On the stretch factor of Delaunary triangulations of points in convex position

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概要：Let S be a set of n points in the plane, and let $DT(S)$ be the planar graph of the Delaunay triangulation of S. For a pair of points $a, b \in S$, denote by $|ab|$ the Euclidean distance between a and b. Denote by $DT(a, b)$ the shortest path in $DT(S)$ between a and b, and let $|DT(a, b)|$ be the total length of $DT(a, b)$. Dobkin *et al.* were the first to show that $DT(S)$ can be used to approximate the complete graph of S in the sense that the stretch factor $\frac{|DT(a,b)|}{|ab|}$ is bounded above by $((1+\sqrt{5})/2)\pi \approx 5.08$. Recently, Xia improved this factor to 1.998. In this paper, we prove that if the points of S are in convex position, then the stretch factor of $DT(S)$ is less than 1.82. A set of points is said to be in *convex position*, if all points form the vertices of a convex polygon.

凸位置にある点集合のドローネ三角形分割の stretch factor について

1. Introduction

Let S be a set of n points in the plane, and let $G(S)$ be such a graph that each vertex corresponds to a point in S and the weight of an edge is the Euclidean distance between its two endpoints. For a pair of points u, v in the plane, denote by uv the line segment connecting u and v, and $|uv|$ the Euclidean distance between u and v. For a pair of points $a, b \in S$, denote by $G(a, b)$ the shortest path in $G(S)$ between a and b, and let $|G(a, b)|$ be the total length of path $G(a, b)$. The graph $G(S)$ is said to approximate the complete graph of S if $\frac{|G(a,b)|}{|ab|}$, called the *stretch factor* of $G(S)$, is bounded above by a constant, independent of S and n. It is then desirable to identify classes of graphs that approximate complete graphs well and have only $O(n)$ edges (in comparison with $O(n^2)$ edges of complete graphs), as these graphs have potential applications in geometric network design problems [3], [7], [8].

Denote by $DT(S)$ the planar graph of the Delaunay triangulation of S. Dobkin *et al.* [5] were the first to give a stretch factor ($((1+\sqrt{5})/2)\pi \approx 5.08$) of Delaunay triangulations to complete graphs, which was later improved to $2\pi/(3\cos(\pi/6)) \approx 2.42$ by Keil and Gutwin [9]. Recently, this factor has been improved to 1.998 by Xia [11]. On the other hand, Xia and Zhang [12] gave a lower bound 1.5932 on the stretch factor of $DT(S)$. Determining the worse possible stretch factor of the Delaunay triangulation has been a long standing open problem in computational geometry.

Cui *et al.* [4] have also studied the stretch factor of $DT(S)$ for the points in convex position. A set of points is said to be in *convex position*, if all points form the vertices of a convex polygon. The currently best known stretch factor in this special situation is 1.88, due to a work of Amani *et al.* on the stretch factor of planar graphs [1]. Notice that the planar graph studied by Amani *et al.* is *not* the Delaunay triangulation of the given point set. (Dumitrescu and Ghosh [6] have also shown that every spanning graph of the vertices of a regular 23-gon has stretch factor at least 1.4308.) Although studying the convex case may not lead to improve upper bounds for the general case, it shows a large possibility in obtaining

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a better upper bound on the stretch factor of $DT(S)$ and may give some intelligent hints for the general case.

In this paper, we prove that $\frac{|DT(a,b)|}{|ab|} < 1.82$ holds for a set S of points in convex position. Our result is obtained by showing that there exists a convex chain between a and b in $DT(S)$ such that it is either contained in a semidisk of diameter ab, or enclosed by segment ab and a simple (convex) chain that consists of a circular arc and one or two line segments. The total length of the simple chain is less than $1.82|ab|$.

2. Preliminary

Without loss of generality, assume that no four points of S are cocircular in the plane. The *Voronoi diagram* for S, denoted by $Vor(S)$, is a partition of the plane into regions, each containing exactly one point in S, such that for each point $p \in S$, every point within its corresponding region, denoted by $Vor(p)$, is closer to p than to any other of S. The boundaries of these Voronoi regions form a planar graph. The *Delaunay triangulation* of S, denoted by $DT(S)$, is the straight-line dual of the Voronoi diagram for S; that is, we connect a pair of points in S if and only if they share a Voronoi boundary. Since $DT(S)$ is a planar graph, it has $O(n)$ edges.

The *bisector* of two points u and v, denoted by $B_{u,v}$, is the perpendicular line through the middle point of segment uv. For a pair of points $a, b \in S$, denote by $DT(a,b)$ the shortest path in $DT(S)$ between a and b, and $|DT(a,b)|$ the total length of path $DT(a,b)$.

We now briefly review an important idea of Dobkin *et al.*'s work [5]. Let $a = a_0, a_1, \ldots, a_m = b$ be the sequence of the points of S, whose Voronoi regions intersect segment ab (Fig. 1). If a Voronoi edge happens to be on segment ab, either of the points defining that Voronoi edge can be chosen as the one on the direct path from a to b. The path obtained in this way is called the *direct path from a to b* [5].

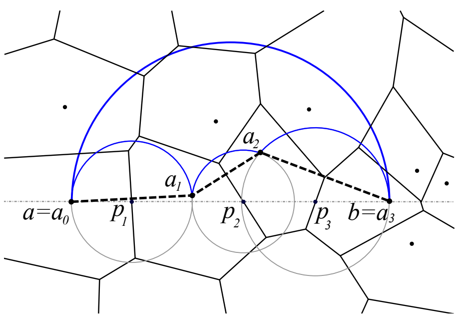

図 1 A one-sided, direct path from a to b.

The direct path from a to b is said to be *one-sided* if all points of the path are to the same side of the line through a and b. See Fig. 1. If the direct path from a to b is one-sided, then it has length at most $\pi|ab|/2$.

Lemma1 (Dobkin et al. [5]) If the direct path from a to b is one-sided, then it has length at most $\pi|ab|/2$.

Let p_i be the intersection point of ab with the Voronoi edge between $Vor(a_{i-1})$ and $Vor(a_i)$, for $1 \le i \le m$. It follows from the definition of the Voronoi diagram that p_i is the center of a circle that passes through a_{i-1} and a_i but contains no points of S in its *interior*, see Fig. 1. All points of the direct path from a to b are thus contained in the circle of diameter ab, no matter whether the path is one-sided or not.

3. The main result

Assume that the set S of given points is in convex position. For a point p in the plane, denote the coordinates of p by $p(x)$ and $p(y)$, respectively. Assume also that the direct path from a to b is not one-sided; otherwise, $\frac{|DT(a,b)|}{|ab|} \le \pi/2$ (≈ 1.58). Without loss of generality, assume that both a and b lie on the x-axis (i.e., $a(y) = b(y) = 0$), with $a(x) < b(x)$.

We say segment ab *properly* intersects a Delaunay triangle if it intersects the interior of the triangle (i.e., segment ab does not intersect only at a vertex of the triangle). Clearly, if a Delaunay triangle does not properly intersect ab, then at least one of its vertices (and two edges incident to that vertex) can be deleted from $DT(S)$, without affecting the value of $\frac{|DT(a,b)|}{|ab|}$. We assume below that ab properly intersects all triangles of $DT(S)$.

Denote by $SA[a,b]$ ($SB[a,b]$) the portion of the *convex chain* of S above (below) the line through a and b. The union of $SA[a,b]$ and $SB[a,b]$ is then the convex hull of the points of S. For a point p on $SA[a,b]$, denote by $SA[a,p]$ ($SA[p,b]$) the portion of $SA[a,b]$ from a to p (from p to b). Analogously, for a point q on $SB[a,b]$, denote by $SB[a,q]$ ($SB[q,b]$) the portion of $SB[a,b]$ from a to q (from q to b). Also, we denote by $SA(a,b)$ ($SB(a,b)$) the open chain of $SA[a,b]$ ($SB[a,b]$).

Denote by C the circle of diameter ab. The main idea of our proof is the following. If the direct path from a to b intersects segment ab an even number times, then $\frac{|DT(a,b)|}{|ab|} \le \pi/2$ (Lemma 3). For the *difficult* case that the direct path from a to b intersects ab an odd number times, we first show that either $SA[a,b]$ or $SB[a,b]$ is contained in the union of two semidisks; one is of diameter ab and the other is of diameter bi, where i is a point on

C. See Fig. 3. Denote by H the semidisk of diameter bi. To bound the length of $DT(a,b)$, we may further draw a tangent from point i to the convex chain of S contained in H. As a final result, either $SA[a,b]$ or $SB[a,b]$ is completely contained in the region bounded by segment ab and a simple (convex) chain that consist of a circular arc of diameter bi and one or two line segments.

Lemma2 Suppose that the first and last segments of the direct path from a to b are below and above the line through a and b, respectively. Then, there exists an angle $\alpha > 0$ such that $|DT(a,b)|/|ab| \le \sin(\alpha) + \pi\cos(\alpha)/2$, $\pi/4 \le \alpha < \pi/2$, or $|DT(a,b)|/|ab| \le \max\{(\sin(\alpha) + \cos(\alpha)(\cos(\alpha)+\alpha)), (\sin(\alpha)+\cos(\alpha)(\sin(2\alpha)+\pi/2-2\alpha))\}$, $\alpha < \pi/4$.

Proof. First, since it is assumed that a and b, with $a(x) < b(x)$, lie on the x-axis, the x-coordinates of points of the direct path from a to b are monotonically increasing (see Lemma 1 of [5]). Assume also that neither $SA[a,b]$ nor $SB[a,b]$ is not completely contained in C; otherwise, $\frac{|DT(a,b)|}{|ab|} \le \pi/2$ and we are done.

Denote by ac and bd the first and last segments of the direct path from a to b respectively, as viewed from a. From the lemma assumption, both ac and bd are contained in C. Extend segments ac and bd until they touch the boundary of C, say, at points c' and d' respectively, see Fig. 2. Since $\angle ac'b = \angle ad'b = \pi/2$, either $\angle c'ad'$ or $\angle c'bd'$ is at least $\pi/2$. In the following, we assume that $\angle c'bd' \ge \pi/2$, or equivalently, $\angle c'bd \ge \pi/2$.

Let i be the intersection point of C with $B_{b,d}$, which is vertically below segment ab. Since $B_{b,d}$ is perpendicular to bd, and since $\angle bc'a = \pi/2$ and $\angle c'bd \ge \pi/2$, $B_{b,d}$ intersects segment ac'. Thus, segment bi intersects ac', and point i is outside of the convex hull of S, see Fig. 2.

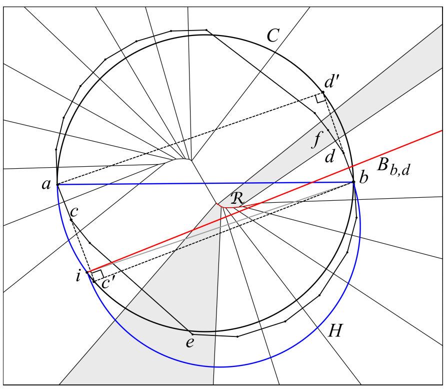

図 2 Illustration of the proof of Lemma 2.

Let e be the first point of $SB[a,b]$ outside of C, as viewed from a, and f the last point of $SA[a,b]$ such that $Vor(e)$ and $Vor(f)$ are adjacent. See Figs. 2 and 3. Then, all the points of $SB[e,b]$ are vertically below segment bi.

Denote by $\mathcal{R}$ the chain formed by all bounded (or finite) edges of regions $Vor(g)$, $g \in SB[e,b)$. See Figs. 2 and 3 for some examples, where $\mathcal{R}$ is shown in dotted and solid line. Let us consider the first subchain of $\mathcal{R}$, which consists of the edges with positive slope, starting from its endpoint on $B_{b,d}$. From the convexity of Voronoi regions, the slopes of edges of that subchain are monotonically decreasing, as viewed from b. Also, $Vor(d)$ is vertically above $B_{b,d}$. Thus, $B_{b,d}$ properly intersects the Voronoi region of the point, which is immediately before b on $SB[a,b]$. (Figs. 2 and 3). Analogously, for two adjacent regions $Vor(p)$ and $Vor(q)$, $p \in SA[f,b)$ and $q \in SB[e,b)$, the bisector $B_{p,q}$ properly intersects the Voronoi region whose defining point is immediately after q on $SB[a,b]$. Since the slopes of edges of the considered subchain are monotonically decreasing, the intersection points of these bisectors $B_{p,q}$ with $B_{b,d}$ are vertically below that subchain of $\mathcal{R}$. In other words, the considered subchain of $\mathcal{R}$ is vertically above $B_{b,d}$ as well as bi.

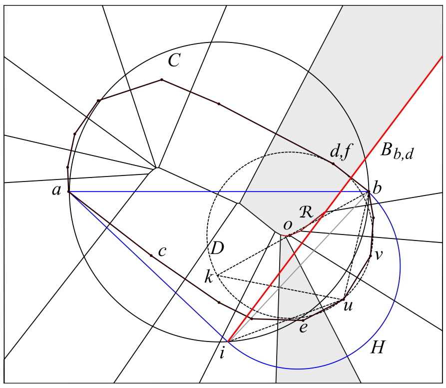

図 3 A situation in which $\angle abi \ge \pi/4$.

Note that $\mathcal{R}$ may have the other (second) subchain that consists of the edges with negative slope. Clearly, this subchain is vertically above bi, too. We now claim that $\mathcal{R}$ has only one subchain consisting of the edges with positive slope and possibly the other subchain consisting of the edges with negative slope. Since the positive slopes of edges on the first subchain of $\mathcal{R}$ are monotonically decreasing, and since each point of $SA[f,b)$ is connected by one or multiple edges of $DT(S)$ to one or several (consecutive) points of $SB[e,b)$, both the x- and y-coordinates of the first subchain of $\mathcal{R}$ are monotonically decreasing, starting from the endpoint of that subchain on $B_{b,d}$. It then follows from the convexity of Voronoi regions that the unbounded edges between $Vor(u)$ and $Vor(v)$,

$u, v \in SB[e, b]$, whose finite vertices are on the first subchain of $\mathcal{R}$, have to monotonically increase their cut angles with segment bi, as viewed from b. (A cut angle of segment bi with the unbounded edge between $Vor(u)$ and $Vor(v)$ is defined as the angle formed by point b, the intersection point of two segments and the infinite point along the unbounded edge.) If $\mathcal{R}$ has the second subchain consisting of edges of negative slope, then the x-coordinates (y-coordinates) of the second subchain are monotonically decreasing (increasing), starting from the common point of the two subchains. Also, the unbounded edges between $Vor(u')$ and $Vor(v')$, $u', v' \in SB[e, b)$, whose finite vertices are on the second subchain of $\mathcal{R}$, excluding the common point of the subchains, increase their cut angles with bi monotonically. Moreover, their cut angles (with bi) have to be larger than $\pi/2$, because of a sign change of slopes of $\mathcal{R}$'s edges. Observe that since all the points of $SB[e, b]$ are vertically below bi, the unbounded Voronoi edges formed by them have the monotonically increasing cut angles with bi, as viewed from b. Hence, the rest edges of $\mathcal{R}$ are all of negative slope, and our claim is proved. (Note that there may exist a region $Vor(w)$, $w \in SB[e, b)$, such that it has some Voronoi edges of positive slope and the others of negative slope.) Therefore, any finite vertex of the Voronoi region whose defining point belongs to $SB[e, b)$ is vertically above $B_{b,d}$.

Let u and v be two adjacent points on $SB[g, b]$ such that u is immediately before v on $SB[g, b]$. Then, $u \neq b$. Since it is assumed that every triangle of $DT(S)$ properly intersects ab, the Delaunay triangle with an edge uv has its third vertex on $SA[f, b)$. We claim that $\angle bui > \pi/2$. Denote by D the circumcircle of the Delaunay triangle with edge uv, centered at a Voronoi vertex o (Fig. 3). Since point o is vertically above $B_{b,d}$, it is vertically above bi, too. By definition of $DT(S)$, point b is on or outside of D. Let k be the intersection point of D with the line through b and o such that k is not contained in segment ob, see Fig. 3. Since no point of S is contained in the interior of D, point k is contained in the convex hull of S. Moreover, since i is outside of the convex hull of S and o is vertically above bi and below ab, point k is contained in the triangle with three verices a, b and i. Hence, segment bi intersects uk. Therefore, $\angle bui > \angle buk \geq \pi/2$.

It follows from our claim that $SB[g, b]$ is contained in the circle of diameter bi. Denote by H the semicircle of diameter bi, which is vertically below bi. From the convexity of S and the definition of points i and g, $SB[a, b]$ is completely contained in the region bounded by ab, ai and H, see Figs. 2 and 3.

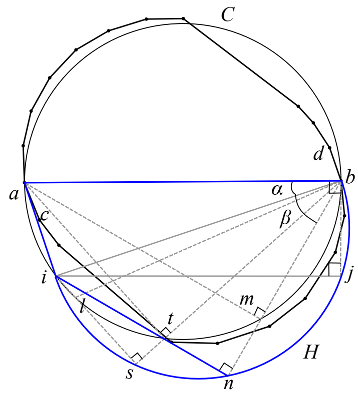

図 4 Illustration of the inequality $\beta \geq 2\alpha$.

Let us now describe a method to bound the length of $DT(a, b)$. Let $\alpha = \angle abi$. If $\alpha \geq \pi/4$, then $|ai| = \sin(\alpha)|ab|$ and $|bi| = \cos(\alpha)|ab|$. A simple argument (as in [10]) shows that the length of $SB[a, b]$, denoted by $|SB[a, b]|$, is less than $(\sin(\alpha) + \pi\cos(\alpha)/2)|ab|$, see Fig. 3. Thus, $|DT(a, b)| \leq |SB[a, b]| \leq (\sin(\alpha) + \pi\cos(\alpha)/2)|ab|$.

Assume below that $\alpha < \pi/4$. Let j be the intersection point of H with the horizontal line through point b, see Fig. 4. So, ij is parallel to ab. Since $\angle bji = \pi/2$, we have $\angle abj = \pi/2$. Thus, the line through b and j is tangent to C. If the whole chain $SB[a, b]$ is vertically above the line through i and j, then $SB[a, b]$ is contained in the convex region bounded by ba, ai, ij and the circular arc $\widehat{jb}$ of diameter bi, with the inscribed angle α. Since $|ij| = \cos(\alpha)|bi| = \cos^2(\alpha)|ab|$ and $\widehat{jb} = \cos(\alpha)\alpha|ab|$, we have $|DT(a, b)| \leq |SB[a, b]| \leq (\sin(\alpha) + \cos(\alpha)(\cos(\alpha) + \alpha))|ab|$, $0 < \alpha < \pi/4$.

Finally, consider the situation in which some portion of $SB[a, b]$ is below ij. To bound the length of $SB[a, b]$, we draw a tangent from point i to the portion of $SB[a, b]$ contained in H. (Recall that i is outside of the convex hull of S.) The tangent intersects H at a point, say, n ($\neq i$). Since a portion of $SB[a, b]$ is below segment ij and the arc of H below ij is x-monotone, we have $n(y) < j(y)$ and $n(x) < j(x)$. Thus, segment bn intersects C at a point, say, m ($\neq b$), see Fig. 4.

Let $\beta = \angle ibn$. Since n and m are on H and C respectively, $\angle bni = \angle bma = \pi/2$. Two segments am and in are thus parallel. Since in is tangent to $SB[a, b]$, it intersects C at a point, say, s ($\neq i$). Thus, two circular arcs $\widehat{ai}$ and $\widehat{ms}$ of C are of the same length. So, we have $\angle sbm = \alpha$. Let t be the intersection point of H with the line through b and s. Since $\angle sti = \angle bns = \pi/2$, we have

$\angle tis = \angle sbn = \alpha$. So, segment is intersects C at a point, say, l ($\neq i$). Hence, $\angle lbs = \alpha$, see Fig. 4. This implies that $\beta = \angle ibm \geq 2\alpha$ and $\angle abm \geq 3\alpha$ as well. Since $\angle jib = \alpha$ and $\alpha + \beta + \angle jib \leq \pi/2$, we obtain $\alpha \leq \pi/8$.

From the discussion made above, $SB[a, b]$ is contained in the convex region bounded by ba, ai, in and the circular arc $\widehat{nb}$ of diameter bi, with the inscribed angle $\pi/2 - \beta$. Since $|ai| = \sin(\alpha)|ab|$, $|in| = \cos(\alpha)\sin(\beta)|ab|$ and $|\widehat{nb}| = \cos(\alpha)(\pi/2 - \beta)|ab|$, we have $|SB[a, b]| \leq (\sin(\alpha)+\cos(\alpha)(\sin(\beta)+\pi/2-\beta))|ab|$. Note that $sin(\beta) - \beta < sin(2\alpha) - 2\alpha$, $0 < 2\alpha \leq \beta < \pi/2$. Thus, $|DT(a, b)| \leq (\sin(\alpha)+\cos(\alpha)(\sin(2\alpha)+\pi/2-2\alpha))|ab|$, $0 < \alpha \leq \pi/8$. By notice the fact that $(\sin(\alpha) + \cos(\alpha)(\sin(2\alpha) + \pi/2 - 2\alpha))$ is a monotonically decreasing function for $\pi/8 \leq \alpha < \pi/4$, the proof is complete. □

Lemma3 Suppose that both the first and last segments of the direct path from a to b are to the same side of the line through a and b. Then, $|DT(a, b)| \leq \pi|ab|/2$.

Proof. Assume that the direct path from a to b is not one-sided; otherwise, $|DT(a, b)|/|ab| \leq \pi/2$ and we are done. Then, the direct path from a to b intersects segment ab an even number times, as its first and last segments are to the same side of the x-axis.

Without loss of generality, assume that both the first and last segments of the direct path from a to b are vertically above ab. Let ce (df) be the first (second) segment of the direct path from a to b, which intersects ab.[*1] Assume also that $c(x) < e(x)$ and $d(x) > f(x)$, see Fig. 5. Denote by u and v two intersection points of ab with $B_{c,e}$ and $B_{d,f}$, respectively. Let l be the leftmost point of the circle of radius uc (or ue), centered at u, and let r be the rightmost point of the circle of radius vd (or vf), centered at v. Since both ce and df belong to the direct path from a to b, segment lr is completely contained in ab, see Fig. 5.

We show below that all points of $SA[a, b]$ are contained in C. Clearly, it suffices to show that all points of $SA[c, d]$ are contained in C. Here, $SA[c, d]$ denotes the portion of $SA[a, b]$ between c and d. Denote by $c = p_1, p_2, \ldots, p_{k+1} = d$ the sequence of points on $SA[a, b]$. Then, $k \geq 2$, and the Voronoi edges defined by all pairs (p_i, p_{i+1}) $(1 \leq i \leq k)$ do no intersect ab. Let us extend these Voronoi edges until they touch ab. Denote by $q_1, q_2, \ldots, q_k$ the extended points on ab such that $|q_i p_i| = |q_i p_{i+1}|$, for all $1 \leq i \leq k$, see Fig. 5.

We first claim that $u(x) < q_i(x) < v(x)$, for all $1 \leq i \leq k$. Notice that the slope of $B_{c,e}$ ($B_{d,f}$) is positive (negative). Assume that (p_i, p_{i+1}), $1 \leq i \leq k$, is a pair of points such that the slope of the common edge between $Vor(p_i)$ and $Vor(p_{i+1})$ is negative. From the convexity of S, we have $c(x) < p_i(x) < p_{i+1}(x) < d(x)$. Since both c and d are on the direct path from a to b, the *lower* vertex, say, w_i, of the common edge between $Vor(p_i)$ and $Vor(p_{i+1})$ is to the right (left) of point u (v). Hence, the line segment extended from that edge intersects ab at a point (i.e., q_i) that is to the right of w_i. So, we have $u(x) < w_i(x) < q_i(x)$. On the other hand, the line through w_i and v intersects $B_{p_i,p_{i+1}}$ at point w_i (see Fig. 5). Since $w_i(y) > v(y)(= 0)$ and $w_i(x) < v(x)$, the slope of the line through w_i and v is negative. Since the (negative) slope of $B_{p_i,p_{i+1}}$ is smaller than that of the line through v and w_i, we have $q_i(x) < v(x)$. If the slope of the common edge between $Vor(p_i)$ and $Vor(p_{i+1})$ is positive, a symmetric argument can also show $u(x) < q_i(x) < v(x)$.

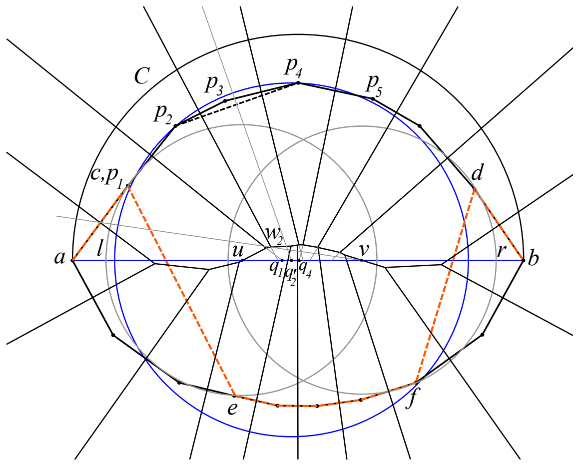

図 5 Illustrating the proof of Lemma 3.

Suppose now that $u(x) < q_1(x) < q_2(x) < \ldots < q_k(x) < v(x)$. In this case, $|q_1 p_2| = |q_1 p_1| < |q_1 u| + |up_1|$. Thus, the leftmost point of the circle of radius $q_1 p_2$, centered at q_1, is to the right of point l on ab. Recall that l is the leftmost point of the circle of radius up_1, centered at u, on ab. Since $u(x) < q_1(x) < q_2(x) < \ldots < q_k(x) < v(x)$, by an analogous argument, the leftmost point of the circle of radius $q_j p_{j+1}$ $(2 \leq j \leq k)$, centered at q_j, on ab is to the right of the leftmost point of the circle of radius $q_{j-1} p_j$, centered at q_{j-1}, on ab. Hence, the leftmost points of all circles of radius $q_i p_{i+1}$, centered at q_i for all $1 \leq i \leq k$, are to the right of point l on ab. Analogously, the rightmost points of all circles of radius $q_i p_i$, centered at q_i for all $1 \leq i \leq k$, are to the left of point r on ab.

Let m be the midpoint of segment lr, and let C' be the

[*1] There exists an instance in which the direct path from a to b intersects ab four times.

circle of radius lm, centered at point m. If q_i $(1 \leq i \leq k)$ is to the left of m on ab, then $|mp_i| < |mq_i| + |q_ip_i| < |ml|$. (The latter inequality comes from the known fact that the leftmost points of the circle of radius q_ip_i, centered at q_i, is to the right of point l on ab.) Moreover, since q_i $(1 \leq i \leq k)$ is to the left of m on ab, both m and p_{i+1} are to the same side of $B_{p_i,p_{i+1}}$. Thus, $|mp_{i+1}| < |mp_i| < |ml|$. Hence, both p_i and p_{i+1} are contained in C'. Analogously, if q_i is to the right of m on ab, both p_i and p_{i+1} are contained in C', too. Therefore, all points $c = p_1, p_2, \dots, p_{k+1} = d$ are contained in C'. Since lr is completely contained in ab, an analogous argument shows that all points of $SA[c,d]$ are contained in C, too.

Finally, consider the situation in which $u(x) < q_1(x) < q_2(x) < \dots < q_k(x) < v(x)$ does not hold. For ease of presentation, let $q_0(x) = u(x)$ and $q_{k+1}(x) = v(x)$. Assume that $[i,j]$ is a maximal interval such that $1 \leq i < j \leq k$ and $q_i(x) > q_j(x)$, see Fig. 5. So, $q_{i-1}(x) < q_i(x)$ and $q_j(x) < q_{j+1}(x)$. Clearly, it is suffices to consider the situation in which $[i,j]$ is the first (or leftmost) maximal interval on $[1,k]$. Since $q_i(x) > q_{i+1}(x)$, both q_i and p_{i+2} are to the same side of $B_{p_{i+1},p_{i+2}}$, and thus $|q_ip_{i+2}| < |q_ip_{i+1}|$. Analogously, since both q_i and p_{i+l}, $l \geq 3$ and $i+l \leq j+1$, are to the same side of $B_{p_{i+l-1},p_{i+l}}$, we have $|q_ip_{i+l}| < |q_ip_{i+l-1}| < \dots < |q_ip_{i+1}|$. This implies that points $p_{i+2}, \dots p_{j+1}$ are all contained in the circle of radius q_ip_{i+1}, centered at q_i. The discussion on $p_{i+2}, \dots p_{j+1}$ is then the same as that on p_{i+1}, and thus, all points $q_{i+1}, \dots, q_{j+1}$ can be ignored. To continue the discussion, we denote by q'_{i+1} the intersection point of ab with $B_{p_{i+1},p_{j+2}}$ $(j+1 \leq k)$, and consider q'_{i+1} as a new point immediately after q_i and before q_{j+2} $(j+1 \leq k)$. Since $[i,j]$ is a maximal interval on $[1,k]$, we then have $q_i(x) < q'_{i+1} < q_{j+2}(x)$. For the instance of Fig. 5, the first maximal interval we considered is $[1,2]$. The intersection point q'_2 of ab with B_{p_2,p_4} is thus taken into consideration, and point p_3 is contained in the circle of radius $p_2q'_2$, centered at q'_2. This process can repeatedly be performed, until an x-monotone sequence of the points q_n or q'_m is obtained. The rest discussion is the same as the situation in which $u(x) < q_1(x) < q_2(x) < \dots < q_k(x) < v(x)$ holds. Again, all points of $SA[c,d]$ are contained in C.

In summary, all points of $SA[a,b]$ are contained in C. From the convexity of S, $|DT(a,b)| \leq |SA[a,b]| \leq \pi|ab|/2$. □

We can now give the main result of this paper.

Theorem1 Suppose that the set S of given points is in convex position, and a and b are two points of S. In the Delaunay triangulation of S, there is a path from a to b such that its length is less than $1.82|ab|$.

Proof. Suppose that the direct path from a to b is not one-sided; otherwise, $|DT(a,b)| \leq \pi|ab|/2$ and we are done. Let $f_1 = \sin(\alpha) + \pi\cos(\alpha)/2$, $\alpha \in [\pi/4, \pi/2)$, and $f_2(\alpha) = \sin(\alpha) + \cos(\alpha)(\cos(\alpha) + \alpha)$ and $f_3(\alpha) = \sin(\alpha) + \cos(\alpha)(\sin(2\alpha) + \pi/2 - 2\alpha)$, $\alpha \in (0, \pi/4)$. It then follows from Lemmas 2 and 3 that $|DT(a,b)|/|ab| \leq \max\{\pi/2, f_1(\alpha), f_2(\alpha), f_3(\alpha)\}$. Since $f'_1(\alpha) = cos(\alpha) - \pi\sin(\alpha)/2 < 0$, $\alpha \in [\pi/4, \pi/2)$, $f_1(\alpha)$ is a monotonically decreasing function. Thus, $f_1(\alpha) \leq f_1(\pi/4) < 1.82$. Moreover, since the function $f_i(\alpha)$ is convex, $i = 2$ or 3, we can obtain $f_i(\alpha) < 1.77$ by letting $f'_i(\alpha) = 0$. □

4. Concluding remarks

We have shown that the stretch factor of the Delaunay triangulation of a set of points in convex position is less than 1.82. We believe that the same stretch factor also holds for the set of points in general position. A possible way might be to examine two different paths between a and b; one above and the other below the line through a and b. These two paths between a and b, although they are non-convex, may give the same stretch factor as $SA[a,b]$ and $SB[a,b]$, which are used for the sets of points in convex position. It is also a challenge open problem to reduce the stretch factor of $DT(S)$ further, so as to close the gap to its lower bound (roughly about 1.60).

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