Nonbipartite Dulmage-Mendelsohn Decomposition for Berge Duality

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Abstract: The *Dulmage-Mendelsohn decomposition* is a classical *canonical decomposition* in matching theory applicable for bipartite graphs and is famous not only for its application in the field of matrix computation, but also for providing a prototypal structure in matroidal optimization theory. The Dulmage-Mendelsohn decomposition is stated and proved using the two color classes of a bipartite graph, and therefore generalizing this decomposition for nonbipartite graphs has been a difficult task. In this paper, we obtain a new canonical decomposition that is a generalization of the Dulmage-Mendelsohn decomposition for arbitrary graphs using a recently introduced tool in matching theory, the *basilica decomposition*. Our result enables us to understand all known canonical decompositions in a unified way. Furthermore, we apply our result to derive a new theorem regarding *barriers*. The duality theorem for the maximum matching problem is the celebrated *Berge formula*, in which dual optimizers are known as barriers. Several results regarding maximal barriers have been derived by known canonical decompositions; however, no characterization has been known for general graphs. In this paper, we provide a characterization of the family of *maximal barriers* in general graphs, in which the known results are developed and unified.

Keywords: matchings, the Dulmage-Mendelsohn decomposition, barriers, the Berge formula

1. Introduction

We establish the Dulmage-Mendelsohn decomposition for general graphs. The *Dulmage-Mendelsohn decomposition* [2–4], or the *DM decomposition* in short, is a classical canonical decomposition in matching theory [17] applicable for bipartite graphs. This decomposition is famous for its application for combinatorial matrix theory, especially for providing an efficient solution for a system of linear equations [1, 4] and is also important in matroidal optimization theory.

Canonical decompositions of a graph are fundamental tools in matching theory [17]. A canonical decomposition partitions a given graph in a way uniquely determined for the graph and describes the structure of maximum matchings using this partition. The classical canonical decompositions are the *Gallai-Edmonds* [5, 6] and *Kotzig-Lovász* decompositions [13–15] in addition to the DM decomposition. The DM and Kotzig-Lovász decompositions are applicable for bipartite graphs and *factor-connected graphs*, respectively. The Gallai-Edmonds decomposition partitions an arbitrary graph into three parts: that is, the so-called $D(G)$, $A(G)$, and $C(G)$ parts. Comparably recently, a new canonical decomposition was proposed: the *basilica decomposition* [8–10]. This decomposition is applicable for arbitrary graphs and contains a generalization of the Kotzig-Lovász decomposition and a refinement the Gallai-Edmonds decomposition. (The $C(G)$ part can be decomposed nontrivially.)

In this paper, we establish an analogue of the DM decomposition for general graphs using the basilica decomposition. Our results accordingly provide a paradigm that enables us to handle any graph and understand the known canonical decompositions in a unified way. In the original theory of DM decomposition, the concept of the *DM components* of a bipartite graph is first defined, and then it is proved that these components form a poset with respect to a certain binary relation.

This theory depends heavily on the two color classes of a bipartite graph and cannot be easily generalized for nonbipartite graphs. In our generalization, we first define a generalization of the DM components using the basilica decomposition. To capture the structure formed by these components in nonbipartite graphs, we introduce a slightly more complex concept: *posets with a transitive forbidden relation*. We then prove that the generalized DM components form a poset with a transitive forbidden relation for certain binary relations.

Furthermore, we apply our generalized DM decomposition to derive a characterization of the family of *maximal barriers* in general graphs. The *Berge formula* is a combinatorial min-max theorem in which maximum matchings are the optimizers of one hand, and the optimizers of the other hand are known as *barriers* [17]. That is, barriers are the dual optimizers of the maximum matchings problem. Barriers are heavily employed as a tool for studying

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matchings. However, not as much is known about barriers themselves [17]. Aside from several observations that are derived rather easily from the Berge formula, several substantial results about (inclusion-wise) maximal barriers have been provided by canonical decompositions.

Our result for maximal barriers proves that our generalization of the DM decomposition has a reasonable consistency with the relationship between each known canonical decomposition and maximal barriers. Each known canonical decomposition can be used to state the structure of maximal barriers. The original DM decomposition provides a characterization of the family of maximal barriers in a bipartite graph in terms of ideals in the poset; minimum vertex covers in bipartite graphs are equivalent to maximal barriers. The Gallai-Edmonds decomposition derives a characterization of the intersection of all maximal barriers (that is, the $A(G)$ part) [17]; this characterization is known as the *Gallai-Edmonds description*. The Kotzig-Lovász decomposition is used for characterizing the family of maximal barriers in factor-connected graphs [17]; this result is known as *Lovász's canonical partition theorem* [16, 17]. The basilica decomposition provides the structure of a given maximal barrier in general graphs, which contains a common generalization of the Gallai-Edmonds description and Lovász's canonical partition theorem. Hence, a generalization of the DM decomposition would be reasonable if it can characterize the family of maximal barriers, and our generalization attains this in a way analogical to the classical DM decomposition, that is, in terms of ideals in the poset with a transitive forbidden relation.

Our results imply a new possibility in matroidal optimization theory. The nonbipartite maximum matching problem and the Berge formula are not captured by submodular function theory [18] today. Submodular function theory is a systematic field of study that captures many well-solved problems in terms of submodular functions and generalizations. In this theory, the bipartite maximum matching problem is an important exemplary problem. According to the Hall-Ore theorem [18], which is the duality theorem for the bipartite maximum matching problem, its dual problem is a special case of the submodular function minimization. The DM decomposition therefore has a special meaning in this theory as it describes the structure of the family of minimizers of a submodular function. The nonbipartite maximum matching problem is also an important well-solved problem, and is even referred to as the archetype of well-solved problems [17, 18]. In fact, the idea of polyhedral combinatorics and some of its central concepts, such as the total dual integrality, have been discovered from the nonbipartite maximum matching problem. However, the nonbipartite maximum matching problem and its duality shown by the Berge formula are not included in submodular function theory today and nor in any of its generalizations. Our nonbipartite DM decomposition may provide a clue to a new epoch of submodular function theory that can be brought in by capturing these concepts.

The remainder of this paper is organized as follows: In Section 2, we provide the basic definitions. In Section 3, we present the preliminary results from the basilica decomposition theory. In Section 4, we introduce the new concept of posets with a transitive forbidden relation. In Section 5, we provide our main result, the nonbipartie DM decomposition. In Section 6, we present preliminary definitions and results regarding barriers. We then prove in Section 7 that our generalization of the DM decomposition can be used to characterize the family of maximal barriers. In Section 8, we show how our results contain the original DM decomposition for bipartite graphs. In Section 9, we prove computational properties.

2. Basic Preliminaries

2.1 General Definitions

For basic notation for sets, graphs, and algorithms, we mostly follow Schrijver [18]. In this section, we explain exceptions or nonstandard definitions. In Section 2, unless otherwise stated, let G be a graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. We treat paths and circuits as graphs. For a path P and vertices x and y from P, xPy denotes the subpath of P between x and y. The singleton set $\{x\}$ is often denoted by just x. We often treat a graph as the set of its vertices.

In the remainder of this section, let $X \subseteq V(G)$. The subgraph of G induced by X is denoted by $G[X]$. The graph $G[V(G) \setminus X]$ is denoted by $G - X$. The contraction of G by X is denoted by G/X. Let $F \subseteq E(G)$. The graph obtained by deleting F from G without removing vertices is denoted by $G - F$. Let H be a subgraph of G. The graph obtained by adding F to H is denoted by $H + F$. Regarding these operations, we identify vertices, edges, subgraphs of the newly created graph with the naturally corresponding items of old graphs.

A *neighbor* of X is a vertex from $V(G) \setminus X$ that is adjacent to some vertex from X. The neighbor set of X is denoted by $N_G(X)$. Let $Y \subseteq V(G)$. The set of edges joining X and Y is denoted by $E_G[X, Y]$. The set $E_G[X, V(G) \setminus X]$ is denoted by $\delta_G(X)$.

A set $M \subseteq E(G)$ is a *matching* if $|\delta_G(v) \cap M| \le 1$ holds for each $v \in V(G)$. For a matching M, we say that M *covers* a vertex v if $|\delta_G(v) \cap M| = 1$; otherwise, we say that M *exposes* v. A matching is *maximum* if it consists of the maximum number of edges. A graph can possess an exponentially large number of matchings. A matching is *perfect* if it covers every vertex. A graph is *factorizable* if it has a perfect matchings. A graph is *factor-critical* if, for each vertex v, $G - v$ is factorizable. A graph with only one vertex is defined to be factor-critical. The number of edges in a maximum matching is denoted by $\nu(G)$. The number of vertices exposed by a maximum matching is denoted by $\mathrm{def}(G)$; that is, $\mathrm{def}(G) := |V(G)| - 2\nu(G)$.

Let $M \subseteq E(G)$. A circuit or path is said to be M-*alternating* if edges in M and not in M appear alternately. The precise definition is the following: A circuit C of G is

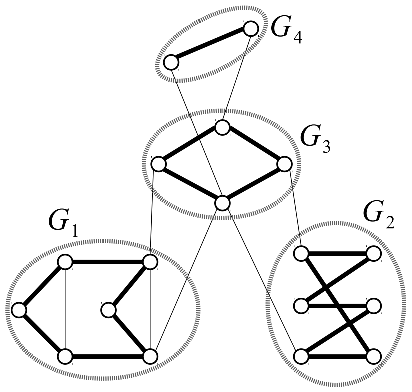

Fig. 1 The factor-components of a graph G: Bold lines indicate allowed edges. This graph has four factor-components $G_1, \ldots, G_4$.

M-alternating if $E(C) \cap M$ is a perfect matching of C. We define the three types of M-alternating paths. Let P be a path with ends s and t. We say that P is *M-forwarding* from s to t if $M \cap E(P)$ is a matching of P that covers every vertex except for t. We say that P is *M-saturated* between s and t if $M \cap E(P)$ is a perfect matching of P. We say that P is *M-exposed* between s and t if $M \cap E(P)$ is a matching of P that covers every vertex except for s and t. Any path with exactly one vertex x is defined to be an M-forwarding path from x to x, and is never treated as an M-exposed path. Any M-forwarding path has an even number of edges, which can be zero, whereas any M-saturated or -exposed path has an odd number of edges.

A path P is an *ear* relative to X if the internal vertices of P are disjoint from X, whereas the ends are in X. A circuit C is an *ear* relative to X if exactly one vertex of C is in X; for simplicity, we call the vertex in $X \cap V(C)$ the *end* of the ear C. We call an ear P relative to X an *M-ear* if $P - X$ is empty or an M-saturated path, and $\delta_P(X) \cap M = \emptyset$.

2.2 Barriers, Gallai-Edmonds Family, and Factor-Components

We now explain the Berge Formula and the definition of barriers. An *odd component* (resp. *even component*) of a graph is a connected component with an odd (resp. even) number of vertices. The number of odd components of $G-X$ is denoted by $q_G(X)$. The set of vertices from odd components (resp. even components) of $G - X$ is denoted by D_X (resp. C_X).

Theorem 2.1 (Berge Formula [17]). *For a graph G, $\mathrm{def}(G)$ is equal to the maximum value of $q_G(X) - |X|$, where X is taken over all subsets of $V(G)$.*

The set of vertices that attains the maximum value in this relation is called a *barrier*. That is, a set of vertices X is a *barrier* if $\mathrm{def}(G) = q_G(X) - |X|$.

The set of vertices that can be exposed by maximum matchings is denoted by $D(G)$. The neighbor set of $D(G)$ is denoted by $A(G)$, and the set $V(G) \setminus D(G) \setminus A(G)$ is denoted by $C(G)$. The following statement about $D(G)$, $A(G)$, and $C(G)$ is the celebrated *Gallai-Edmonds structure theorem* [5, 6, 17].

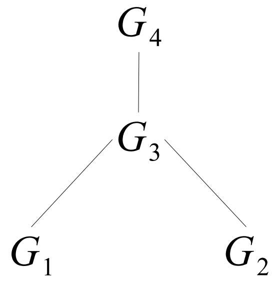

Fig. 2 The Hasse diagram of the poset $(\mathcal{G}(G), \triangleleft)$.

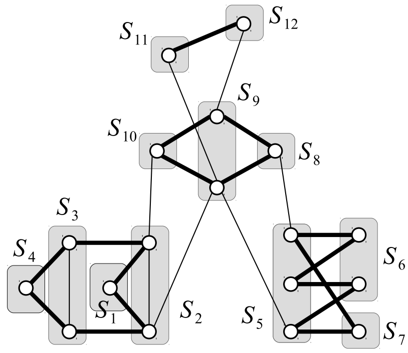

Fig. 3 The general Kotzig-Lovász decomposition of G: $\mathcal{P}(G)$ has 12 members $S_1, \ldots, S_{12}$.

Theorem 2.2 (Gallai-Edmonds Structure Theorem). *For any graph G,*
(i) *$A(G)$ is a barrier for which $D_{A(G)} = D(G)$ and $C_{A(G)} = C(G)$;*
(ii) *each odd component of $G - A(G)$ is factor-critical; and,*
(iii) *any edge in $E_G[A(G), D(G)]$ is allowed.*

An edge is *allowed* if it is contained in some maximum matching. Two vertices are *factor-connected* if they are connected by a path whose edges are allowed. A subgraph is *factor-connected* if any two vertices are factor-connected. A maximal factor-connected subgraph is called a *factor-connected component* or *factor-component*. A graph consists of its factor-components and edges joining them that are not allowed. The set of factor-components of G is denoted by $\mathcal{G}(G)$.

A factor-component C is *inconsistent* if $V(C) \cap D(G) \neq \emptyset$. Otherwise, C is said to be *consistent*. We denote the sets of consistent and inconsistent factor-components of G by $\mathcal{G}^+(G)$ and $\mathcal{G}^-(G)$, respectively. The next property is easily confirmed from the Gallai-Edmonds structure theorem.

Fact 2.3. *A subgraph C of G is an inconsistent factor-component if and only if C is a connected component of $G[D(G) \cup A(G)]$. Any consistent factor-component has the vertex set contained in $C(G)$.*

That is, the structure of inconsistent factor-components are rather trivial under the Gallai-Edmonds structure theorem.

3. Basilica Decomposition of Graphs

3.1 Central Concepts

We now introduce the basilica decomposition of graphs [9, 10]. The theory of basilica decomposition is made up of the

three central concepts:
(i) a canonical partial order between factor-components (Theorem 3.1),
(ii) the general Kotzig-Lovász decomposition (Theorem 3.2), and
(iii) an interrelationship between the two (Theorem 3.3).

In Section 3.1, we explain these three concepts and give the definition of the basilica decomposition. Every statement in the following is from Kita [9,10]. *1 In the following, let G be a graph unless otherwise stated.

Definition 3.1. *A set* $X \subseteq V(G)$ *is said to be* separating *if there exist* $H_1, \ldots, H_k \in \mathcal{G}(G)$*, where* $k \geq 1$*, such that* $X = V(H_1) \cup \cdots \cup V(H_k)$*. For* $G_1, G_2 \in \mathcal{G}(G)$*, we say* $G_1 \triangleleft G_2$ *if there exists a separating set* $X \subseteq V(G)$ *with* $V(G_1) \cup V(G_2) \subseteq X$ *such that* $G[X]/G_1$ *is a factor-critical graph.*

Theorem 3.1. *For a graph* G*, the binary relation* $\triangleleft$ *is a partial order over* $\mathcal{G}(G)$*.*

Definition 3.2. *For* $u, v \in V(G) \setminus D(G)$*, we say* $u \sim_G v$ *if* u *and* v *are identical or if* u *and* v *are factor-connected and satisfy* $\mathrm{def}(G - u - v) > \mathrm{def}(G)$*.*

Theorem 3.2. *For a graph* G*, the binary relation* $\sim_G$ *is an equivalence relation.*

We denote as $\mathcal{P}(G)$ the family of equivalence classes determined by $\sim_G$. This family is known as the *general Kotzig-Lovász decomposition* or just the *Kotzig-Lovász decomposition* of G. From the definition of $\sim_G$, for each $H \in \mathcal{G}(G)$, the family $\{S \in \mathcal{P}(G) : S \subseteq V(H)\}$ forms a partition of $V(H) \setminus D(G)$. We denote this family by $\mathcal{P}_G(H)$.

Let $H \in \mathcal{G}(G)$. The sets of strict and nonstrict upper bounds of H are denoted by $\mathcal{U}_G(H)$ and $\mathcal{U}^*_G(H)$, respectively. The sets of vertices $\bigcup\{V(I) : I \in \mathcal{U}_G(H)\}$ and $\bigcup\{V(I) : I \in \mathcal{U}^*_G(H)\}$ are denoted by $U_G(H)$ and $U^*_G(H)$, respectively.

Theorem 3.3. *Let* G *be a graph, and let* $H \in \mathcal{G}(G)$*. Then, for each connected component* K *of* $G[U_G(H)]$*, there exists* $S \in \mathcal{P}_G(H)$ *such that* $N_G(K) \cap V(H) \subseteq S$*.*

Under Theorem 3.3, for $S \in \mathcal{P}_G(H)$, we denote by $\mathcal{U}_G(S)$ the set of factor-components that are contained in a connected component K of $G[U_G(H)]$ with $N_G(K) \cap V(H) \subseteq S$. The set $\bigcup\{V(I) : I \in U_G(H)\}$ is denoted by $U_G(S)$. We denote $U_G(H) \setminus S \setminus U_G(S)$ by ${}^\top U_G(S)$.

Theorem 3.3 integrates the two structures given by Theorems 3.1 and 3.2 into a structure of graphs that is reminiscent of an architectural building. We call this integrated structure the *basilica decomposition* of a graph.

See Figures 1, 2, and 3 for an example of the basilica decomposition.

3.2 Remark on Inconsistent Factor-Components

Inconsistent factor-components in a graph have a trivial structure regarding the basilica decomposition. The next statement is easily confirmed from Fact 2.3 and the Gallai-Edmonds structure theorem.

Fact 3.4. *Let* G *be a graph. Any inconsistent component is minimal in the poset* $(\mathcal{G}(G), \triangleleft)$*. For any* $H \in \mathcal{G}^-(G)$*, if* $V(H) \cap A(G) \neq \emptyset$*, then* $\mathcal{P}_G(H) = \{V(H) \cap A(G)\}$*; otherwise,* $\mathcal{P}_G(H) = \emptyset$*.*

For simplicity, even for $H \in \mathcal{G}^-(G)$ with $V(H) \cap A(G) = \emptyset$, we treat as if $V(H) \cap A(G)$ is a member of $\mathcal{P}(G)$. That is, we let $\mathcal{P}_G(H) = \{V(H) \cap A(G)\}$ and ${}^\top U_G(V(H) \cap A(G)) = {}^\top U_G(\emptyset) = V(H) \cap D(G) = V(H)$.

Under Fact 3.4, the substantial information provided by the basilica decomposition lies in the consistent factor-components.

3.3 Additional Properties

In this section, we present some properties of the basilica decomposition that are used in later sections. The next lemma can be found in Kita [11].

Lemma 3.5. *Let* G *be a graph, and let* M *be a maximum matching of* G*. Let* $H \in \mathcal{G}^+(G)$*,* $S \in \mathcal{P}_G(H)$*, and* $s \in S$*.*
(i) *For any* $t \in S$*, there is an* M*-forwarding path from* s *to* t*, whose vertices are contained in* $S \cup {}^\top U_G(S)$*; however, there is no* M*-saturated path between* s *and* t*.*
(ii) *For any* $t \in {}^\top U_G(S)$*, there exists an* M*-saturated path between* s *and* t *whose vertices are contained in* $S \cup {}^\top U_G(S)$*.*
(iii) *For any* $t \in U_G(S)$*, there is an* M*-forwarding path from* t *to* s*, whereas there is no* M*-forwarding path from* s *to* t *or* M*-saturated path between* s *and* t*.*

The first part of the next lemma is provided in Kita [12], and the second part can be easily proved from Lemma 3.5.

Lemma 3.6. *Let* G *be a graph, and let* M *be a maximum matching of* G*. Let* $S \in \mathcal{P}(G)$*. If there is an* M*-ear relative to* $S \cup {}^\top U_G(S)$ *that has internal vertices, then the ends of this ear are contained in* S*.*

4. Poset with Transitive Forbidden Relation

We now introduce the new concept of *posets with a transitive forbidden relation*, which serves as a language to describe the nonbipartite DM decomposition.

Definition 4.1. *Let* X *be a set, and let* $\preceq$ *be a partial order over* X*. Let* $\smile$ *be a binary relation over* X *such that,*
(i) *for each* $x, y, z \in X$*, if* $x \preceq y$ *and* $y \smile z$ *hold, then* $x \smile z$ *holds (transitivity);*
(ii) *for each* $x \in X$*,* $x \smile x$ *does not hold (nonreflexivity); and,*
(iii) *for each* $x, y \in X$*, if* $x \smile y$ *holds, then* $y \smile x$ *also holds (symmetry).*

We call this poset endowed with this additional binary relation a poset with a transitive forbidden relation *or* TFR poset *in short, and denote this by* $(X, \preceq, \smile)$*. We call a pair of two elements* x *and* y *with* $x \smile y$ forbidden.

*1 The essential part of the structure described by the basilica decomposition lies in the factorizable graph $G[C(G)]$. Therefore, statements for factorizable graphs [9,10] can be straightforwardly generalized for arbitrary graphs under the Gallai-Edmonds structure theorem.

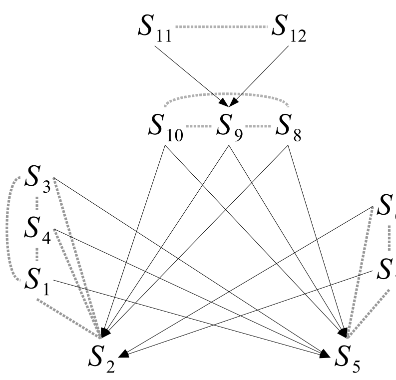

Fig. 4 The nonbipartite Dulmage-Mendelsohn decomposition of G: For each immediate compatible pair, an arrow points from the lower element to the upper element. The two elements from each immediate forbidden pair are connected by a gray broken line.

Let $(X, \preceq, \smile)$ be a TFR poset. For two elements $x, y \in X$ with $x \smile y$, we say that $x \overset{\star}{\smile} y$ if, there is no $z \in X \setminus \{x, y\}$ with $x \preceq z$ and $z \smile y$. We call such a forbidden pair of x and y *immediate*. A TFR poset can be visualized in a similar way to an ordinary posets. We represent $\preceq$ just in the same way as the Hasse diagrams and depict $\smile$ by indicating every immediate forbidden pairs.

Definition 4.2. *Let P be a TFR poset $(X, \preceq, \smile)$. A lower or upper ideal Y of P is* legitimate *if no elements $x, y \in Y$ satisfy $x \smile y$. Otherwise, we say that Y is* illegitimate. *Let Y be a consistent lower or upper ideal, and let Z be the subset of $X \setminus Y$ such that, for each $x \in Z$, there exists $y \in Y$ with $x \smile y$. We say that Y is* spanning *if $Y \cup Z = X$.*

5. DM Decomposition for General Graphs

We now provide our new results of the DM decomposition for general graphs. In this section, unless otherwise stated, let G be a graph.

Definition 5.1. *A* Dulmage-Mendelsohn component*, or a* DM component *in short, is a subgraph of the form $G[S \cup {}^{\top}U_G(S)]$, where $S \in \mathcal{P}(G)$, endowed with S as an attribute known as the* base. *For a DM component C, the base of C is denoted by $\pi(C)$. Conversely, for $S \in \mathcal{P}(G)$, $K(S)$ denotes the DM components whose base is S. We denote by $\mathcal{D}(G)$ the set of DM components of G.*

Hence, distinct DM components can be equivalent as a subgraph of G. Each member from $\mathcal{P}(G)$ serves as an identifier of a DM component.

Definition 5.2. *A DM component C is said to be* inconsistent *if $\pi(C) \in \mathcal{P}_G(H)$ for some $H \in \mathcal{G}^-(G)$; otherwise, C is said to be* consistent. *The sets of consistent and inconsistent DM components are denoted by $\mathcal{D}^+(G)$ and $\mathcal{D}^-(G)$, respectively.*

Under Fact 3.4, any $H \in \mathcal{D}^-(G)$ is equal to an inconsistent factor-component as a subgraph of G, and $\pi(H) = V(H) \cap A(G)$ and $V(H) \setminus \pi(H) = V(H) \cap D(G)$.

Definition 5.3. *We define binary relations $\preceq^\circ$ and $\preceq$ over $\mathcal{D}(G)$ as follows: for $D_1, D_2 \in \mathcal{D}(G)$, we let $D_1 \preceq^\circ D_2$ if $D_1 = D_2$ or if $N_G({}^{\top}U_G(S_1)) \cap S_2 \neq \emptyset$; we let $D_1 \preceq D_2$ if there exist $C_1, \ldots, C_k \in \mathcal{D}(G)$, where $k \geq 1$, such that $\pi(C_1) = \pi(D_1)$, $\pi(C_k) = \pi(D_2)$, and $C_i \preceq^\circ C_{i+1}$ for each $i \in \{1, \ldots, k\} \setminus \{k\}$.*

Definition 5.4. *We define binary relations $\smile^\circ$ and $\smile$ over $\mathcal{D}(G)$ as follows: for $D_1, D_2 \in \mathcal{D}(G)$, we let $D_1 \smile^\circ D_2$ if $\pi(D_2) \subseteq V(D_1) \setminus \pi(D_1)$ holds; we let $D_1 \smile D_2$ if there exists $D' \in \mathcal{D}(G)$ with $D_1 \preceq D'$ and $D' \smile^\circ D_2$.*

In the following, we prove that $(\mathcal{D}(G), \preceq, \smile)$ is a TFR poset, which gives a generalization of the DM decomposition. The next lemma is easily observed from Facts 2.3 and 3.4.

Lemma 5.1. *If C is an inconsistent DM component of a graph G, then there is no $C' \in \mathcal{D}(G) \setminus \{C\}$ with $C \preceq C'$ or $C \smile C'$.*

We first prove that $\preceq$ is a partial order over $\mathcal{D}(G)$. We provide Lemmas 5.2 and 5.3 and thus prove Theorem 5.5.

Lemma 5.2. *Let G be a graph, let M be a maximum matching of G, and let $D_1, \ldots, D_k \in \mathcal{D}(G)$, where $k \geq 1$, be DM components with $D_1 \preceq^\circ \cdots \preceq^\circ D_k$ no two of which share vertices and for which $D_k \in \mathcal{D}^+(G)$ holds. Then, for any $s \in \pi(D_1)$ and any $t \in \pi(D_k)$ (resp. $t \in V(D_k) \setminus \pi(D_k)$), there is an M-forwarding path from s to t (resp. M-saturated path between s and t) whose vertices are contained in $V(D_1) \dot\cup \cdots \dot\cup V(D_k)$.*

Proof. For each $i \in \{1, \ldots, k\} \setminus \{k\}$, let $t_i \in {}^{\top}U_G(\pi(D_i))$ and $s_{i+1} \in \pi(D_{i+1})$ be vertices with $t_i s_{i+1} \in E(G)$. Let $s_1 := s$ and $t_k := t$. According to Lemma 3.5, for each $i \in \{1, \ldots, k\} \setminus \{k\}$, there is an M-saturated path P_i between s_i and t_i with $V(P_i) \subseteq V(D_i)$; additionally, there is an M-forwarding path P_k from s_k to t with $V(P_k) \subseteq V(D_k)$. Thus, $P_1 + \cdots + P_k + \{t_i s_{i+1} : i = 1, \ldots, k-1\}$ is a desired M-forwarding path from s to t. The claim for $t \in V(D_k) \setminus \pi(D_k)$ can also be proved in a similar way using Lemma 3.5. $\square$

Lemma 5.2 yields Lemma 5.3:

Lemma 5.3. *Let G be a graph, let M be a maximum matching of G, and let $D_1, \ldots, D_k$, where $k \geq 2$, be DM components with $D_1 \preceq^\circ \cdots \preceq^\circ D_k$ such that $\pi(D_i) \neq \pi(D_{i+1})$ for any $i \in \{1, \ldots, k-1\}$. Then, for any $i, j \in \{1, \ldots, k\}$ with $i \neq j$, $V(D_i) \cap V(D_j) = \emptyset$.*

Proof. Suppose that the claim fails. Then, there exist $p, q \in \{1, \ldots, k-1\}$ with $p \leq q$ such that $D_p, \ldots, D_{q+1}$ are mutually disjoint except that $V(D_p) \cap V(D_{q+1}) \neq \emptyset$. Then, Lemma 5.1 implies $D_p, \ldots, D_{q+1} \in \mathcal{D}^+(G)$. If $\pi(D_{q+1}) \subseteq V(D_p)$ holds, then let $t_{q+1} \in {}^{\top}U_G(\pi(D_{q+1}))$; otherwise, let $t_{q+1} \in {}^{\top}U_G(\pi(D_{q+1})) \cap V(D_p)$. Let $t_p \in {}^{\top}U_G(\pi(D_p))$ and $s_{p+1} \in \pi(D_{p+1})$ be vertices with $t_p s_{p+1} \in E(G)$, and let Q be an M-saturated path Q between s_{p+1} and t_q taken under Lemma 5.2. Then, $t_p s_{p+1} + Q + t_q s_{q+1} + s_{q+1} P t_{q+1}$ contains an M-ear relative to D_p one of whose ends is t_p. This contradicts Lemma 3.6. $\square$

Combining Lemmas 5.2 and 5.3, the next lemma can be stated, which we will use for proving Lemma 5.8.

Lemma 5.4. *Let G be a graph. Let $C_1, C_2 \in \mathcal{D}(G)$ with $C_1 \preceq C_2$, and let $D_1, \ldots, D_k \in \mathcal{D}(G)$, where $k \geq 1$, be DM components with $C_1 = D_1$, $C_2 = D_k$, and $D_1 \preceq^\circ \cdots \preceq^\circ D_k$. Then, for any $s \in \pi(D_1)$ and any $t \in \pi(D_k)$ (resp. $t \in V(D_k) \setminus \pi(D_k)$), there is an M-forwarding path from s to t (resp. M-saturated path between s and t) whose vertices are contained in $V(D_1)\dot{\cup} \cdots \dot{\cup} V(D_k)$.*

Reflexivity and transitivity of $\preceq$ are obvious from the definition, and antisymmetry is now proved by Lemma 5.3:

Theorem 5.5. *Let G be a graph. Then, $\preceq$ is a partial order over $\mathcal{D}(G)$.*

In the following, we prove the properties required for $\smile$ to form a TFR poset $(\mathcal{D}(G), \preceq, \smile)$. We provide Lemmas 5.6, 5.7, and 5.8, and thus prove Theorem 5.9.

Lemma 5.6. *Let G be a graph, and let M be a maximum matching of G. Let $s, t \in V(G)$, and let S be the member of $\mathcal{P}(G)$ with $s \in S$. Let P be an M-forwarding path P from s to t or an M-saturated path between s and t. If $t \in S \cup {}^\top U_G(S)$ holds, then $P - E(G[S \cup {}^\top U_G(S)])$ is empty; otherwise, $P - E(G[S \cup {}^\top U_G(S)])$ is a path.*

Proof. Suppose that the claim fails. The connected components of $P - E(G[S \cup {}^\top U_G(S)])$ except for the one that contains s are M-ears relative to $S \cup {}^\top U_G(S)$ with internal vertices. Let S' be the set of the ends of these M-ears. From Lemma 3.6, we have $S' \subseteq S$. Trace P from s, and let s' be the first vertex in S'. Then, sPr is an M-saturated path between s and s', which contradicts $s \sim_G s'$. This proves the claim. □

Lemma 5.6 derives the next lemma with Lemmas 3.5 and 3.6.

Lemma 5.7. *Let G be a graph, and let M be a maximum matching of G. Let $s, t \in V(G)$, and let S and T be the members from $\mathcal{P}(G)$ with $s \in S$ and $t \in T$, respectively.*

(i) *If there is no M-saturated path between s and t, whereas there is an M-forwarding path from s to t, then $K(S) \preceq K(T)$ holds.*
(ii) *If there is an M-saturated path between s and t, then $K(S) \smile K(T)$ holds.*

Proof. Let P be an M-saturated between s and t. We proceed by induction on the number of edges in P. By Lemma 5.6, $P - E(K(S))$ is an M-exposed path; let x be the end of $P - E(K(S))$ other than t. Let $y \in V(P)$ be the vertex with $xy \in E(P)$, and let $R \in \mathcal{P}(G)$ with $y \in R$. By Lemma 3.5, we have $K(S) \preceq^\circ K(R)$. The subpath yPt is M-saturated between y and t. Therefore, the induction hypothesis implies $K(R) \smile K(T)$. Thus, $K(S) \smile K(T)$ is proved. Statement (i) can also be proved in a similar way. □

The symmetry of $\smile$ can now be proved from Lemmas 5.4 and 5.7.

Lemma 5.8. *For a graph G, the binary relation $\smile$ is symmetric, that is, if $D_1 \smile D_2$ holds for $D_1, D_2 \in \mathcal{D}(G)$, then $D_2 \smile D_1$ holds.*

Theorem 5.5 and Lemma 5.8 now prove Theorem 5.9:

Theorem 5.9. *For a graph G, the triple $(\mathcal{D}(G), \preceq, \smile)$ is a TFR poset.*

Proof. Under Theorem 5.5, it now suffices to prove the conditions for $\smile$. Nonreflexivity and transitivity are obvious from the definition. Symmetry is proved by Lemma 5.8. □

For a graph G, the TFR poset $(\mathcal{D}(G), \preceq, \smile)$ is uniquely determined. We denote this TFR poset by $\mathcal{O}(G)$. We call this canonical structure that $\mathcal{O}(G)$ describes the *nonbipartite Dulmage-Mendelsohn (DM) decomposition* of G. We show in Section 8 that this is a generalization of the classical DM decomposition for bipartite graphs.

Remark 5.1. *As mentioned previously, a DM component is identified by its base. Therefore, the nonbipartite DM decomposition is essentially the relations between the members of $\mathcal{P}(G)$. In Figure 4, we provide an example of the nonbipartite DM decomposition for the graph G from Figures 1, 2, and 3.*

Remark 5.2. *Our result is distinct from the result by Iwata [7]. This can also be confirmed from the example graph G in Figure 1.*

Immediate forbidded pairs in $\mathcal{O}(G)$ can be characterized as follows:

Theorem 5.10. *Let G be a graph. Let $S, T \in \mathcal{P}(G)$. Then, $K(S)$ and $K(T)$ are immediate forbidden pairs if and only if S and T are contained in the same factor-component.*

Proof. The necessity is obvious. For proving sufficiency, let H_1 and H_2 be the factor-components that contain S and T, respectively. Obviously, $T \subseteq {}^\top U_G(S)$ holds. Hence, if the claim fails, then $T \subseteq U_G(S')$ holds for some $S' \in \mathcal{P}_G(H_1)$. From Lemmas 3.5 and 5.7, this implies $K(T) \preceq K(S')$. As $S' \smile^\circ S$ holds, $K(T)$ and $K(S)$ are not immediate, which is a contradiction. □

6. Preliminaries on Maximal Barriers

6.1 Classical Properties of Maximal Barriers

We now present some preliminary properties of maximal barriers to be used in Section 7. A barrier is *maximal* if it is inclusion-wise maximal. A barrier X is *odd-maximal* if it is maximal with respect to D_X; that is, for no $Y \subseteq D_X$, $X \cup Y$ is a barrier. A maximal barrier is an odd-maximal barrier.

The next two propositions are classical facts. See Lovász and Plummer [17].

Proposition 6.1. *Let G be a graph, and let $X \subseteq V(G)$ be a barrier. Then, X is an odd-maximal barrier if and only if every odd component of $G - X$ are factor-critical.*

Proposition 6.2. *Let G be a graph. An odd-maximal barrier X is a maximal barrier if and only if $C_X = \emptyset$.*

6.2 Generalization of Lovász's Canonical Partition Theorem

In this section, we explain a known theorem about the structure of a given odd-maximal barrier [11]. This theorem is a generalization of *Lovász's canonical partition theorem* [11, 16, 17] for general graphs, which is originally for factor-connected graphs. This theorem contains the classical result about the relationship between maximal barriers and the Gallai-Edmonds decomposition, which states that $A(G)$ of a graph G is the intersection of all maximal barriers [17].

Theorem 6.3 (Kita [11]). *Let G be a graph and $X \subseteq V(G)$ be an odd-maximal barrier of G. Then, there exist $S_1, \ldots, S_k \in \mathcal{P}(G)$, where $k \geq 1$, such that $X = S_1 \dot\cup \cdots \dot\cup S_k$ and $D_X = {}^\top U_G(S_1) \dot\cup \cdots \dot\cup {}^\top U_G(S_k)$. The odd components of $G - X$ are the connected components of $G[{}^\top U_G(S_i)]$, where i is taken over all $\{1, \ldots, k\}$.*

The next statement can be derived from Theorem 6.3 as a corollary.

Corollary 6.4. *Let G be a graph. For each $S \in \mathcal{P}(G)$, $G[{}^\top U_G(S)]$ consists of $|S| + \mathrm{def}(G[S \cup {}^\top U_G(S)])$ connected components, which are factor-critical. If $S \in \mathcal{P}_G(H)$ holds for some $H \in \mathcal{G}^+(G)$, then $\mathrm{def}(G[S \cup {}^\top U_G(S)]) = 0$; otherwise, $\mathrm{def}(G[S \cup {}^\top U_G(S)]) > 0$. Let $\mathcal{S} := \bigcup\{S \in \mathcal{P}_G(H) : H \in \mathcal{G}^-(G)$ and $V(H) \cap X \neq \emptyset\}$. Then, $\Sigma_{S \in \mathcal{S}} \mathrm{def}(G[S \cup {}^\top U_G(S)]) = \mathrm{def}(G)$.*

7. Canonical Characterization of Maximal Barriers

We now derive the characterization of the family of maximal barriers in general graphs, using the nonbipartite DM decomposition. In this section, unless otherwise stated, let G be a graph. It is a known fact that a graph has an exponentially many number of maximal barriers, however the family of maximal barriers can be fully characterized in terms of ideals of $\mathcal{O}(G)$.

Definition 7.1. *For $\mathcal{I} \subseteq \mathcal{D}(G)$, the* normalization *of $\mathcal{I}$ is the set $\mathcal{I} \cup \mathcal{D}^-(G)$. A set $\mathcal{I}' \subseteq \mathcal{D}(G)$ is said to be* normalized *if $\mathcal{I}' = \mathcal{I} \cup \mathcal{D}^-(G)$ for some $\mathcal{I} \subseteq \mathcal{D}(G)$.*

From Lemma 5.1, note that the normalization of an upper ideal is an upper ideal; the normalization of a legitimate upper ideal is legitimate.

From Theorem 6.3, the next lemma characterizes the family of odd-maximal barriers, which can be proved rather easily from Theorem 3.2.

Lemma 7.1. *Let G be a graph. A set of vertices $X \subseteq V(G)$ is an odd-maximal barrier if and only if there exists a legitimate normalized upper ideal $\mathcal{I}$ of the TFR poset $\mathcal{O}(G)$ such that $X = \bigcup\{\pi(C) : C \in \mathcal{I}\}$.*

From Lemma 7.1 and Proposition 6.2, the family of maximal barriers is now characterized:

Theorem 7.2. *Let G be a graph. A set of vertices $X \subseteq V(G)$ is a maximal barrier if and only if there exists a spanning legitimate normalized upper ideal $\mathcal{I}$ of the TFR poset $\mathcal{O}(G)$ such that $X = \bigcup\{\pi(C) : C \in \mathcal{I}\}$.*

8. Original DM Decomposition for Bipartite Graphs

In this section, we explain the original DM decomposition for bipartite graphs, and prove this from our result in Section 5. In the remainder of this section, unless stated otherwise, let G be a bipartite graph with color classes A and B, and let $W \in \{A, B\}$.

Definition 8.1. *The binary relations $\leq_W^\circ$ and $\leq_W$ over $\mathcal{G}(G)$ are defined as follows: for $G_1, G_2 \in \mathcal{G}(G)$, let $G_1 \leq_W^\circ G_2$ if $G_1 = G_2$ or if $E_G[W \cap V(G_2), V(G_1) \setminus W] \neq \emptyset$; let $G_1 \leq_W G_2$ if there exist $H_1, \ldots, H_k \in \mathcal{G}(G)$, where $k \geq 1$, such that $H_1 = G_1$, $H_k = G_2$, and $H_1 \leq_W^\circ \cdots \leq_W^\circ H_k$.*

Note that $G_1 \leq_A G_2$ holds if and only if $G_2 \leq_B G_1$ holds.

Theorem 8.1 (Dulmage and Mendelsohn [2–4, 17]). *Let G be a bipartite graph with color classes A and B, and let $W \in \{A, B\}$. Then, the binary relation $\leq_W$ is a partial order over $\mathcal{G}(G)$.*

We call the poset $(\mathcal{G}(G), \leq_W)$ proved by Theorem 8.1 the *Dulmage-Mendelsohn decomposition* of a bipartite graph G. It is easily confirmed, e.g., from the Gallai-Edmonds structure theorem that $\mathcal{G}_A^-(G) \cap \mathcal{G}_B^-(G) = \emptyset$ and that any $C \in \mathcal{G}_B^-(G)$ is minimal with respect to $\leq_A$.

Additionally, bipartite graphs have a trivial structure regarding the basilica decomposition:

(i) For each $H \in \mathcal{G}^+(G)$, $\mathcal{P}_G(H) = \{V(H) \cap A, V(H) \cap B\}$. For each $H \in \mathcal{G}_W^-(G)$, $\mathcal{P}_G(H) = \{V(H) \cap W\}$.
(ii) For any $H_1, H_2 \in \mathcal{G}(G)$ with $H_1 \neq H_2$, $H_1 \lhd H_2$ does not hold.

Under these properties, we define $\mathcal{D}^W(G)$ as the set $\{C \in \mathcal{D}(G) : \pi(C) \subseteq W\}$.

Define a mapping $f_W : \mathcal{G}^+(G) \cup \mathcal{G}_W^-(G) \to \mathcal{D}^W(G)$ as $f_W(C) := K(V(C) \cap W)$ for $C \in \mathcal{G}^+(G)$. The next statement is now obvious.

Observation 8.2. *The mapping f_W is a bijection; and, for any $C_1, C_2 \in \mathcal{G}(G)$, $C_1 \leq_W C_2$ holds if and only if $f(C_1) \preceq f(C_2)$ holds.*

According to Theorem 5.9 and Observation 8.2, the system $(\mathcal{G}^+(G) \cup \mathcal{G}_W^-(G), \leq_W)$ is a poset. Thus, this proves Theorem 8.1.

9. Algorithmic Properties

Given a graph G, its basilica decomposition can be computed in $O(|V(G)| \cdot |E(G)|)$ time [9, 10]. Assume that the basilica decomposition of G is given. From the definition of $\preceq^\circ$, the poset $(\mathcal{D}(G), \preceq)$ can be computed in $O(|\mathcal{P}(G)| \cdot |E(G)|)$ time, and accordingly, in $O(|V(G)| \cdot |E(G)|)$ time. According to the definition of $\smile^\circ$, given the poset $(\mathcal{D}(G), \preceq)$, the TFR poset $\mathcal{O}(G)$ can be obtained in $O(|V(G)|)$ time. Therefore, the next thereom can be stated.

Theorem 9.1. *Given a graph G, the TFR poset $\mathcal{O}(G)$ can be computed in $O(|V(G)| \cdot |E(G)|)$ time.*

Acknowledgments This study was supported by JSPS KAKENHI 15J09683.

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