j)}\}$;
   CutMinInterval($S(Z_j), Z_j, \mathcal{D}'_{S(Z_j)}$);
  $S' = S - S((z^*_1, z^*_J])$; $D' = D - (z^*_1, z^*_J]$;
if $S' \neq \emptyset$ **then**
  $\mathcal{D}_{S'} = \emptyset$;
  for each $D_i \in \mathcal{D}_S$ with $i \in S'$ **do**
   $D'_i = D_i - (z^*_1, z^*_J]$; $\mathcal{D}_{S'} = \mathcal{D}_{S'} + \{D'_i\}$;
  Perform contraction of all $Z_1, Z_2, \ldots, Z_J$;
  Let D', $\mathcal{D}_{S'}$ become D'', $\mathcal{D}'_{S'}$ after contraction;
  CutMinInterval($S', D'', \mathcal{D}'_{S'}$);
  Perform inverse contraction of all $Z_1, Z_2, \ldots, Z_J$;
}

Based on Corollary 4.1, Lemma 4.2 and Lemma 4.4, we can show that Mechanism 4.1 correctly finds, in $O(n^3)$ time, an allocation $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ of the cake C to n players N with $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ such that $A_i = A_{i_1} + A_{i_2} + \cdots + A_{i_{k_i}} \subseteq C_i$ for each player $i \in N$. Envy-freeness and truthfulness of Mechanism 4.1 can be obtained by induction on the number of calls on Procedure CutCake($P, D, \mathcal{D}_P$) by Corollary 4.1. Truthfulness of Mechanism 4.1 can be also shown in a similar way as in papers [2], [6]. We can show that the number of cuts is at most $2(n-1)$ in a similar way as in paper [6].

Thus, we have the following theorem.

Theorem 4.1 Mechanism 4.1 is envy-free and truthful, and the number of cuts made by Mechanism 4.1 on the cake is at most $2(n-1)$.

References

[1] R. Alijani, M. Farhadi, M. Ghodsi, M. Seddighin, and A.S. Tajik, Envy-free mechanisms with minimum number of cuts, *Proc. of 31st AAAI Conference on Artificial Intelligence*, pp. 312–318, 2017.
[2] Y. Chen, J. K. Lai, D. C. Parkes, and A. D. Procaccia, Truth, justice, and cake cutting, *Games and Economic Behavior*, Vol.77, pp. 284–297, 2013.
[3] X. Deng, Q. Qi, and A. Saberi, Algorithmic solutions for envy-free cake cutting, *Operations Research*, Vol.60, pp. 1461–1476, 2012.
[4] P. Hall, On representations of subsets, *Journal of London Mathematical Society*, Vol.10, pp.26–30, 1934.
[5] A.D. Procaccia, Cake cutting: not just child's play, *Communications of ACM*, Vol.56, pp.78–87, 2013.
[6] M. Seddighin, M. Farhadi, M. Ghodsi, R. Alijani, and A.S. Tajik, Expand the shares together: envy-free mechanisms with a small number of cuts, *Algorithmica*, Vol. 81, pp. 1728–1755, 2019.
[7] H. Steinhaus, The problem of fair division, *Econometrica*, Vol. 16, pp. 101–104, 1948.