An Improved Algorithm for Uniform Page Migration on Euclidean Space

Akira Matsubayashi[1,a)]

Abstract: This report addresses the classical page migration problem on Euclidean space. It is known that there is a 2.75-competitive deterministic online page migration algorithm on Euclidean space under the uniform model. We extend this algorithm and improve the competitive ratio to 2.5723. Although there still exists a slight gap between our competitive ratio and the currently known lower bound 2.5, our result significantly narrows the previous gap. We also demonstrate that our analysis of the competitive ratio is nearly tight.

Keywords: page migration, Euclidean space, online algorithm, competitive analysis

1. Introduction

The *page migration problem* is one of the most classic online problems extensively studied for 30 years. In this problem, we manage the position of a data object, called a *page*, in a metric space M. Online requests are a sequence of points in M. Every time we receive a request $r \in M$, we are required to serve the request r by paying a cost of the distance between r and the current position of the page. Then, we are asked to determine the next position of the page. I.e., we may move the page to another position by paying the cost of the distance between the current and next positions multiplied by the page size D. Typically D is a positive integer because the page generally models a collection of D data pieces of the unit size that can be accessed at a time. The objective is to minimize the total sum of service costs and the migration costs. Although this problem was introduced for an application of efficiently managing of multiple cache memories of a multiprocessor system, we can apply the problem to more general frameworks of dynamically allocating a date object shared on a network of various kinds. See [2] for a survey such as other variants of the page migration problem. The formal definition of the page migration problem, as well as the definitions of other notions, such as online algorithms and competitive ratio, are provided in Sect. 2.

In this report, we focus on Euclidean space as the underlying metric space. Considering continuous metric spaces should be important to model wireless networks. Several results on n-dimensional real space $\mathbb{R}^n$ were proved by Chrobak et al. [5]. In particular, they presented a $(2+\frac{1}{2D})$-competitive deterministic algorithm on $\mathbb{R}^n$ under L^1 norm, and a $c(D)$-competitive deterministic algorithm on $\mathbb{R}^n$ under any norm, where $c(D)$ is a function such that $c(1) = 2.8$ and tends to $\frac{3+\sqrt{5}}{2} \approx 2.6180$ as D grows large. Since they also proved that $2+\frac{1}{2D}$ is a lower bound on $\mathbb{R}^1$ (hence, even on $\mathbb{R}^n$), their algorithm under L^1 metric is optimal. However, it is open for over 20 years whether we can close the gap between upper and lower bounds under L^p norm with $p \geq 2$. Recently, under *uniform model*, i.e., $D = 1$, and L^2 norm, the competitive ratio was improved from $c(1) = 2.8$ to 2.75 [6]. Although the uniform model is a significant restriction, we know an interesting case (actually, only three points) such that the competitive ratio is not monotonic with respect to D [10]. This would imply different principles between small and large D in the page migration problem. Therefore, studying the uniform model, as well as large D, would be important for our deeper understanding of the page migration problem and possibly other problems.

In this report, we further improve this ratio to 2.5723 on $\mathbb{R}^n$ under L^2 norm and uniform model. Our ratio narrows the previous gap between 2.75 and $2 + \frac{1}{2} = 2.5$ for $D = 1$ significantly, although a slight gap still remains. We do not know whether the general lower bound can be improved; however, we demonstrate that our analysis of the competitive ratio is nearly tight.

The algorithm of [6] is extremely simple. It maintains two points, called *counters* here, always located at two of previous requests, and keeps its page at the midpoint of the counters. The similar idea (often implicitly) was used in many other algorithms. Actually, $(2 + \frac{1}{2D})$-competitive algorithms under L^1 norm works in a similar way for $D = 1$. We modify in our algorithm the way of maintaining the counters by introducing a new rule to locate a counter not necessarily at a previous request. Previous counter-based algorithms "count" the number of requests that are thought to be important to determine the new page position. Our algorithm generalize the idea to "mark" positions that are thought to be important to determine the new page position, without regard to previous requests. Specific definition of our algorithm is provided in Sect. 3

1 Division of Electrical Engineering and Computer Science, Kanazawa University Kakuma-machi, Kanazawa, 920–1192 Japan
a) mbayashi@t.kanazawa-u.ac.jp

1.1 Other related results

Competitive analysis of the page migration problem was first studied by Black and Sleator [4]. They presented tight 3-competitive deterministic algorithms on trees, uniform networks, and their Cartesian products. The first lower bound larger than 3 for general networks under uniform model was found by Chrobak et al. [5], who proved that there is a network (of arbitrarily large size) on which no deterministic algorithm achieves a competitive ratio smaller than $85/27 \approx 3.148$. This lower bound was improved to 3.1639 [8]. The first lower bound of $3 + \Omega(1)$, where Ω-notation is with respect to D, under non-uniform model was presented in [9]. Although the best deterministic algorithm on general metric spaces had been 4.086-competitive for over 20 years [1], very recently, this bound is improved to 4 by Bienkowski et al. [3]. For uniform model, there exists a better, $(2 + \sqrt{2}) \approx 3.4142$-competitive algorithm [8]. For three points, there are 3-competitive algorithms for $D = 1, 2$ [5], [10] and $(3 + \Theta(1/D))$-competitive algorithm for any $D \geq 3$ [10].

For randomized algorithms, there is $c(D)$- and 3-competitive algorithms against oblivious and adaptive online adversaries, respectively, for any metric space [11]. The latter algorithm is optimal. For trees [5] and uniform networks [7], there are tight $(2 + \frac{1}{2D})$-competitive algorithms against oblivious adversaries.

2. Definitions and Notations

Let M be a metric space with metric $\|uv\|$ for $p, q \in M$. In the *page migration problem*, we are given requests $r_1, \ldots, r_k \in M$, the initial position $s_0 \in M$ of the page of size $D \in \mathbb{Z}^+$, and asked to find $s_1, \ldots, s_k$ such that the total cost $\sum_{i=1}^{k}(\|s_{i-1}r_i\| + D\|s_{i-1}s_i\|)$ is minimized. An online algorithm Alg receives the requests online, i.e., r_i after r_{i-1}, pays the cost $\|s_{i-1}r_i\|$ to serve r_i, and then determines the next position s_i of the page before r_{i+1} arrives. The decision of s_i is irrevocable, i.e., Alg cannot change s_i after r_i arrives. The migration costs $D\|s_{i-1}s_i\|$. We denote an optimal offline algorithm, i.e, which knows all the requests in advance, by Opt. We denote the cost of an algorithm A by C_A. For a deterministic online algorithm Alg, if there exists a constant α, i.e., independent of the number of requests, such that $C_{\text{Alg}} \leq \rho \cdot C_{\text{Opt}} + \alpha$ for any inputs, then Alg is said to be *ρ-competitive*.

In the remainder of the report, we consider n-dimensional Euclidean space with $n \geq 2$ as the metric space. I.e., $\|uv\|$ is the Euclidean distance (L^2 norm) between u and v.

3. Algorithm and Competitiveness

3.1 Algorithm Mark

Our Algorithm, called *Mark*, maintains two points p and q on Euclidean Space as our markers, in such a way that our page s is always located at the midpoint of p and q. Markers p and q may be located at the same point; in particular, they coincide initially. Suppose we have markers p, q and our page s at the midpoint of the markers. After a request r is issued, we perform the following steps:

(1) Case $\|pr\| \geq \|qr\|$.

  (a) If

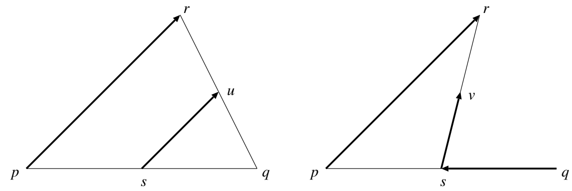

Fig. 1 Algorithm Mark.

$$\|sr\| - \frac{\rho - 1}{2}\|pr\| - \left(\frac{\rho}{2} - 1\right)\|qr\| \leq \frac{5}{2}\|sr\| - \frac{\rho}{2}(\|pr\| + \|qr\|) \quad (1)$$

  with $\rho = 2.5753$, then move p to r, and s to the midpoint u of r and q.

  (b) Otherwise, move p to r, q to s, and then s to the midpoint v of r and the original s.

  See Fig. 1 for these operations.

(2) Case $\|pr\| < \|qr\|$. We interchange p and q and perform Step 1.

Actually, Mark is obtained from the algorithm of [6] (Step (a)) by adding Inequality (1) and Step (b).

3.2 Competitiveness

We prove the competitiveness of Mark using the standard potential function method. The basic framework of our proof (such as our potential function and events) is the same as that of [6]. Our potential function is defined as

$$\Phi := \frac{\rho}{2}(\|tp\| + \|tq\|) - \left(\frac{\rho}{2} - 1\right)\|pq\|,$$

where t is the page of Opt. Our goal is to prove that

$$f := \Delta C_{\text{Mark}} + \Delta\Phi - \rho \cdot \Delta C_{\text{Opt}} \leq 0 \quad (2)$$

for any event of the following types:

- Any Opt's migration.
- Services of Opt and Mark for a request r, together with migration of Mark after the request r.

Here, we use the symbol Δ to denote the amount changed by an event. By the symmetry of Φ with respect to p and q, and by the definition of Mark, we may assume $\|pr\| \geq \|qr\|$ and prove $f \leq 0$ only for Step (1).

Lemma 1 ([6]) *For any Opt's migration, it follows that $f \leq 0$.*

Proof Suppose that Opt moves its page from t to t'. Then, $\Delta C_{\text{Mark}} = 0$, $\Delta\Phi = \frac{\rho}{2}(\|t'p\| - \|tp\| + \|t'q\| - \|tq\|)$, and $\Delta C_{\text{Opt}} = \|tt'\|$. Therefore, it follows that

$$\begin{aligned} f &= \frac{\rho}{2}(\|t'p\| - \|tp\| + \|t'q\| - \|tq\|) - \rho \cdot \|tt'\| \\ &\leq \frac{\rho}{2}(\|tt'\| + \|tt'\|) - \rho \cdot \|tt'\| = 0. \end{aligned}$$

□

We prove $f \leq 0$ for events of the other type by a series of lemmas.

Lemma 2 *For any point t, it follows that*

$$\|tp\| + \|tq\| + \|tr\| - \|ts\| \geq \|pr\| + \|qr\| - \|sr\|. \quad (3)$$

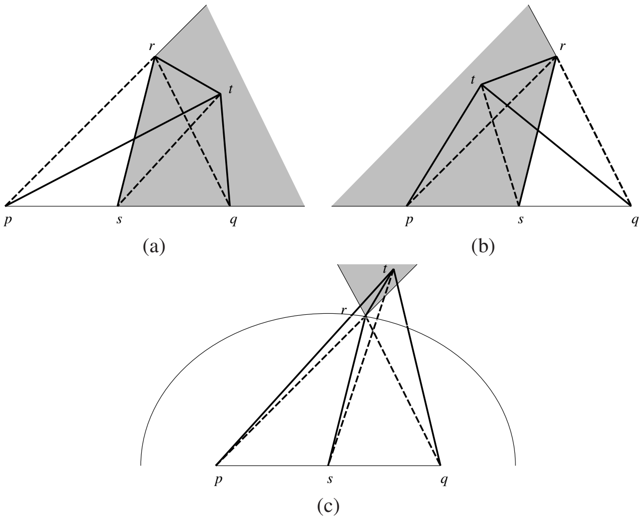

Fig. 2 Regions for t.

Proof If $s = p = q$, then (3) follows by the triangle inequality $\|ts\| + \|tr\| \geq \|sr\|$. Moreover, if r is on the line segment pq, then we obtain (3) by the triangle inequalities $\|tp\| + \|tq\| \geq \|pq\|$ and $\|tr\| - \|ts\| \geq -\|sr\|$. We therefore assume neither $s = p = q$ nor r is on the line segment pq. We note that the assumption $s = p = q$ implies $p \neq q$.

We fix p, q, r, and s. We regard t to be variable but fix $\|tp\|$ and $\|tq\|$ such that $\|tp\| + \|tq\| \geq \|pq\|$. Then, $\|ts\|$ is also fixed, and t lies at the distance $\|tp\| \sin \angle tpq = \|tq\| \sin \angle tqp$ to the line pq. If the dimension of Euclidean space is more than 2, then t is on the circumference C, on the plane orthogonal to the line pq, with radius $\|tp\| \sin \angle tpq = \|tq\| \sin \angle tqp$ and a center on the line pq. Since $\|tp\|$, $\|tq\|$, and $\|ts\|$ are fixed, $\|tp\| + \|tq\| + \|tr\| - \|ts\|$ is minimized with the minimum $\|tr\|$, i.e., t at the point closer to r of the two intersections of C and the plane P containing p, q, and r. Therefore, it is sufficient to consider t on the half-plane H of P divided by the line pq and containing r.

We divide the half-plane H into three regions of t as shown in Fig. 2 and prove (3) for each case of the regions containing t.

Case Fig. 2 (a). We can observe that

$$\|tq\| + \|tr\| \geq \|qr\| \text{ and } \|tp\| + \|sr\| \geq \|ts\| + \|pr\|.$$

Summing these inequalities, we have (3).

Case Fig. 2 (b). We can observe that

$$\|tp\| + \|tr\| \geq \|pr\| \text{ and } \|tq\| + \|sr\| \geq \|ts\| + \|qr\|.$$

Summing these inequalities, we have (3).

Case Fig. 2 (c). We can observe that

$$\|tr\| + \|sr\| \geq \|ts\|.$$

Because t is outside the ellipse consisting of points with the sum of the distances to p and q equal to $\|pr\| + \|qr\|$, it follows that

$$\|tp\| + \|tq\| \geq \|pr\| + \|qr\|.$$

Summing the obtained inequalities, we have (3).

□

Lemma 3 *For services of Opt and Mark for a request r, together with migration of Mark after the request r, $f = \min\{f_1, f_2\}$, where*

$$f_1 := \|sr\| - \frac{\rho - 1}{2}\|pr\| - \left(\frac{\rho}{2} - 1\right)\|qr\| + \left(\frac{\rho}{2} - 1\right)\|pq\|, \text{ and}$$
$$f_2 := \frac{5}{2}\|sr\| - \frac{\rho}{2}\left(\|pr\| + \|qr\|\right) + \left(\frac{\rho}{2} - 1\right)\|pq\|.$$

Proof Suppose that Mark and Opt locates their pages at s and t, respectively, at the point that the request r is issued. Then, $\Delta C_{\mathrm{Opt}} = \|tr\|$.

If Inequality (1) follows, i.e., $f_1 \leq f_2$, then p is moved to r, and then s is moved to the midpoint u of r and q. Hence, $\Delta C_{\mathrm{Mark}} = \|sr\| + \|su\|$ and $\Delta\Phi = \frac{\rho}{2}(\|tr\| - \|tp\|) - (\frac{\rho}{2} - 1)(\|qr\| - \|pq\|)$. Therefore, it follows that

$$\begin{aligned}
f &= \|sr\| + \|su\| + \frac{\rho}{2}\left(\|tr\| - \|tp\|\right) - \left(\frac{\rho}{2} - 1\right)\left(\|qr\| - \|pq\|\right) - \rho\|tr\| \\
&= \|sr\| + \frac{\|pr\|}{2} - \frac{\rho}{2}\left(\|tr\| + \|tp\|\right) - \left(\frac{\rho}{2} - 1\right)\left(\|qr\| - \|pq\|\right) \\
&\leq \|sr\| - \frac{\rho - 1}{2}\|pr\| - \left(\frac{\rho}{2} - 1\right)\|qr\| + \left(\frac{\rho}{2} - 1\right)\|pq\| \\
&= f_1 = \min\{f_1, f_2\}.
\end{aligned}$$

Here, $\|su\| = \|pr\|/2$ since s and r are midpoints of the line segments pq and qr.

If Inequality (1) is not satisfied, i.e., $f_1 > f_2$, then p, q, and s are moved to r, s, and the midpoint v of s and r, respectively. Hence, $\Delta C_{\mathrm{Mark}} = \|sr\| + \|sv\|$ and $\Delta\Phi = \frac{\rho}{2}(\|tr\| - \|tp\| + \|ts\| - \|tq\|) - (\frac{\rho}{2} - 1)(\|sr\| - \|pq\|)$. Therefore, it follows that

$$\begin{aligned}
f &= \|sr\| + \|sv\| + \frac{\rho}{2}\left(\|tr\| - \|tp\| + \|ts\| - \|tq\|\right) \\
&\quad - \left(\frac{\rho}{2} - 1\right)\left(\|sr\| - \|pq\|\right) - \rho\|tr\| \\
&\leq \frac{3}{2}\|sr\| - \frac{\rho}{2}\left(\|tr\| + \|tp\| + \|tq\| - \|ts\|\right) - \left(\frac{\rho}{2} - 1\right)\left(\|sr\| - \|pq\|\right) \\
&\leq \frac{3}{2}\|sr\| - \frac{\rho}{2}\left(\|pr\| + \|qr\| - \|sr\|\right) - \left(\frac{\rho}{2} - 1\right)\left(\|sr\| - \|pq\|\right) \\
&\qquad \text{[by Lemma 2]} \\
&\leq \frac{5}{2}\|sr\| - \frac{\rho}{2}\left(\|pr\| + \|qr\|\right) + \left(\frac{\rho}{2} - 1\right)\|pq\| \\
&= f_2 = \min\{f_1, f_2\}.
\end{aligned}$$

□

We want to prove $\min\{f_1, f_2\} < 0$. To this end, we set xy-coordinates on the plane with s at the origin and p and q on the x-axis. We may assume without loss of generality that $q = (-1, 0)$, $p = (1, 0)$, and $r = (\ell \cos\theta, \ell \sin\theta)$ with $\ell \geq 0$ and $0 \leq \theta \leq \pi/2$.

Lemma 4 ([6]) *For any $\ell \geq 0$, there exists $0 \leq \tau \leq \pi/2$ such that if $r \neq q$, then $\frac{\partial f_1}{\partial\theta} < 0$ for $0 \leq \theta < \tau$, and $\frac{\partial f_1}{\partial\theta} \geq 0$ for $\tau \leq \theta \leq \pi/2$.*

Proof It follows that

$$\|pr\| = \sqrt{(\ell\cos\theta + 1)^2 + (\ell\sin\theta)^2} = \sqrt{\ell^2 + 2\ell\cos\theta + 1},$$
$$\|qr\| = \sqrt{(\ell\cos\theta - 1)^2 + (\ell\sin\theta)^2} = \sqrt{\ell^2 - 2\ell\cos\theta + 1}.$$

Hence,

$$\frac{\partial \|pr\|}{\partial \theta} = \frac{-\ell \sin\theta}{\sqrt{\ell^2 + 2\ell\cos\theta + 1}} = -\frac{\ell\sin\theta}{\|pr\|},$$
$$\frac{\partial \|qr\|}{\partial \theta} = \frac{\ell \sin\theta}{\sqrt{\ell^2 - 2\ell\cos\theta + 1}} = \frac{\ell\sin\theta}{\|qr\|}.$$

Since

$$f_1 = \ell - \frac{\rho-1}{2}\|pr\| - \left(\frac{\rho}{2} - 1\right)\|qr\| + \rho - 2,$$

we have

$$\frac{\partial f_1}{\partial \theta} = -\frac{\rho-1}{2}\left(-\frac{\ell\sin\theta}{\|pr\|}\right) - \left(\frac{\rho}{2}-1\right)\frac{\ell\sin\theta}{\|qr\|}$$
$$= \frac{\rho-1}{2}\cdot\frac{\ell\sin\theta}{\|qr\|}\left(\frac{\|qr\|}{\|pr\|} - \frac{\rho-2}{\rho-1}\right).$$

The term $\frac{\|qr\|}{\|pr\|} = \frac{\sqrt{\ell^2-2\ell\cos\theta+1}}{\sqrt{\ell^2+2\ell\cos\theta+1}}$ monotonically increases from $\frac{|\ell-1|}{\ell+1}$ to 1 as θ increases from 0 to $\pi/2$. If $\frac{|\ell-1|}{\ell+1} \geq \frac{\rho-2}{\rho-1}$, then $\frac{\partial f_1}{\partial\theta} \geq 0$ for any $0 \leq \theta \leq \pi/2$, i.e., the lemma holds for $\tau = 0$. Otherwise, since $0 < \frac{\rho-2}{\rho-1} < 1$, there exists $0 < \tau < \pi/2$ such that $\frac{\sqrt{\ell^2-2\ell\cos\tau+1}}{\sqrt{\ell^2+2\ell\cos\tau+1}} = \frac{\rho-2}{\rho-1}$. It follows for such τ that $\frac{\partial f_1}{\partial\theta} < 0$ for $0 \leq \theta < \tau$, and $\frac{\partial f_1}{\partial\theta} \geq 0$ for $\tau \leq \theta \leq \pi/2$. □

Lemma 5 *For any $\ell \geq 0$, it follows that if $r \neq q$, then $\frac{\partial f_2}{\partial\theta} \leq 0$.*

Proof Since

$$f_2 = \frac{5}{2}\ell - \frac{\rho}{2}(\|pr\| + \|qr\|) + \rho - 2,$$

we have

$$\frac{\partial f_2}{\partial\theta} = -\frac{\rho}{2}\left(-\frac{\ell\sin\theta}{\|pr\|} + \frac{\ell\sin\theta}{\|qr\|}\right) = \frac{\rho}{2}\cdot\ell\sin\theta\left(\frac{1}{\|pr\|} - \frac{1}{\|qr\|}\right).$$

This is at most 0 by $\|pr\| \geq \|qr\|$. □

Lemma 6 $\min\{f_1, f_2\} < 0$ *for* $\rho = 2.5753$.

Proof By Lemmas 4 and 5, $\min\{f_1, f_2\}$ is maximized if r lies on the x-axis or yields $f_1 = f_2$. If r lies on the x-axis, then

$$\min\{f_1, f_2\} \leq f_1 = \ell - \frac{\rho-1}{2}(\ell+1) - \left(\frac{\rho}{2}-1\right)|\ell-1| + \rho - 2$$
$$\leq \ell - \frac{\rho-1}{2}(\ell+1) + \rho - 2$$
$$= \frac{2-\rho}{2}\cdot\ell + \frac{\rho-3}{2},$$

which is negative since $2 < \rho < 3$.

We assume r yields $f_1 = f_2$. In the proof for this case, we need quite involved derivation of formulae. Taking priority of comprehensibility of ideas, we omit some derivation of formulae.

A relation between ℓ and θ of points such that $f_1 = f_2$ is provided by

$$\frac{3}{2}\|sr\| - \frac{1}{2}\|pr\| - \|qr\|$$
$$= \frac{3}{2}\ell - \frac{1}{2}\sqrt{\ell^2 + 2\ell\cos\theta + 1} - \sqrt{\ell^2 - 2\ell\cos\theta + 1} = 0.$$

Solving this with respect to $\cos\theta$, we can obtain

$$\cos\theta = \frac{6}{25}\left(\sqrt{\ell^2+10} - \ell\right) + \frac{3}{10\ell}.$$

Substituting $3\|sr\| - 2\|qr\|$ for $\|pr\|$ and $\frac{6}{25}(\sqrt{\ell^2+10} - \ell + \frac{3}{10\ell})$ for $\cos\theta$ in f_1, we can obtain the following equation g of point such that $f_1 = f_2$:

$$g := \|sr\| - \frac{\rho-1}{2}\|pr\| - \left(\frac{\rho}{2}-1\right)\|qr\| + \rho - 2$$
$$= \|sr\| - \frac{\rho-1}{2}(3\|sr\| - 2\|qr\|) - \left(\frac{\rho}{2}-1\right)\|qr\| + \rho - 2$$
$$= -\frac{3\rho-5}{2}\|sr\| + \frac{\rho}{2}\|qr\| + \rho - 2$$
$$= -\frac{3\rho-5}{2}\ell + \frac{\rho}{2}\sqrt{\ell^2 - 2\ell\cos\theta + 1} + \rho - 2$$
$$= -\frac{3\rho-5}{2}\ell + \frac{\rho}{2}\sqrt{\ell^2 - 2\ell\left\{\frac{6}{25}\left(\sqrt{\ell^2+10} - \ell\right) + \frac{3}{10\ell}\right\} + 1}$$
$$+ \rho - 2$$
$$= -\frac{3\rho-5}{2}\ell + \frac{\rho}{10}\sqrt{37\ell^2 - 12\ell\sqrt{\ell^2+10} + 10} + \rho - 2$$

We note that g in the last form, in which $\cos\theta$ is replaced with a function of ℓ, has a value for all ℓ. However, it is not the case in the original form of g including $\cos\theta$ even for $\ell > 0$, due to $\cos\theta \leq 1$. Actually, $\frac{d}{d\ell}\cos\theta = \frac{6}{25}(\frac{\ell}{\ell^2+10} - 1) - \frac{3}{10\ell^2} < 0$ and $\cos\theta = 1$ (i.e., $\theta = 0$) at $\ell = 3/4$. This means that any point r with $f_1 = f_2$ has $\ell \geq 3/4$.

Verifying $g \to -\infty$ as $\ell \to \infty$ and $\frac{d^2 g}{d\ell^2} < 0$ for $\ell > \sqrt{2/7} \approx 0.535$, we see that g has a unique maximum value. Setting $\rho \geq 2.5753$ bounds the maximum value to a negative number. □

Theorem 1 *Mark is* 2.5753*-competitive.*

Proof By Lemmas 1, 3, and 6, $f \leq 0$ for any event and $\rho = 2.5753$. The potential function Φ is non-negative because $\Phi = \frac{\rho}{2}(\|tp\|+\|tq\|) - (\frac{\rho}{2}-1)\|pq\| \geq \frac{\rho}{2}\|pq\| - (\frac{\rho}{2}-1)\|pq\| = \|pq\| \geq 0$. Moreover, $\Phi = 0$ initially since $p = q = s = t$. Summing f overall events, we obtain $C_{\mathrm{Mark}} \leq \rho \cdot C_{\mathrm{Opt}}$. □

4. Lower Bound of Algorithm Mark

In this section, we demonstrate that our analysis of the competitiveness of Mark is nearly tight.

Theorem 2 *If Mark is ρ-compteitive, then $\rho \geq 2.5672$.*

Proof Let P be a plane with setting xy-coordinates arbitrarily. We assume without loss of generality that the initial positions of p, q, and s are at $(1, 0)$. Suppose that the first request is issued at $(-1, 0)$. Then, by the definition of Mark, p and s are moved to $(-1, 0)$ and $(0, 0)$, respectively. We issue the second request at the point r such that $\|qr\| = 2$ and r is on the point on g defined in the proof of Lemma 6, i.e., $f_1 = f_2$ for r. Since $f_1 = f_2$, Mark moves p to r and its page to the midpoint u of q and r. We generate the third request at $r' = (1, 0)$. We can observe by symmetry that $f_1 = f_2$ for r'. Hence, Mark moves p to r' and the page to the midpoint v of q and r'. See Fig. 3 for locations of u, v, r, and r'. We issue requests hereafter r and r' alternately. After the first request, Mark shuttles p between r and r', and its page between

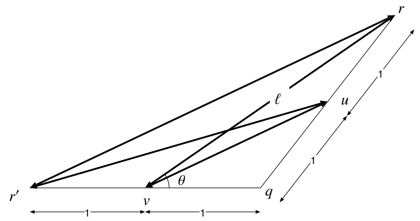

Fig. 3 Locations of u, v, r, and r'.

u and v.

Suppose that we issue k requests at each of r and r'. Let $\ell = \|vr\| = \|ur'\|$ and $r = (\ell\cos\theta, \ell\sin\theta)$ with $0 \le \theta \le \pi/2$. It follows that

$$C_{\mathrm{Mark}} = 3 + (\ell + \|uv\|)\,k = 3 + \left(\ell + \frac{\|rr'\|}{2}\right)k.$$

The optimal offline cost C_{Opt} is at most the offline cost of keeping its page at the initial position r'. Hence, it follows that

$$C_{\mathrm{Opt}} \le 2 + \|rr'\| \cdot \frac{k}{2}.$$

Combining these, we have

$$\begin{aligned}\frac{C_{\mathrm{Mark}}}{C_{\mathrm{Opt}}} &\ge \frac{3 + \left(\ell + \frac{\|rr'\|}{2}\right)k}{2 + \|rr'\| \cdot \frac{k}{2}} \to \frac{2\ell}{\|rr'\|} + 1 \quad (k \to \infty)\\ &= \frac{2\ell}{\sqrt{(\ell\cos\theta + 1)^2 + (\ell\sin\theta)^2}} + 1\\ &= \frac{2\ell}{\sqrt{\ell^2 + 2\ell\cos\theta + 1}} + 1 \qquad (4)\end{aligned}$$

Since $\|qr\| = 2$, it follows that

$$\|qr\| = \sqrt{(\ell\cos\theta - 1)^2 + (\ell\sin\theta)^2} = \sqrt{\ell^2 - 2\ell\cos\theta + 1} = 2,$$

which yields

$$\cos\theta = \frac{\ell^2 - 3}{2\ell}. \qquad (5)$$

On the other hand, as estimated in the proof of Lemma 6, we have

$$\cos\theta = \frac{6}{25}\left(\sqrt{\ell^2 + 10} - \ell\right) + \frac{3}{10\ell}. \qquad (6)$$

Solving (5) and (6), we obtain $\ell \approx 2.3204$ and $\cos\theta \approx 0.5137$. Substituting these values for ℓ and $\cos\theta$ in (4), we can verify that $C_{\mathrm{Mark}}/C_{\mathrm{Opt}} \ge 2.5672$. □

References

[1] Bartal, Y., Charikar, M. and Indyk, P.: On Page Migration and Other Relaxed Task Systems, *Theoret. Comput. Sci.*, Vol. 268, No. 1, pp. 43–66 (2001).

[2] Bienkowski, M.: Migrating and Replicating Data in Networks, *Comput. Sci. Res. Dev.*, Vol. 27, No. 3, pp. 169–179 (2012).

[3] Bienkowski, M., Byrka, J. and Mucha, M.: Dynamic Beats Fixed: On Phase-Based Algorithms for File Migration, *44th International Colloquium on Automata, Languages, and Programming (ICALP 2017)* (Chatzigiannakis, I., Indyk, P., Kuhn, F. and Muscholl, A., eds.), Leibniz International Proceedings in Informatics (LIPIcs), Vol. 80, Schloss Dagstuhl–Leibniz-Zentrum für Informatik, Dagstuhl, Germany, pp. 13:1–13:14 (2017).

[4] Black, D. L. and Sleator, D. D.: Competitive Algorithms for Replication and Migration Problems, Technical Report CMU-CS-89-201, Department of Computer Science, Carnegie Mellon University (1989).

[5] Chrobak, M., Larmore, L. L., Reingold, N. and Westbrook, J.: Page Migration Algorithms Using Work Functions, *J. Algorithms*, Vol. 24, No. 1, pp. 124–157 (1997).

[6] Khorramian, A. and Matsubayashi, A.: Uniform Page Migration Problem in Euclidean Space, *Algorithms*, Vol. 9, No. 3, pp. 57:1–7 (2016).

[7] Lund, C., Reingold, N., Westbrook, J. and Yan, D.: Competitive On-Line Algorithms for Distributed Data Management, *SIAM J. Comput.*, Vol. 28, No. 3, pp. 1086–1111 (1999).

[8] Matsubayashi, A.: Uniform Page Migration on General Networks, *Int. J. Pure Appl. Math.*, Vol. 42, No. 2, pp. 161–168 (2008).

[9] Matsubayashi, A.: A 3+Ω(1) Lower Bound for Page Migration, *Proceedings of the 3rd International Symposium on Computing and Networking (CANDAR 2015)*, pp. 314–320 (2015).

[10] Matsubayashi, A.: Asymptotically Optimal Online Page Migration on Three Points, *Algorithmica*, Vol. 71, No. 4, pp. 1035–1064 (2015).

[11] Westbrook, J.: Randomized Algorithms for Multiprocessor Page Migration, *SIAM J. Comput.*, Vol. 23, No. 5, pp. 951–965 (1994).