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Dual-context Modal Logic as Left Adjoint of Fitch-style Modal Logic

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Abstract: In the stream of studies on intuitionistic modal logic, we can find mainly three kinds of natural deduction systems. For logical aspects, adding axiom schemata is a simple and popular way to construct a system. The Curry-Howard correspondence, however, gives us a connection between logic and computer science. From the viewpoint of programming languages, two more important systems, called a dual-context system and a Fitch-style system, have been proposed. While dual-context systems for S4 are heavily used in the field of staged computation, a dual-context system for K is also studied more recently. In our previous studies, categorical semantics for Fitch-style modal logic is proposed and usefulness of levels is noticed. This paper observes an interesting fact that the box modality of the dual-context system is in fact a left adjoint of that of the Fitch-style system. In order to show the statement, we embed both the two systems, which are refined with levels, into the adjoint calculus that equips an adjunction a priori. Moreover, the adjunction is refined with polarity and the adjoint calculus is extended to polarized logic.

Keywords: modal logic, lambda-calculus, adjunction, staged computation

1. Introduction

Natural deduction systems are a kind of deductive systems in logic. A part of importance of natural deduction systems is a relationship with programming languages. Such a relationship is called a *Curry-Howard correspondence*. It is known that intuitionistic modal logic corresponds to staged computation via a Curry-Howard correspondence. The natural deduction systems for intuitionistic modal logic have been studied widely and mainly three kinds have been proposed. From the logical point of view, Kripke-style semantics and Hilbert-style systems with axioms are commonly accepted for variations of intuitionistic modal logic. However, studies on natural deduction systems and corresponding calculi for intuitionistic modal logic are diverse: *Gentzen-style*, *dual-context* and *Fitch-style* are main streams.

A Gentzen-style calculus was first proposed by Bellin et al. [2], and was later refined by one of the authors [11], [12]. Dual-context calculi [5], [8] have been developed mainly for intuitionistic S4 (IS4), because the first dual-context calculus [1] was dedicated to intuitionistic linear logic with the exponential modality. After several studies on S4, a dual-context calculus for intuitionistic K (IK) appeared in Ref. [2]. Recently, generalized dual-context calculi [13] have been provided to accommodate various modal logic, K, T, K4 and so on. In this paper, our study focuses on the intuitionistic fragment of the simplest modal logic K for generality. The third style is Fitch-style [6], [16]. Despite a long history, computational meaning of Fitch-style natural deduction systems is not well established. Since both the dual-context and Fitch-style have multiple contexts and similar judgments, we compare them in this study.

It is known that a judgment in the Fitch-style system can be translated to a judgment in Gentzen-style system, and vice versa. A survey paper [9] by de Paiva and Ritter might be helpful for understanding Fitch-style.

$$\frac{\Gamma_{m-1} ; \cdots ; \Gamma_1 ; \Gamma_0 \vdash A \quad \text{in Fitch-style}}{\vdash \Box(\Gamma_{m-1} \to \cdots \Box(\Gamma_1 \to \Box(\Gamma_0 \to A))) \quad \text{in Gentzen-style}}$$

A proof relevant translation is discussed in our work [19] via the semantics. A ccc with a normal monoidal endofunctor G is an instance of our semantics of the Fitch-style calculus. The following is a rough sketch of the semantics of the Fitch-style.

$$[\![\Gamma_1 ; \Gamma_0 \vdash M : A]\!] : [\![\Gamma_1]\!] \to G([\![\Gamma_0]\!] \to [\![A]\!])$$

On the other hand, a model for Gentzen-style and dual-context is believed to be a ccc with a monoidal endofunctor [2], [5], [10], [12], [13]. If we write F for a model of the dual-context calculus, semantics of the dual-context is as follows.

$$[\![\Gamma_1 ; \Gamma_0 \vdash M : A]\!] : F[\![\Gamma_1]\!] \times [\![\Gamma_0]\!] \to [\![A]\!]$$

If we assume that a common judgment has the same meaning both in the Fitch-style and in the dual-context, F is expected to be a left adjoint of G. It is a naive idea of this study. Such an adjunction is used to give semantics to the Fitch-style calculus by Clouston [7], though he focuses on the diamond modality rather than the box modality of the dual-context side.

In this paper, we show an adjunction by embedding both the Fitch-style and dual-context calculi into the adjoint calculus [3], [4], which was proposed in order to represent monoidal

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adjunctions. To achieve our aim, we refine the Fitch-style and the dual-context with levels along the line of our previous work [19]. Due to the assignment of levels, it can be shown that the union of the 2-level calculi exactly corresponds to the adjoint calculus. After the 2-level case, we extend the result to multi-level. If we consider *contextual modalities* of the Fitch-style, our results induce a decomposition of a contextual S4 modality [18].

Moreover, we extend the adjoint calculus to a *polarized* calculus, and refine the adjunction of the Fitch-style and the dual-context. Our polarization is based on the lecture notes by Pfenning [20] on the call-by-push-value calculus [15]. We also provide both the call-by-name and call-by-value big-step semantics to the adjoint calculus, and compare the adjoint calculus with the polarized adjoint calculus.

2. Intuitionistic Modal Logic

In this section, we introduce three kinds of deductive systems for intuitionistic modal logic. This paper focuses on the box fragment without disjunctions. Conjunctions are not excluded in our study, but we ignore them except for the last section because of the syntactic simplicity.

2.1 Gentzen-style

In this paper, the natural deduction of the intuitionistic propositional logic means the simply typed λ-calculus.

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.\, M : A \to B}$$

$$\frac{\Gamma \vdash N : A \qquad \Gamma \vdash M : A \to B}{\Gamma \vdash MN : B}$$

This paper follows some conventions about the λ-calculus. Especially, a context is a set of type declaration of variables, and hence the exchange rule does not appear explicitly.

The usual β-η-equality is given by the following.

$$(\lambda x.\, M)N = M[N/x]$$

$$\lambda x.\, Mx = M \quad \text{if} \quad x \notin \mathrm{FV}(M)$$

As usual, the equality is formally defined on judgments, but we omit contexts and types through the paper.

Theorems of intuitionistic modal logic IK is usually given by adding the following rules to the intuitionistic propositional logic.

$$\frac{\vdash A}{\vdash \Box A} \qquad \frac{}{\vdash \Box(A \to B) \to \Box A \to \Box B}$$

The consequence of the latter rule is often called K. Systems shown in this paper are equivalent to this logic with respect to the provability.

Although we can assign terms directly on the above derivations, it does not look a natural deduction in the usual sense. In Gentzen-style formulation [2], [11], the two rules for $\Box$ are combined into a single rule as follows.

$$\frac{\Gamma \vdash \Box A \quad (\forall A \in \Gamma') \qquad \Gamma' \vdash B}{\Gamma \vdash \Box B}$$

The logical equivalence of this system and IK is not so difficult. While term assignment enables us to consider computational meaning, logical harmony between introduction and elimination is broken in the Gentzen-style system. In some sense, the dual-context system can be obtained from splitting this rule into introduction and elimination.

Although the equality on proofs of Gentzen-style are important in staged computation, connection with this study is rather weak. So, we do not refer any more in this paper. Whereas the Gentzen-style calculus naively represents a cartesian closed category with a lax monoidal endofunctor, the following Fitch-style calculus represents an infinitely enriched category [19].

2.2 Fitch-style

In this paper, we use the symbol Δ as a sequence of contexts, while the symbol Γ denotes a context. To avoid confusion, we use semicolons as separators between contexts instead of commas. Without loss of generality, we assume that any variable does not occur in different two contexts of Δ. We write $\cdot$ for the empty context not the empty sequence of contexts. We may implicitly omit or add empty contexts in the left-most part of Δ.

The Fitch-style calculus in this study is based on the formulation [17] of IK by one of the authors. In Fitch-style, a judgment has a form $\Delta \vdash^{l} M : A$, where l is a level ranging over natural numbers. In this formulation, Δ means a context categorized into levels. When a judgment $\Gamma_{m-1} ; \cdots ; \Gamma_0 \vdash^{l} M : A$ is derivable, Γ_i can be considered a context of the level $i + l$. A level corresponds to a stage in staged computation, and terms of a level $l + 1$ can handle codes of the level l.

We can extend the simply typed λ-calculus to categorized contexts straightforwardly, just replacing Γ with Δ. The formal definition is included in **Fig. 1**. The specific rules for the box of Fitch-style are given as follows.

$$\frac{\Delta ; \cdot \vdash^{l} M : A}{\Delta \vdash^{l+1} {`}M : \Box A} \qquad \frac{\Delta \vdash^{l+1} M : \Box A}{\Delta ; \Gamma \vdash^{l} {,}M : A}$$

While the intuitionistic propositional part keeps a level, the box rules change levels in derivations. Both the rules move the focus in contexts syntactically, but do almost nothing semantically. These rules are called *necessitation* and *denecessitation*, respectively. Necessitation with the empty context is a well-known property if we ignore levels.

Remark 1. Levels naturally arises from the semantics, and are helpful for understanding the staging. However, they are not essential for provability as logic. If we forget levels, another Fitch-style system for IK can be obtained, which is nothing but a calculus proposed by Clouston [7]. It can be seen easily that a formula is provable in the Fitch-style system without levels iff it is provable in our Fitch-style system at some level.

The logical equivalence between the Fitch-style system and IK is not trivial, but logical aspects of Fitch-style have been studied for a long time. Especially, Borghuis' work [6] is a monumental achievement. The axiom K can be derived as follows.

$$\vdash^{1} \lambda y.\, \lambda x.\, {`}({,}y\, {,}x) : \Box(A \to B) \to \Box A \to \Box B$$

If we discuss the logical provability of our calculus formally, we have to remove levels as mentioned in the above remark.

From a viewpoint of the Curry-Howard correspondence, also

$$\frac{}{\Delta, x : A \vdash^{l} x : A} \qquad \frac{\Delta, x : A \vdash^{l} M : B}{\Delta \vdash^{l} \lambda x.\, M : A \to B} \qquad \frac{\Delta \vdash^{l} N : A \qquad \Delta \vdash^{l} M : A \to B}{\Delta \vdash^{l} MN : B}$$

$$\frac{\Delta\, ;\, \cdot \vdash^{l} M : A}{\Delta \vdash^{l+1} \text{‘}M : \Box A} \qquad \frac{\Delta \vdash^{l+1} M : \Box A}{\Delta\, ;\, \Gamma \vdash^{l} ,M : A}$$

Fig. 1 Typing rules of Fitch-style calculus.

$$(\lambda x.\, M)N = M[N/x] \qquad \lambda x.\, Mx = M \text{ if } x \notin \mathrm{FV}(M)$$

$$,\text{‘}M = M \qquad \text{‘},M = M$$

Fig. 2 Equality of Fitch-style calculus.

the equality should be discussed. Following Lisp, ‘M can be regarded as a quasi-quoted code. As you expect, ,M is an unquoted term. Such an idea can be characterized by the following equations.

$$,\text{‘}M = M$$
$$\text{‘},M = M$$

One can be referred to **Fig. 2** for the full equations. Despite the simple axiomatization, the Fitch-style calculus is enough rich for capturing unbounded iteration of categorical enrichment.

There are various notations and naive meaning of quotation and unquotation in previous studies. In traditional staged computation, boxing and unboxing may be preferable. According to Clouston [7], operational intuition can be explained as the words, *shutting* and *opening*.

In the last of this subsection, we define a terminology: the m-level system denotes the subsystem where each level occurring in a derivation is less than m and the sum of a proof level and the depth of its context stack is exactly m. Hence, only the levels 0 and 1 occur in the 2-level Fitch-style system. In 2-level systems, we often call 0-level object-level, and call 1-level meta-level.

2.3 Dual-context

Studies on dual-context systems were motivated by linear logic in early days. Since the ! modality of linear logic is a kind of S4 modality, dual-context system can be applied to non-linear IS4 [5] straightforwardly. Although the dual-context formulation of IK is not trivial, we adopt Kavvos' calculus [13].

A judgment in the dual-context calculus has two kinds of contexts, a *modal context* and a *normal context*. The form of judgments are very much like that of Fitch-style, but the derivation rules are quite different. The $\Box$ modality of the dual-context system is characterized by the following two rules.

$$\frac{\cdot\, ;\, \Gamma' \vdash M : A}{\Gamma'\, ;\, \Gamma \vdash \mathtt{meta}\, M : \Box A}$$

$$\frac{\Gamma'\, ;\, \Gamma \vdash N : \Box A \qquad \Gamma', x : A\, ;\, \Gamma \vdash M : B}{\Gamma'\, ;\, \Gamma \vdash \mathtt{let\ meta}\, x\ \mathtt{be}\ N\ \mathtt{in}\ M : B}$$

The first rule seems similar to the necessitation rule of the Fitch-style in the case that all contexts are empty.

The specific equations are as follows.

$$\mathtt{let\ meta}\, x\ \mathtt{be\ meta}\, N\ \mathtt{in}\ M = M[N/x]$$
$$\mathtt{let\ meta}\, x\ \mathtt{be}\ M\ \mathtt{in\ meta}\, x = M$$

If we consider a reduction system of the β-part, the strong normalization and confluence theorems have been shown by Kavvos [13].

We refine this calculus for our aim. First, we assign levels to judgments. Level assignment is a key idea of this study. Only one rule changes levels in derivations, and other rules just keep levels.

$$\frac{\cdot\, ;\, \Gamma' \vdash^{l+1} M : A}{\Gamma'\, ;\, \Gamma \vdash^{l} \mathtt{meta}\, M : \Box A}$$

Levels are just auxiliary and meaningless as logic. If we have a proof in the dual-context system, appropriate levels can be assigned to all judgments occurring in the proof. In this sense, the level assignment does not restrict the system as logic.

Contrary to the name "dual", in fact, we can make the number of contexts unbounded without a heavy modification. We say that such a generalized system is a *multi-context* system. In order to obtain the multi-context system, we replace all Γ' with Δ in typing rules.

$$\frac{\Delta \vdash M : A}{\Delta\, ;\, \Gamma \vdash \mathtt{meta}\, M : \Box A}$$

$$\frac{\Delta\, ;\, \Gamma \vdash N : \Box A \qquad \Delta, x : A\, ;\, \Gamma \vdash M : B}{\Delta\, ;\, \Gamma \vdash \mathtt{let\ meta}\, x\ \mathtt{be}\ N\ \mathtt{in}\ M : B}$$

Although shift to multi-context is independent of level assignment, we use the mixed variant mainly. The full definition of the multi-context calculus with levels is described in **Fig. 3** and **Fig. 4**.

It is a little surprising that the multi-context system is not stronger than the dual-context system w.r.t. the provability. A reason depends on the fact that the following derivations are admissible in the dual-context.

$$\frac{\Gamma_1, A\, ;\, \Gamma_0 \vdash B}{\Gamma_1\, ;\, \Box A, \Gamma_0 \vdash B}$$

Due to these admissible derivations, we can reduce contexts higher than 1 to a 1-level context. Although the two judgments have the same meaning, the upper is preferable for programmers in general. Since l-level $\Box A$ essentially means $(l+1)$-level A, we usually expect A to be extracted from $\Box A$. Similarly, in the multi-context system, we can explicitly regard l-level $\Box\Box A$ as $(l+2)$-level A. In this sense, we say that the multi-context calculus is a refinement of the dual-context calculus. We can remark that only dual-contexts occur in a typing derivation for any closed term in the multi-context calculus even if its type includes $\Box\Box A$. Instead, a morphism from $\Box\Box A$ to B in a model has three kinds of syntactic expressions in the multi-context calculus. Since we believe the multi-context has the whole essence of the dual-context, we intentionally confuse the word "multi-context" with "dual-context".

$$\frac{}{\varDelta, x : A \vdash^{l} x : A} \qquad \frac{\varDelta, x : A \vdash^{l} M : B}{\varDelta \vdash^{l} \lambda x.\, M : A \to B} \qquad \frac{\varDelta \vdash^{l} N : A \qquad \varDelta \vdash^{l} M : A \to B}{\varDelta \vdash^{l} MN : B}$$

$$\frac{\varDelta \vdash^{l+1} M : A}{\varDelta \,;\, \varGamma \vdash^{l} \mathtt{meta}\, M : \Box A} \qquad \frac{\varDelta \,;\, \varGamma \vdash^{l} N : \Box A \qquad \varDelta, x : A \,;\, \varGamma \vdash^{l} M : B}{\varDelta \,;\, \varGamma \vdash^{l} \mathtt{let\ meta}\, x\ \mathtt{be}\ N\ \mathtt{in}\ M : B}$$

Fig. 3 Typing rules of multi-context calculus.

$$(\lambda x.\, M)N = M[N/x] \qquad \lambda x.\, Mx = M \ \text{ if } \ x \notin \mathrm{FV}(M)$$

$$\mathtt{let\ meta}\, x\ \mathtt{be\ meta}\, N\ \mathtt{in}\ M = M[N/x] \qquad \mathtt{let\ meta}\, x\ \mathtt{be}\ M\ \mathtt{in\ meta}\, x = M$$

Fig. 4 Equality of multi-context calculus.

In order to use higher-level contexts more efficiently, we can consider additional elimination rules. The +2 case is as follows.

$$\frac{\varDelta \,;\, \varGamma_1 \,;\, \varGamma_0 \vdash \Box\Box A \qquad \varDelta, A \,;\, \varGamma_1 \,;\, \varGamma_0 \vdash B}{\varDelta \,;\, \varGamma_1 \,;\, \varGamma_0 \vdash B}$$

Such rules are semantically sound, but not admissible in the current multi-context system. The multi-context adjoint calculus given in the next section admits and justifies these rules.

In later sections, we write $\sqsubset$ for $\Box$ of the dual-context and multi-context systems, while $\sqsupset$ is used for the Fitch-style. The notation comes from that the composite $\sqsubset\sqsupset$ is the S4 box modality. If we restrict levels to less than 2, the logic becomes strictly weaker. We first compare such weakened calculi and later compare the full-level calculi.

3. Calculus with Adjunction

In this section, we reconstruct an additive variant of the *adjoint calculus* [4] via Fitch-style and dual-context systems. Since a model of the adjoint calculus is a monoidal adjunction between two cartesian closed categories, we can claim that the pair of the Fitch-style and the dual-context is an adjunction. In addition, because the adjoint calculus includes the dual-context IS4 system as a subcalculus, the IS4 modality decomposes into the two IK modalities. In the last subsection, we discuss contextual modalities and show that a contextual dual-context IS4 system decomposes into the dual-context IK system and the contextual Fitch-style IK system.

3.1 2-Level Adjunction

An adjoint calculus consists of two worlds. Each type belongs to a world, and worlds are mutually exclusive. In a typical situation, one of two worlds is intuitionistic multiplicative exponential linear logic and the other is intuitionistic logic. This case is also known as LNL (linear non-linear) logic [3]. The LNL calculus has been provided by Benton to investigate the exponential modality of linear logic. Benton's LNL calculus is strongly related to S4 modal logic naturally from its origin.

In this paper, we shift linear logic to ordinary logic in a usual manner. There is no technical barrier against changing the structural rules, syntactically nor semantically. (We still say "linear" even if the linearity is dropped.) The derivation rules are formally defined in **Fig. 5**. The adjoint calculus has two kinds of judgments, $\varTheta \,;\, \varGamma \vdash^{0} M : A$ and $\varTheta \vdash^{1} M : \varPhi$, where $\varTheta$ is a context for level 1 variables. We use 0 and 1 for Benton's original $\mathcal{L}$ and C, respectively. Hence, 0 refers to the linear world, 1 refers to the classical world. In the figure, A, B are used for 0-level types, and $\varPhi$, $\varPsi$ are for 1-level types.

It is remarkable that $\sqsupset A$ belongs to the world 1 for any A of the world 0, and conversely, $\sqsubset\varPhi$ belongs to the world 0 for any $\varPhi$ of the world 1. So, $\sqsubset$ and $\sqsupset$ transfer types between the worlds. Indeed, in categorical semantics, a world corresponds to a cartesian closed category, and the interpretations of $\sqsubset$ and $\sqsupset$ form a monoidal adjunction. In Benton's original paper, the letters F and G are used for the modalities according to the conventional manner to describe an adjunction. The equality on judgments can be derived from this semantics, which is described in **Fig. 6**.

One can see that Fig. 6 is just the union of Fig. 2 and Fig. 4. Moreover, Fig. 5 consists of the special cases of Fig. 1 and Fig. 3, that is, the 2-level Fitch-style system and the 2-level dual-context system. The following result immediately follows from this observation.

Theorem 1. *The* 2*-level Fitch-style calculus for IK is a subcalculus of the adjoint calculus. Also the* 2*-level dual-context calculus for IK is another subcalculus of the adjoint calculus. Moreover, the modality of the dual-context is a left adjoint of the modality of the Fitch-style in the adjoint calculus.*

It is well-known that the composite $\sqsubset\sqsupset$ is a monoidal comonad. We write ! for $\sqsubset\sqsupset$ following linear logic. The following derivation about ! is admissible in the adjoint logic.

$$\frac{\dfrac{\dfrac{}{\varTheta, x : \sqsupset A \,;\, \varGamma \vdash^{0} ,x : A}}{\varTheta \,;\, \cdot \vdash^{0} M : A}}{\varTheta \,;\, \varGamma \vdash^{0} \mathtt{meta}\,`M : !A}$$

$$\frac{\varTheta \,;\, \varGamma \vdash^{0} N : !A \qquad \varTheta, x : \sqsupset A \,;\, \varGamma \vdash^{0} M : B}{\varTheta \,;\, \varGamma \vdash^{0} \mathtt{let\ meta}\, x\ \mathtt{be}\ N\ \mathtt{in}\ M : B}$$

If we rewrite outer contexts replacing $x : \sqsupset A$ with $x : A$, we can get another dual-context system including only $\vdash^{0}$. In fact, this system corresponds to IS4 in the sense of Curry-Howard. When the 0-level monoidal structure is multiplicative, the calculus is known as DILL [1]. Therefore, if we start from the dual-context calculus for IS4, it can decompose into the Fitch-style and the dual-context, both of which corresponds to IK. We shall revisit this fact later around contextual modalities.

3.2 Multi-Level Adjunction

We have investigated the 2-level calculi in the previous subsection. The result is extended to multi-level in this subsection.

For that purpose, we extend the adjoint calculus to multi-level and multi-contexts. As mentioned before, all the worlds are conceptually disjoint. However, since they can be completely distinguishable due to syntactically assigned levels, we use the same

$$\frac{}{\Theta\,;\Gamma, x : A \vdash^0 x : A} \qquad \frac{\Theta\,;\Gamma, x : A \vdash^0 M : B}{\Theta\,;\Gamma \vdash^0 \lambda x.\, M : A \to B} \qquad \frac{\Theta\,;\Gamma \vdash^0 N : A \qquad \Theta\,;\Gamma \vdash^0 M : A \to B}{\Theta\,;\Gamma \vdash^0 MN : B}$$

$$\frac{}{\Theta, x : \Phi \vdash^1 x : \Phi} \qquad \frac{\Theta, x : \Phi \vdash^1 M : \Psi}{\Theta \vdash^1 \lambda x.\, M : \Phi \to \Psi} \qquad \frac{\Theta \vdash^1 N : \Phi \qquad \Theta \vdash^1 M : \Phi \to \Psi}{\Theta \vdash^1 MN : \Psi}$$

$$\frac{\Theta\,;\,\cdot\, \vdash^0 M : A}{\Theta \vdash^1 \text{`}M : \sqsupset A} \qquad \frac{\Theta \vdash^1 M : \sqsupset A}{\Theta\,;\Gamma \vdash^0 {,}M : A}$$

$$\frac{\Theta \vdash^1 M : \Phi}{\Theta\,;\Gamma \vdash^0 \mathtt{meta}\, M : \sqsubset\Phi} \qquad \frac{\Theta\,;\Gamma \vdash^0 N : \sqsubset\Phi \qquad \Theta, x : \Phi\,;\Gamma \vdash^0 M : B}{\Theta\,;\Gamma \vdash^0 \mathtt{let\ meta}\, x\ \mathtt{be}\ N\ \mathtt{in}\ M : B}$$

Fig. 5 Typing rules of 2-level additive adjoint calculus.

$$(\lambda x.\, M)N = M[N/x] \qquad\qquad \lambda x.\, Mx = M \ \text{ if } \ x \notin \mathrm{FV}(M)$$

$$,\text{`}M = M \qquad\qquad \text{`},M = M$$

$$\mathtt{let\ meta}\, x\ \mathtt{be\ meta}\, N\ \mathtt{in}\ M = M[N/x] \qquad\qquad \mathtt{let\ meta}\, x\ \mathtt{be}\ M\ \mathtt{in\ meta}\, x = M$$

Fig. 6 Equality of additive adjoint calculus.

propositional letters in every world. The multi-level adjoint calculus is given in Fig. 6 and **Fig. 7**.

Of course, if we restrict this calculus to 2-level, we can obtain the usual adjoint logic in the previous subsection. A categorical model of this multi-level adjoint calculus is an infinite sequence of ccc's and monoidal adjunctions. Such models are a straightforward extension of models of the 2-level adjoint calculus. The main result can be extended to multi-level without any difficulty.

Theorem 2. *The Fitch-style calculus for IK is a subcalculus of the multi-level adjoint calculus. Also the multi-context calculus for IK is another subcalculus of the multi-level adjoint calculus. Moreover, for each level, the modality of the dual-context is a left adjoint of the modality of the Fitch-style in the multi-level adjoint calculus.*

3.3 Contextual Modalities

In Fitch-style formulation, the notion of *contextual modality* [18] can be introduced straightforwardly. Roughly speaking, a contextual modal formula $[\Gamma]A$ means $\Box(\Gamma \to A)$. In some sense, contextual modalities internalize hypothetical judgments: whereas the ordinary modality asserts the truth of a proposition under no hypothesis, the contextual modality permits an assertion under a hypothesis. The following rules characterize the contextual modality of IK in Fitch-style.

$$\frac{\Delta\,;\Gamma \vdash^l B}{\Delta \vdash^{l+1} [\Gamma]B}$$

$$\frac{\Delta\,;\Gamma' \vdash^l A \quad (\forall A \in \Gamma) \qquad \Delta \vdash^{l+1} [\Gamma]B}{\Delta\,;\Gamma' \vdash^l B}$$

In the introduction, we have mentioned a logical correspondence between Fitch-style and Gentzen-style. In fact, such a translation can be justified in the Fitch-style system itself, and explains the meaning of contextual modalities.

$$\frac{\Gamma_{m-1}\,;\cdots;\Gamma_1\,;\Gamma_0 \vdash^0 A}{\vdash^m [\Gamma_{m-1}]\cdots[\Gamma_1][\Gamma_0]A}$$

A term assignment was proposed by one of the author [17], but we do not show details here. This Fitch-style contextual system is equivalent to the ordinary Fitch-style system not only as logic but also as a calculus. In the categorical semantics, for objects A and B in a C-enriched category, $[A]B$ means the hom-object in C. So, from the semantic point of view, the contextual modalities are more fundamental than the ordinary box modality.

On the other hand, a dual-context system with a contextual modality is provided by Nanevski et al. [18] for IS4. In their formulation, an additional form $A[\Gamma]$ is required in 1-level contexts. We write $![\Gamma]A$ for a contextual modal formula in order to distinguish the modality from the contextual modality of K.

$$\frac{\Theta, B[\Gamma]\,;\Gamma' \vdash^0 A \quad (\forall A \in \Gamma)}{\Theta, B[\Gamma]\,;\Gamma' \vdash^0 B}$$

$$\frac{\Theta\,;\Gamma \vdash^0 M : A}{\Theta\,;\Gamma' \vdash^0 ![\Gamma]A}$$

$$\frac{\Theta\,;\Gamma' \vdash^0 ![\Gamma]A \qquad \Theta, A[\Gamma]\,;\Gamma' \vdash^0 B}{\Theta\,;\Gamma' \vdash^0 B}$$

Other trivial 0-level rules are omitted. Since computational meaning of this S4 calculus is not trivial, keywords `true` and `valid` are assigned to various parts in the original paper [18]. We reconstruct it simply with an adjunction.

If we are reminded of the construction of the dual-context IS4 system from the adjoint calculus, we can see that the contextual IS4 modality decomposes into the ordinary IK modality and the contextual IK modality. More formally, the following statement holds.

Theorem 3. *When we read $![\Gamma]A$ and $A[\Gamma]$ as $\sqsubset[\Gamma]A$ and $[\Gamma]A$ respectively, all the typing rules of the contextual dual-context IS4 system are admissible in the 2-level contextual adjoint calculus, which consists of the contextual Fitch-style IK system and the ordinary dual-context IK system.*

Proof. The first rule is derived from the contextual unquotation rule of the Fitch-style.

$$\frac{\Theta, [\Gamma]B\,;\Gamma' \vdash^0 A \quad (\forall A \in \Gamma) \qquad \dfrac{}{\Theta, [\Gamma]B \vdash^1 [\Gamma]B}}{\Theta, [\Gamma]B\,;\Gamma' \vdash^0 B}$$

The second and third rules can be derived in the same manner as the non-contextual case. □

4. Polarity and Adjunction

We have seen that modalities in dual-context and Fitch-style systems are regarded as left and right adjoints. Then, one ques-

$$\frac{}{\varDelta, x : A \vdash^{l} x : A} \qquad \frac{\varDelta, x : A \vdash^{l} M : B}{\varDelta \vdash^{l} \lambda x.\, M : A \to B} \qquad \frac{\varDelta \vdash^{l} N : A \qquad \varDelta \vdash^{l} M : A \to B}{\varDelta \vdash^{l} MN : B}$$

$$\frac{\varDelta \,;\, \cdot \vdash^{l} M : A}{\varDelta \vdash^{l+1} \text{‘}M : \sqsupset A} \qquad \frac{\varDelta \vdash^{l+1} M : \sqsupset A}{\varDelta \,;\, \varGamma \vdash^{l} {,}M : A}$$

$$\frac{\varDelta \vdash^{l+1} M : A}{\varDelta \,;\, \varGamma \vdash^{l} \mathtt{meta}\, M : \sqsubset A} \qquad \frac{\varDelta \,;\, \varGamma \vdash^{l} N : \sqsubset A \qquad \varDelta, x : A \,;\, \varGamma \vdash^{l} M : B}{\varDelta \,;\, \varGamma \vdash^{l} \mathtt{let}\ \mathtt{meta}\, x\ \mathtt{be}\ N\ \mathtt{in}\ M : B}$$

Fig. 7 Typing rules of multi-level additive adjoint calculus.

tion naturally arises: is it possible to explain the difference between those two modalities especially from the syntactic point of view? An answer to this question is found in operational semantics.

4.1 Polarized Adjoint Calculus

Looking at the typing rules of 2-level adjoint calculus, one sees that the terms for $\sqsupset$ have a similarity with projections or tupling and those for $\sqsubset$ have that with terms performing pattern matching, such as `case` of sum types. In fact, such syntactic distinction can be clearly formalized using the notion of *polarity*. In a polarized type theory, types are classified into two categories, *positive types* and *negative types*. Negative types include arrow types and product types. Terms assigned to elimination rules of negative types are likely to be rather "direct", for example, πM and MN. Positive types including sum types, on the other hand, are given apparently "indirect" terms, such as $\mathtt{case}\ M\ \mathtt{of}\ \{i.x_i.\, N_i\}$. Operational behaviors of eliminations of positive types are typically understood as deconstruction of terms with *pattern matching*.

One non-trivial phenomenon is that products belong to both categories. This is because there are two equivalent definitions for elimination of product types, one in favor of construction (a,b) and the other deconstruction (c).

$$\frac{\varGamma \vdash M : A \times B}{\varGamma \vdash \pi_{\mathrm{L}} M : A}\ \text{(a)} \qquad \frac{\varGamma \vdash M : A \times B}{\varGamma \vdash \pi_{\mathrm{R}} M : B}\ \text{(b)}$$

$$\frac{\varGamma \vdash M : A \times B \qquad \varGamma, x : A, y : B \vdash N : C}{\varGamma \vdash \mathtt{let}\ (x, y)\ \mathtt{be}\ M\ \mathtt{in}\ N : C}\ \text{(c)}$$

It is therefore usual to add two different types that behave like products, defining the negative one with (a,b) and the positive one with (c). Polarities of types can also be understood from the denotational perspective: negative types are interpreted as right adjoints and positive types are as left adjoints. This perspective also gives another explanation of why products may be both negative and positive.

The notion of polarity can be then clearly applied to adjoint calculus. As presented in **Table 1**, we can find an evident similarity between the differences of positive/negative types and dual-context/Fitch-style modalities, both from the syntactic and semantic viewpoints. Motivated by this observation, we attempt to polarize the adjoint calculus and indeed characterize dual-context modality as positive types and Fitch-style modality as negative types.

Polarization is investigated in many papers. Among them, we base our theory on Levy's call-by-push-value [15]. Levy's insight is that positive types can be thought of *values* and negative types *computations*. In the call-by-push-value calculus, any terms are strictly classified as either values or computations following the idea of polarities, which leads us to a refinement of traditional functional languages that subsumes call-by-name and call-by-value. Our attempt to polarize the adjoint calculus can be then considered as a multi-stage extension of the call-by-push-value calculus. In this sense, the calculus we introduce here may be called either *polarized adjoint calculus* or *staged call-by-push-value*, though we prefer the first here.

Table 1 Polarities and modalities.

	Positive	Negative
Denotations	Left adjoints	Right adjoints
Terms	$\mathtt{let}\ (x, y)\ \mathtt{be}\ M\ \mathtt{in}\ N$	$\pi_i M$

	Dual-context	Fitch-style
Denotations	Left adjoints	Right adjoints
Terms	$\mathtt{let}\ \mathtt{meta}\, x\ \mathtt{be}\ M\ \mathtt{in}\ N$	${,}M$

The typing rules of polarized adjoint calculus are presented in **Fig. 8**. Our syntax is taken not from Levy's but from Pfenning's [20]. Judgments are any of the following forms.

$$\varTheta \,;\, \varGamma^{+} \vdash^{0} M : A^{+}$$
$$\varTheta \,;\, \varGamma^{+} \vdash^{0} V : A^{-}$$
$$\varTheta \vdash^{1} P : \varPsi$$

In this section, we distinguish meta-level and object-level by notations for readability. Meta-level variables, terms, and types are denoted by $\alpha, \beta, \cdots$, $P, Q, \cdots$, and $\varPhi, \varPsi, \cdots$. In addition, $M, N, \cdots$ and $V, W, \cdots$ are used for object-level terms according as polarities. We may write $A_0^- + A_1^-$ for $\sum_{i \in \{0,1\}} A_i^-$ and 0 for $\sum_{i \in \emptyset} A_i^-$.

Shift operators $\uparrow$ and $\downarrow$ change the polarities of object-level terms. The ingredients required for this polarization include the followings:

- We only polarize the object-level, because we consider the interaction of staging and effectful computation is of special interests.
- The premise M of quotation ‘M should be negative, as quotation must be performable regardless of the effects involved in the object-level term.
- $\sqsubset$-ed types are positive, since they have pattern-matching syntax.

By these arguments, the composite $\sqsubset\sqsupset$ sends negative terms to positive terms. Interestingly, this behavior coincides with the linear exponential modality ! in polarized linear logic [14]. Since ! is a special kind of the S4 modality, this result justifies our polarization.

For the polarized adjoint calculus, we give the operational big-step semantics in **Fig. 9** instead of the equality. Here, M denotes a closed object-level computation (i.e., a term of a negative type) and T denotes a terminal computation, which is defined as follows.

$$\frac{}{\Theta ; \Gamma^+, x : A^+ \vdash^0 x : A^+} \qquad \frac{\Theta ; \Gamma^+, x : A^+ \vdash^0 M : B^-}{\Theta ; \Gamma^+ \vdash^0 \lambda x. M : A^+ \to B^-} \qquad \frac{\Theta ; \Gamma^+ \vdash^0 V : A^+ \qquad \Theta ; \Gamma^+ \vdash^0 M : A^+ \to B^-}{\Theta ; \Gamma^+ \vdash^0 MV : B^-}$$

$$\frac{\Theta ; \Gamma^+ \vdash^0 M_i : A_i^- \quad (\forall i \in I)}{\Theta ; \Gamma^+ \vdash^0 \{i. M_i\}_{i\in I} : \prod_{i\in I} A_i^-} \qquad \frac{\Theta ; \Gamma^+ \vdash^0 M : \prod_{i\in I} A_i^-}{\Theta ; \Gamma^+ \vdash^0 \pi_i M : A_i^-}$$

$$\frac{\Theta ; \Gamma^+ \vdash^0 V : A^+ \qquad \Theta ; \Gamma^+ \vdash^0 W : B^+}{\Theta ; \Gamma^+ \vdash^0 (V, W) : A^+ \times B^+} \qquad \frac{\Theta ; \Gamma^+ \vdash^0 V : A^+ \times B^+ \qquad \Theta ; \Gamma^+, x : A^+, y : B^+ \vdash^0 M : C^-}{\Theta ; \Gamma^+ \vdash^0 \mathtt{let}\ (x, y)\ \mathtt{be}\ V\ \mathtt{in}\ M : C^-}$$

$$\frac{}{\Theta ; \Gamma^+ \vdash^0 () : 1} \qquad \frac{\Theta ; \Gamma^+ \vdash^0 V : 1 \qquad \Theta ; \Gamma^+ \vdash^0 M : A^-}{\Theta ; \Gamma^+ \vdash^0 \mathtt{let}\ ()\ \mathtt{be}\ V\ \mathtt{in}\ M : A^-}$$

$$\frac{\Theta ; \Gamma^+ \vdash^0 V : A_i^+}{\Theta ; \Gamma^+ \vdash^0 \iota_i V : \sum_{i\in I} A_i^+} \qquad \frac{\Theta ; \Gamma^+ \vdash^0 V : \sum_{i\in I} A_i^+ \qquad \Theta ; \Gamma^+, x_i : A_i^+ \vdash^0 M_i : C^- \quad (\forall i \in I)}{\Theta ; \Gamma^+ \vdash^0 \mathtt{case}\ V\ \mathtt{of}\ \{i.x_i. M_i\}_{i\in I} : C^-}$$

$$\frac{\Theta ; \Gamma^+ \vdash^0 V : A^+}{\Theta ; \Gamma^+ \vdash^0 \mathtt{val}\, V : \uparrow A^+} \qquad \frac{\Theta ; \Gamma^+ \vdash^0 M : \uparrow A^+ \qquad \Theta ; \Gamma^+, x : A^+ \vdash^0 N : B^-}{\Theta ; \Gamma^+ \vdash^0 \mathtt{let}\ \mathtt{val}\, x\ \mathtt{be}\ M\ \mathtt{in}\ N : B^-}$$

$$\frac{\Theta ; \Gamma^+ \vdash^0 M : A^-}{\Theta ; \Gamma^+ \vdash^0 \mathtt{thunk}\, M : \downarrow A^-} \qquad \frac{\Theta ; \Gamma^+ \vdash^0 V : \downarrow A^-}{\Theta ; \Gamma^+ \vdash^0 \mathtt{force}\, V : A^-}$$

$$\frac{\Theta ; \cdot \vdash^0 M : A^-}{\Theta \vdash^1 \text{‘}M : \sqsupset A^-} \qquad \frac{\Theta \vdash^1 P : \sqsupset A^-}{\Theta ; \Gamma^+ \vdash^0 ,P : A^-}$$

$$\frac{\Theta \vdash^1 P : \Phi}{\Theta ; \Gamma^+ \vdash^0 \mathtt{meta}\, P : \sqsubset \Phi} \qquad \frac{\Theta ; \Gamma^+ \vdash^0 V : \sqsubset \Phi \qquad \Theta, \alpha : \Phi ; \Gamma^+ \vdash^0 M : A^-}{\Theta ; \Gamma^+ \vdash^0 \mathtt{let}\ \mathtt{meta}\, \alpha\ \mathtt{be}\ V\ \mathtt{in}\ M : A^-}$$

$$\frac{}{\Theta, \alpha : \Phi \vdash^1 \alpha : \Phi} \qquad \frac{\Theta, \alpha : \Phi \vdash^1 P : \Psi}{\Theta \vdash^1 \lambda\alpha. P : \Phi \to \Psi} \qquad \frac{\Theta \vdash^1 P : \Phi \to \Psi \qquad \Theta \vdash^1 Q : \Phi}{\Theta \vdash^1 PQ : \psi}$$

Fig. 8 Typing rules of polarized adjoint calculus.

$$\frac{}{\lambda x. M \Downarrow \lambda x. M} \qquad \frac{M \Downarrow \lambda x. N \qquad N[V/x] \Downarrow T}{MV \Downarrow T}$$

$$\frac{}{\{i. M_i\}_{i\in I} \Downarrow \{i. M_i\}_{i\in I}} \qquad \frac{M \Downarrow \{i. N_i\}_{i\in I} \qquad N_i \Downarrow T}{\pi_i M \Downarrow T}$$

$$\frac{M[V, W/x, y] \Downarrow T}{\mathtt{let}\ (x, y)\ \mathtt{be}\ (V, W)\ \mathtt{in}\ M \Downarrow T} \qquad \frac{M \Downarrow T}{\mathtt{let}\ ()\ \mathtt{be}\ ()\ \mathtt{in}\ M \Downarrow T}$$

$$\frac{M_i[V/x_i] \Downarrow T}{\mathtt{case}\ \iota_i V\ \mathtt{of}\ \{i.x_i. M_i\}_{i\in I} \Downarrow T}$$

$$\frac{}{\mathtt{val}\, V \Downarrow \mathtt{val}\, V} \qquad \frac{M \Downarrow \mathtt{val}\, V \qquad N[V/x] \Downarrow T}{\mathtt{let}\ \mathtt{val}\, x\ \mathtt{be}\ M\ \mathtt{in}\ N \Downarrow T}$$

$$\frac{M \Downarrow T}{\mathtt{force}\ \mathtt{thunk}\, M \Downarrow T}$$

$$\frac{P \to_\beta^* \text{‘}M \qquad M \Downarrow T}{,P \Downarrow T} \qquad \frac{M[P/\alpha] \Downarrow T}{\mathtt{let}\ \mathtt{meta}\, \alpha\ \mathtt{be}\ \mathtt{meta}\, P\ \mathtt{in}\ M \Downarrow T}$$

Fig. 9 Big-step semantics of polarized adjoint calculus.

$$T ::= \mathtt{val}\, V \mid \{i. M_i\}_{i\in I} \mid \lambda x. M$$

Note that, because of subject reduction, any computation does not reduce to a value. $\to_\beta^*$ in the premise of $,P \Downarrow T$ denotes the reflexive transitive closure of β-reductions for meta-level abstractions. The intention behind this definition is that we are only interested in object-level computations and all meta-level β-redexes are reasonably assumed to be reduced beforehand.

4.2 From Adjoint Calculus to Polarized Adjoint Calculus

We show that polarized adjoint calculus indeed works as a *subsuming language*, in the sense that we can faithfully embed (additive) adjoint calculus into polarized adjoint calculus without breaking observational equivalence in effectful settings. To do this, we first define the call-by-name semantics of adjoint calculus by **Fig. 10**. Following Levy, bool and sum types are added to the adjoint calculus here. The typing rules of bool/sum types are omitted because they are routine. Similarly to the polarized one, we assumed meta-level reductions should operate before object-level evaluations start.

To show that a language is a refinement of another language, it is natural to construct a translation from the source language to the target language. However, it is argued by Levy that a term translation function from call-by-name to call-by-push-value does not behave very well. Instead, we introduce a *relation* on terms from the call-by-name adjoint calculus to the polarized adjoint calculus (**Fig. 11**). The translation is extended also to meta-level terms, because we have to treat terms involving meta-level terms, i.e., $,P$. Meta-level terms are translated in the obvious way. If we define the translation on object-level types properly, $M \mapsto^{\mathrm{n}} M'$ and $\Theta ; \Gamma \vdash M : A$ imply $\widetilde{\Theta}^- ; \downarrow\widetilde{\Gamma}^- \vdash M' : \widetilde{A}^-$.

$$\widetilde{A \to B}^- := \downarrow\widetilde{A}^- \to \widetilde{B}^-$$
$$\widetilde{\textstyle\sum A_i}^- := \uparrow(\textstyle\sum \downarrow\widetilde{A_i}^-)$$
$$\widetilde{\mathtt{bool}}^- := \uparrow(1 + 1)$$

Our translation is justified by the following propositions. Note that we need to shift polarities for preservation of substitutions.

Proposition 4. *If $M \mapsto^{\mathrm{n}} M'$ and $N \mapsto^{\mathrm{n}} N'$ then $M[N/x] \mapsto^{\mathrm{n}} M'[\mathtt{thunk}\, N'/x]$.*

Lemma 5. *If $P \to_\beta Q$ with respect to meta-level arrows and $P \mapsto^{\mathrm{n}} P'$, then there exists Q' such that $P' \to_\beta Q'$ with respect to*

$$\frac{}{\lambda x.\, M \Downarrow \lambda x.\, M} \qquad \frac{M \Downarrow \lambda x.\, L \qquad L[N/x] \Downarrow V}{MN \Downarrow V}$$

$$\frac{}{\iota_i M \Downarrow \iota_i M} \qquad \frac{M \Downarrow \iota_i L \qquad N_i[L/x_i] \Downarrow V}{\texttt{case}\ M\ \texttt{of}\ \{i.x_i.\, N_i\}_{i\in I} \Downarrow V}$$

$$\frac{}{\texttt{true} \Downarrow \texttt{true}} \qquad \frac{}{\texttt{false} \Downarrow \texttt{false}}$$

$$\frac{M \Downarrow \texttt{true} \qquad N \Downarrow V}{\texttt{if}\ M\ \texttt{then}\ N\ \texttt{else}\ L \Downarrow V} \qquad \frac{M \Downarrow \texttt{false} \qquad L \Downarrow V}{\texttt{if}\ M\ \texttt{then}\ N\ \texttt{else}\ L \Downarrow V}$$

$$\frac{P \to_\beta^* \text{‘}N \qquad N \Downarrow V}{{,}P \Downarrow V}$$

$$\frac{}{\texttt{meta}\, P \Downarrow \texttt{meta}\, P} \qquad \frac{M \Downarrow \texttt{meta}\, P \qquad N[P/\alpha] \Downarrow V}{\texttt{let meta}\, \alpha\ \texttt{be}\ M\ \texttt{in}\ N \Downarrow V}$$

Fig. 10 Big-step semantics of call-by-name adjoint calculus.

$$\frac{M \mapsto^{\mathrm{n}} M'}{M \mapsto^{\mathrm{n}} \texttt{force}\,\texttt{thunk}\, M'}$$

$$\frac{}{x \mapsto^{\mathrm{n}} \texttt{force}\, x} \qquad \frac{}{\texttt{true} \mapsto^{\mathrm{n}} \texttt{val}\, \iota_1()} \qquad \frac{}{\texttt{false} \mapsto^{\mathrm{n}} \texttt{val}\, \iota_0()}$$

$$\frac{M \mapsto^{\mathrm{n}} M' \qquad N_1 \mapsto^{\mathrm{n}} N_1' \qquad N_0 \mapsto^{\mathrm{n}} N_0'}{\texttt{if}\ M\ \texttt{then}\ N_1\ \texttt{else}\ N_0 \mapsto^{\mathrm{n}} \texttt{let val}\, z\ \texttt{be}\ M'\ \texttt{in case}\ z\ \texttt{of}\ \{i.\_.\, N_i'\}_{i<2}}$$

$$\frac{M \mapsto^{\mathrm{n}} M'}{\iota_i M \mapsto^{\mathrm{n}} \texttt{val}\, \iota_i(\texttt{thunk}\, M')}$$

$$\frac{M \mapsto^{\mathrm{n}} M' \qquad N_i \mapsto^{\mathrm{n}} N_i'}{\texttt{case}\ M\ \texttt{of}\ \{i.x_i.\, N_i\} \mapsto^{\mathrm{n}} \texttt{let val}\, z\ \texttt{be}\ M'\ \texttt{in case}\ z\ \texttt{of}\ \{i.x_i.\, N_i'\}}$$

$$\frac{M \mapsto^{\mathrm{n}} M'}{\lambda x.\, M \mapsto^{\mathrm{n}} \lambda x.\, M'} \qquad \frac{M \mapsto^{\mathrm{n}} M' \qquad N \mapsto^{\mathrm{n}} N'}{MN \mapsto^{\mathrm{n}} M'(\texttt{thunk}\, N')}$$

$$\frac{}{\alpha \mapsto^{\mathrm{n}} \alpha} \qquad \frac{P \mapsto^{\mathrm{n}} P'}{\lambda\alpha.\, P \mapsto^{\mathrm{n}} \lambda\alpha.\, P'} \qquad \frac{P \mapsto^{\mathrm{n}} P' \qquad Q \mapsto^{\mathrm{n}} Q'}{PQ \mapsto^{\mathrm{n}} P'Q'}$$

$$\frac{M \mapsto^{\mathrm{n}} M'}{\text{‘}M \mapsto^{\mathrm{n}} \text{‘}M'} \qquad \frac{P \mapsto^{\mathrm{n}} P'}{{,}P \mapsto^{\mathrm{n}} {,}P'}$$

$$\frac{P \mapsto^{\mathrm{n}} P'}{\texttt{meta}\, P \mapsto^{\mathrm{n}} \texttt{val}\,\texttt{meta}\, P'} \qquad \frac{M \mapsto^{\mathrm{n}} M' \qquad N \mapsto^{\mathrm{n}} N'}{\texttt{let meta}\, \alpha\ \texttt{be}\ M\ \texttt{in}\ N \mapsto^{\mathrm{n}} \texttt{let val}\, z\ \texttt{be}\ M'\ \texttt{in let meta}\, \alpha\ \texttt{be}\ z\ \texttt{in}\ N'}$$

Fig. 11 Relation between cbn adjoint calculus and polarized adjoint calculus.

meta-level arrows and $Q \mapsto^{\mathrm{n}} Q'$.

Proposition 6. $\mapsto^{\mathrm{n}}$ *is a bisimulation:*

(1) If $M \Downarrow V$ *and* $M \mapsto^{\mathrm{n}} M'$*, then there exists* T *such that* $M' \Downarrow T$ *and* $V \mapsto^{\mathrm{n}} T$.

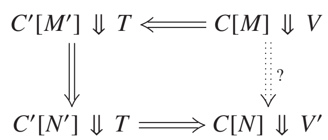
(2) If $M' \Downarrow T$ *and* $M \mapsto^{\mathrm{n}} M'$*, then there exists* V *such that* $M \Downarrow V$ *and* $V \mapsto^{\mathrm{n}} T$.

Proof. For *(1)*, similarly to Levy's proof, perform induction primarily on $M \Downarrow V$ and secondarily on $M \mapsto^{\mathrm{n}} M'$. The main difference with the original proof is the case of ${,}P$. For that case, use the previous lemma and the fact that any closed term P of the $\sqsupset$ type is normalizable to a term of the form ‘M. *(2)* is shown by induction on $M' \Downarrow T$ and the lemma. □

This proposition is straightforwardly extended to various effectful settings such as non-terminating computation and computation with outputs. Then the proposition enables us to show that observational equivalence, or more generally observational inequality, is reflected by the translation relation in the following sense:

if $M' \lesssim N'$ then $M \lesssim N$ for any $M \mapsto^{\mathrm{n}} M'$ and $N \mapsto^{\mathrm{n}} N'$.

In order to prove observational inequality, we need to show that for any context $C[\cdot]$ of some base type $C[M] \Downarrow V$ implies $C[N] \Downarrow V$. Because the translation relation $\mapsto^{\mathrm{n}}$ is a bisimulation, any $M \Downarrow V$ in adjoint calculus is sent to and pulled back from polarized adjoint calculus. Finally it suffices to check that for any terminal computation T, any value V such that $V \mapsto^{\mathrm{n}} T$ is uniquely determined, and this is easy.



We also show that the call-by-value adjoint calculus is similarly translated into polarized adjoint calculus. Again, we follow the arguments by Levy discussed in Ref. [15]. The big-step semantics for the call-by-value is obtained by replacing the rules for the call-by-name with the followings.

$$\frac{M \Downarrow \lambda x.\, L \qquad N \Downarrow W \qquad L[W/x] \Downarrow V}{MN \Downarrow V} \qquad \frac{M \Downarrow W}{\iota_i M \Downarrow \iota_i W}$$

Also, we define a translation using not functions but relations to let it preserve substitutions. Unlike the call-by-name translation, for call-by-value translation, we are required to make a distinction between values and computations on treating terms. The translation is thus comprised of two relations $\mapsto^{\mathrm{v}}$ and $\mapsto^{\texttt{val}}$ (**Fig. 12**).

Again, we define the translation on types for call-by-value as follows.

$$\widetilde{A \to B}^{+} := {\downarrow}(\widetilde{A}^{+} \to {\uparrow}\widetilde{B}^{+})$$
$$\widetilde{\textstyle\sum A_i}^{+} := \textstyle\sum \widetilde{A_i}^{+}$$
$$\widetilde{\texttt{bool}}^{+} := 1 + 1$$

Then, $M \mapsto^{\mathrm{v}} M'$ and $\Theta\,;\,\Gamma \vdash M : A$ imply $\widetilde{\Theta}^{+}\,;\,\widetilde{\Gamma}^{+} \vdash M' : {\uparrow}\widetilde{A}^{+}$.

$$\frac{M \mapsto^{\mathtt{v}} \mathtt{let\ val}\ x\ \mathtt{be\ val}\ V\ \mathtt{in\ val}\ \iota_i x}{M \mapsto^{\mathtt{v}} \mathtt{val}\ \iota_i V}$$

$$\frac{}{x \mapsto^{\mathtt{v}} \mathtt{val}\ x} \qquad \frac{}{\mathtt{true} \mapsto^{\mathtt{v}} \mathtt{val}\ \iota_1()} \qquad \frac{}{\mathtt{false} \mapsto^{\mathtt{v}} \mathtt{val}\ \iota_0()}$$

$$\frac{M \mapsto^{\mathtt{v}} M' \qquad N_1 \mapsto^{\mathtt{v}} N_1' \qquad N_0 \mapsto^{\mathtt{v}} N_0'}{\mathtt{if}\ M\ \mathtt{then}\ N_1\ \mathtt{else}\ N_0 \mapsto^{\mathtt{v}} \mathtt{let\ val}\ z\ \mathtt{be}\ M'\ \mathtt{in\ case}\ z\ \mathtt{of}\ \{i.\_.\ N_i'\}_{i<2}}$$

$$\frac{M \mapsto^{\mathtt{v}} M'}{\iota_i M \mapsto^{\mathtt{v}} \mathtt{let\ val}\ z\ \mathtt{be}\ M'\ \mathtt{in\ val}\ \iota_i z}$$

$$\frac{M \mapsto^{\mathtt{v}} M' \qquad N_i \mapsto^{\mathtt{v}} N_i'}{\mathtt{case}\ M\ \mathtt{of}\ \{i.x_i.\ N_i\} \mapsto^{\mathtt{v}} \mathtt{let\ val}\ z\ \mathtt{be}\ M'\ \mathtt{in\ case}\ z\ \mathtt{of}\ \{i.x_i.\ N_i'\}}$$

$$\frac{M \mapsto^{\mathtt{v}} M'}{\lambda x.\ M \mapsto^{\mathtt{v}} \mathtt{val\ thunk}\ \lambda x.\ M'} \qquad \frac{M \mapsto^{\mathtt{v}} M' \qquad N \mapsto^{\mathtt{v}} N'}{MN \mapsto^{\mathtt{v}} \mathtt{let\ val}\ f\ \mathtt{be}\ M'\ \mathtt{in\ let\ val}\ x\ \mathtt{be}\ N'\ \mathtt{in}\ (\mathtt{force}\ f)x}$$

$$\frac{}{\alpha \mapsto^{\mathtt{v}} \alpha} \qquad \frac{P \mapsto^{\mathtt{v}} P'}{\lambda\alpha.\ P \mapsto^{\mathtt{v}} \lambda\alpha.\ P'} \qquad \frac{P \mapsto^{\mathtt{v}} P' \qquad Q \mapsto^{\mathtt{v}} Q'}{PQ \mapsto^{\mathtt{v}} P'Q'}$$

$$\frac{M \mapsto^{\mathtt{v}} M'}{`M \mapsto^{\mathtt{v}} `M'} \qquad \frac{P \mapsto^{\mathtt{v}} P'}{,P \mapsto^{\mathtt{v}} ,P'}$$

$$\frac{P \mapsto^{\mathtt{v}} P'}{\mathtt{meta}\ P \mapsto^{\mathtt{v}} \mathtt{val\ meta}\ P'} \qquad \frac{M \mapsto^{\mathtt{v}} M' \qquad N \mapsto^{\mathtt{v}} N'}{\mathtt{let\ meta}\ \alpha\ \mathtt{be}\ M\ \mathtt{in}\ N \mapsto^{\mathtt{v}} \mathtt{let\ val}\ z\ \mathtt{be}\ M'\ \mathtt{in\ let\ meta}\ \alpha\ \mathtt{be}\ z\ \mathtt{in}\ N'}$$

$$\frac{}{x \mapsto^{\mathtt{val}} x} \qquad \frac{}{\mathtt{true} \mapsto^{\mathtt{val}} \iota_1()} \qquad \frac{}{\mathtt{false} \mapsto^{\mathtt{val}} \iota_0()}$$

$$\frac{V \mapsto^{\mathtt{val}} V'}{\iota_i V \mapsto^{\mathtt{val}} \iota_i V'}$$

$$\frac{M \mapsto^{\mathtt{v}} M'}{\lambda x.\ M \mapsto^{\mathtt{val}} \mathtt{thunk}\ \lambda x.\ M'} \qquad \frac{P \mapsto^{\mathtt{v}} P'}{\mathtt{meta}\ P \mapsto^{\mathtt{val}} \mathtt{meta}\ P'}$$

Fig. 12 Relations between cbv adjoint calculus and polarized adjoint calculus.

We then claim a result analogous to the one for call-by-name. The proof is a straightforward extension in Ref. [15]. The definition of safe terms should be extended to include, for example, all meta-terms consisting only of object-terms in the form of $`S$ where S is safe.

Proposition 7. $\mapsto^{\mathtt{v}}$ *and* $\mapsto^{\mathtt{val}}$ *satisfy the following bisimulation-like condition.*

(1) If $M \Downarrow V$ *and* $M \mapsto^{\mathtt{v}} M'$*, then there exists* V' *such that* $M' \Downarrow \mathtt{val}\ V'$ *and* $V \mapsto^{\mathtt{val}} V'$*.*

(2) If $M' \Downarrow \mathtt{val}\ V'$ *and* $M \mapsto^{\mathtt{v}} M'$*, then there exists* V *such that* $M \Downarrow V$ *and* $V \mapsto^{\mathtt{val}} V'$*.*

5. Concluding Remarks

For intuitionistic normal modal logic, kinds of deductive systems have been proposed. In this paper, we focuses on two natural deduction systems, the Fitch-style system and the dual-context system. Fitch-style and dual-context have been developed separately, and any direct relationship was not known. This paper provides a clear connection between Fitch-style and dual-context: the box of the dual-context is a left adjoint of the box of the Fitch-style. Since any monoidal adjunction derives a monoidal comonad, the composite of those two modalities becomes a S4 modality. As a corollary of our results, we can obtain decomposition of Nanevski et al.'s contextual S4 modality.

Moreover, we have shown that polarity makes the distinction between the two modalities clearer. Our result claims that the Fitch-style box accepts a negative type and the dual-context box creates a positive type in the polarized adjoint calculus. This result is compatible with the fact that the linear exponential sends a negative formula to a positive formula in polarized linear logic. The underlying idea on the polarized adjoint calculus is the comparison of a monoidal adjunction with a Freyd category, which is a model of the call-by-push-value calculus.

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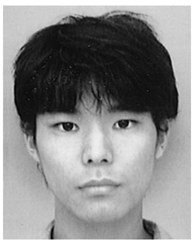

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