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Hitori Numbers

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Abstract: Hitori is a popular "pencil-and-paper" puzzle defined as follows. In n-hitori, we are given an $n \times n$ rectangular grid in which each square is labeled with a positive integer, and the goal is to paint a subset of the squares so that the following three rules are satisfied: Rule 1) No row or column has a repeated unpainted label; Rule 2) Painted squares are never (horizontally or vertically) adjacent; Rule 3) The unpainted squares are all connected (via horizontal and vertical connections). The grid is called an *instance* of n-hitori if it has a unique solution. In this paper, we introduce *hitori number* and *maximum hitori number* which are defined as follows: For every integer n, hitori number $h(n)$ is the minimum number of different integers used in an instance where the minimum is taken over all the instances of n-hitori. For every integer n, maximum hitori number $\bar{h}(n)$ is the maximum number of different integers used in an instance where the maximum is taken over all the instances of n-hitori. We then prove that $\lceil (2n-1)/3 \rceil \leq h(n) \leq 2\lceil n/3 \rceil + 1$ for $n \geq 2$ and $\lceil (4n^2 - 4n + 11)/5 \rceil \leq \bar{h}(n) \leq (4n^2 + 2n - 2)/5$ for $n \geq 3$.

Keywords: Hitori, pencil-and-paper puzzle, unique solution

1. Introduction

"Pencil-and-paper" puzzles are those which require the use of a pencil and paper. Sudoku is one of the most famous "pencil-and-paper" puzzles. There are many studies on Sudoku from the viewpoint of mathematics. Since there are so many instances of Sudoku, one may wonder how many patterns of solutions of 9×9 Sudoku exist. Jarvis [4] shows that there are about 5.5×10^9 essentially different solutions of 9×9 Sudoku. There is another piece of research concerning the construction of instances. In Sudoku, the fewer the number of given hints the harder the instance becomes. Then one may wonder what is the minimum number of given hints. McGuire et al. [5] show that 17 is the minimum number of given hints over all instances of 9×9 Sudoku.

In this paper, we study Hitori which is similar to Sudoku from the viewpoint of mathematics. Hitori is also a "pencil-and-paper" puzzle proposed by Takeyutaka in 1990, and popularized by a Japanese publisher Nikoli [1]. In n-hitori, we are given an $n \times n$ grid in which each square is labeled with an integer. (See **Fig. 1** (a).) We often call such a grid an n-hitori. The goal is to paint a subset of the n^2 squares so that the following three rules are satisfied:

Rule 1 (No Repeated Labels, NRL):
No row or column has a repeated unpainted label;

Rule 2 (Isolated Painted Squares, IPS):
Painted squares are never (horizontally or vertically) adjacent;

Rule 3 (Connected Unpainted Squares, CUS):
The unpainted squares are all connected (via horizontal and vertical connections).

The grid is called an *instance* of n-hitori if it has a unique solution. Note that there is no instance of 1-hitori. Figure 1 (b) illustrates the unique solution for the instance in Fig. 1 (a) while the paints in Fig. 1 (c), (d), and (e) are not solutions. As proved in Ref. [3], it is NP-complete to decide whether a given n-hitori has a solution or not. Gander and Hofer [2] give an algorithm that solves hitori by using a SAT solver.

In this paper, we introduce and investigate new combinatorial characteristics of Hitori, named *hitori number* and *maximum hitori number*. See two instances of 12-hitori given in **Fig. 2** (a) and (b). Each of the two instances has a unique solution displayed as gray squares, but, the numbers of different integers used in grids have a great difference. One in Fig. 2 (a) uses nine different integers while the other in Fig. 2 (b) uses 108 different integers. Considering many instances, we can observe that there are a large variety of the numbers of different integers used in instances. We then have the following question: what is the smallest (largest) number of different integers that can be used to construct an instance?

Clearly, we cannot make an instance with few integers. Consider, for example, a simple case where $n = 4$. **Figure 3** (a) illustrates an 4-hitori with only one integer. Since there are four squares labeled with '1' in the top row, Rule 1 (NRL) implies that we must paint at least three of them. However, any such

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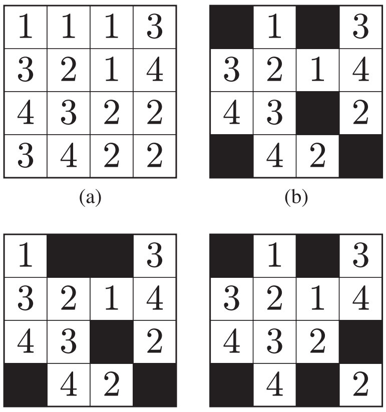


Fig. 1 (a) An instance of 4-hitori. (b) The solution for (a). (c) The paint violates Rule 1 (NRL), because there are two unpainted '1's in the top row. (d) The paint violates Rule 2 (IPS), because there are two adjacent painted squares in the top row. (e) The paint violates Rule 3 (CUS), because the rightmost unpainted square in the bottom row is isolated.

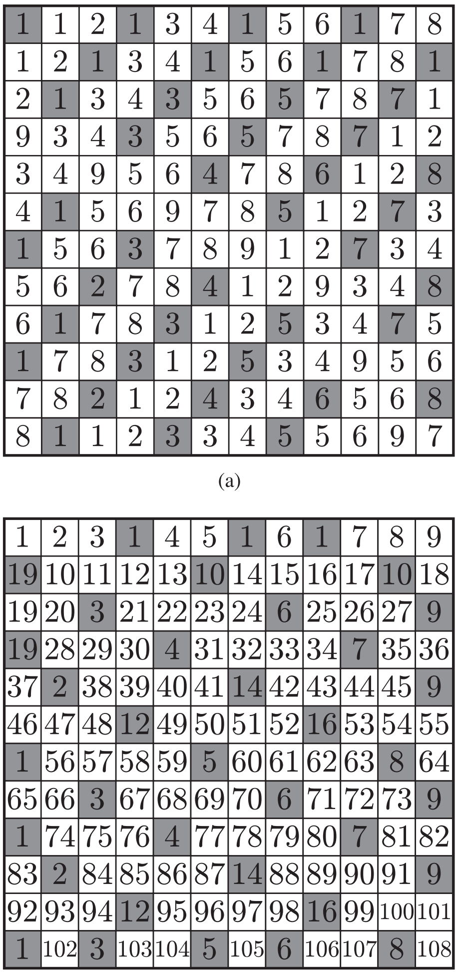


Fig. 2 Instances of 12-hitori. The gray squares display the unique solutions. (a) An instance with nine different integers. (b) An instance with 108 different integers.

paint clearly violates Rule 2 (IPS), and hence this grid has no solution. Similarly, we can show that any 4-hitori with only two different integers has no solution. On the other hand, the instance in Fig. 3 (b) consists of three different integers, and has a unique solution as in Fig. 3 (c).

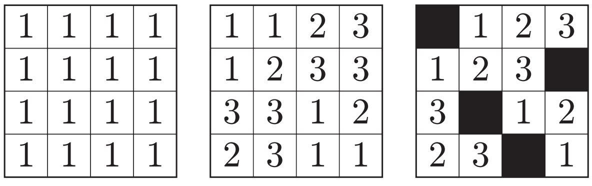


Fig. 3 (a) The 4 × 4 grid with only one integer. (b) An instance with three different integers. (c) The solution for (b).

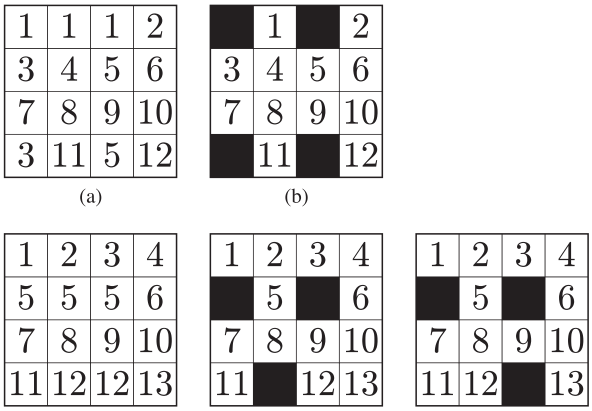


Fig. 4 (a) An instance with 12 different integers. (b) The solution for (a). (c) The 4 × 4 grid with 13 integers. (d)(e) Two solutions of (c).

Based on this observation, we define *hitori number* $h(n)$ for every positive integer n, $n \geq 2$, as the minimum number of different integers used in an instance, where the minimum is taken over all the instances of n-hitori. Recall that there is no instance of 1-hitori. By the above discussion, we have $h(4) = 3$. In this paper, we give lower and upper bounds on $h(n)$:

$$\left\lceil \frac{2n-1}{3} \right\rceil \leq h(n) \leq 2\left\lceil \frac{n}{3} \right\rceil + 1$$

for every integer n, $n \geq 2$. In other words, there is no instance of n-hitori using less than $\lceil (2n-1)/3 \rceil$ different integers, while there is an instance of n-hitori using $2\lceil n/3 \rceil + 1$ different integers. Note that for any integer $n \geq 2$, the gap between our upper and lower bounds is at most two, and hence these bounds are very close.

On the other hand, **Fig. 4** (a) illustrates a 4-hitori with 12 integers, and has a unique solution as in Fig. 4 (b). On the other hand, a grid in Fig. 4 (c) with 13 integers has at least two solutions as in Fig. 4 (d) and (e). In fact, we cannot make an instance of 4-hitori with 13 integers. Recall that any instance of n-hitori must have a unique solution. Similarly, we cannot make an instance of 4-hitori with more than 13 integers.

Based on this observation, we define *maximum hitori number* $\bar{h}(n)$ for every positive integer n, $n \geq 2$, as the maximum number of different integers used in an instance, where the maximum is taken over all the instances of n-hitori. By the above discussion, we have $\bar{h}(4) = 12$. In this paper, we give lower and upper bounds on $\bar{h}(n)$; we prove that

$$\left\lceil \frac{4n^2-4n+11}{5} \right\rceil \leq \bar{h}(n) \leq \frac{4n^2+2n-2}{5}$$

for every integer n, $n \geq 3$. In other words, there is no instance of n-hitori using more than $(4n^2+2n-2)/5$ different integers, while there is an instance of n-hitori using $\lceil (4n^2 - 4n + 11)/5 \rceil$ different integers.

The results on hitori number and maximum hitori number imply the following interesting fact. For 100-hitori, we have $67 \leq h(100) \leq 69$ and $7{,}924 \leq \bar{h}(100) \leq 8{,}000$. Thus, we can say that there is an instance of 100-hitori using 69 different integers, while there is an instance of 100-hitori using 7,924 different integers.

The rest of this paper is organized as follows. In Section 2, we define some terms relating to Hitori. In Section 3, we first present the lower bound of a hitori number, and then give the upper bound. In Section 4, we first present the upper bound of a maximum hitori number, and then give the lower bound. In Section 5, we conclude with some remarks.

2. Definitions

For each positive integer n, we denote $\{0, 1, \ldots, n-1\}$ by $[n]$, and $[n] \times [n]$ by $[n]^2$. In n-hitori, we are given an $n \times n$ grid where each square is labeled with an integer. We often call such a grid an n-hitori. For each pair of $i \in [n]$ and $j \in [n]$, we denote by (i, j) the square on the ith row and the jth column of the grid, and by $H_{i,j}$ the integer in (i, j). We say a square is in the *outer perimeter* if the square is in the 0th row, the $(n-1)$th row, the 0th column, or the $(n-1)$th column. (See **Fig. 5**.) We call the squares $(0, 0)$, $(0, n-1)$, $(n-1, 0)$, $(n-1, n-1)$ the *corner squares*. A square is a *side square* if the square is in the outer perimeter but not a corner square.

Let $S = (S_P, S_U)$ be a partition of $[n]^2$. We say that S is a *feasible partition* for an $n \times n$ grid if S satisfies the rules 2 (IPS) and 3 (CUS) by painting square (i, j) for every $(i, j) \in S_P$ and unpainting square (i, j) for every $(i, j) \in S_U$. We say that S is a *solution* for the grid if S satisfies the three rules by painting square (i, j) for every $(i, j) \in S_P$ and unpainting square (i, j) for every $(i, j) \in S_U$. Let S be a feasible partition. For each $(i, j) \in S_U$, we say that (i, j) is *paintable* if $S' = (S_P \cup \{(i, j)\}, S_U \setminus \{(i, j)\})$ is also a feasible partition. Let S be a solution. For each $(i, j) \in S_P$, we say that (i, j) is *decolorable* if $S' = (S_P \setminus \{(i, j)\}, S_U \cup \{(i, j)\})$ is also a solution.

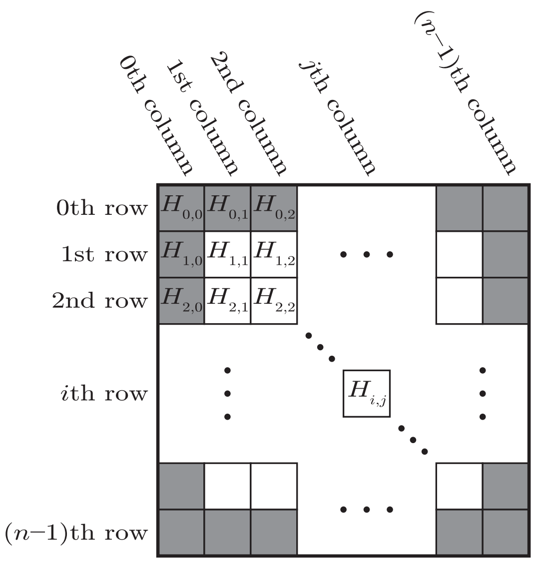


Fig. 5 Notation for the squares of a grid H. The squares in the outer perimeter are colored in gray.

An $n \times n$ grid is called an *instance* of n-hitori if it has a unique solution. Note that there is no instance of 1-hitori. We define $\mathcal{H}_n$ as the set of the instances of n-hitori.

For an instance H, we define $\alpha(H)$ as the number of different integers used in H; that is

$$\alpha(H) = |\{H_{i,j} \mid (i, j) \in [n]^2\}|.$$

We define the *hitori number* $h(n)$ as

$$h(n) = \min_{H \in \mathcal{H}_n} \alpha(H).$$

We define the *maximum hitori number* $\bar{h}(n)$ as

$$\bar{h}(n) = \max_{H \in \mathcal{H}_n} \alpha(H).$$

3. Hitori Number

In this section, we prove that $\lceil (2n-1)/3 \rceil \leq h(n) \leq 2\lceil n/3 \rceil + 1$. In Sections 3.1 and 3.2, we give the lower bound and the upper bound, respectively.

3.1 Lower Bound

In this section, we prove the following theorem.

Theorem 1. *For every integer n, $n \geq 2$, every instance $H \in \mathcal{H}_n$ satisfies*

$$\alpha(H) \geq \left\lceil \frac{2n-1}{3} \right\rceil. \tag{1}$$

Let H be an arbitrary instance in $\mathcal{H}_n$, and let $S = (S_P, S_U)$ be the solution for H. If we have

$$|S_U| \geq \frac{2n^2 - n - 2}{3}, \tag{2}$$

then the pigeonhole principle implies that at least one row of the n rows has at least

$$\left\lceil \frac{(2n^2 - n - 2)/3}{n} \right\rceil = \left\lceil \frac{2n-1}{3} - \frac{2}{3n} \right\rceil$$

unpainted squares. Since S is a solution for H, the unpainted squares in the same row must have different integers from each other. Thus at least $\lceil (2n-1)/3 - 2/3n \rceil$ different integers are used in H. For any integer $n \geq 2$, we have

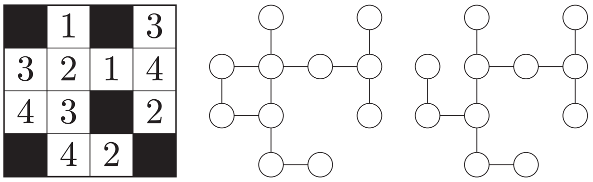
$$\left\lceil \frac{2n-1}{3} - \frac{2}{3n} \right\rceil = \left\lceil \frac{2n-1}{3} \right\rceil,$$

and hence Eq. (1) holds.

In the rest of the proof, we verify Eq. (2). For every $(i, j) \in [n]^2$, let $A(i, j) = \{(i', j') \mid (i', j')$ is adjacent to (i, j) horizontally or vertically$\}$. We then have $2 \leq |A(i, j)| \leq 4$. For every pair of $(i, j) \in [n]^2$ and $(i', j') \in A(i, j)$, we say that the boundary between (i, j) and (i', j') is a *wall* if either "$(i, j) \in S_P$ and $(i', j') \in S_U$" or "$(i, j) \in S_U$ and $(i', j') \in S_P$." We denote by $w(i, j)$ the number of walls around (i, j). Consider the undirected graph G having S_U as its vertex set, and the following edge set:

$$E = \{((i, j), (i', j')) \mid (i, j) \in S_U, (i', j') \in S_U, (i', j') \in A(i, j)\}.$$

Note that, Rule 3 (CUS) guarantees that G is connected. Let T be an arbitrary spanning tree of G. (See **Fig. 6**.) For each $(i, j) \in S_U$,



(a) (b) (c)

Fig. 6 (a) Solution $S = (S_P, S_U)$. (b) The graph $G = (S_U, E)$. (c) A spanning tree T of G.

let $d(i, j)$ be the degree of (i, j) in T. Since T is a tree, the number of edges in T is $|S_U| - 1$. Thus, we have

$$\sum_{(i,j)\in S_U} d(i, j) = 2|S_U| - 2. \tag{3}$$

Clearly we have $w(i, j) \leq 4 - d(i, j)$ for each $(i, j) \in S_U$, and hence Eq. (3) implies that the total number of the walls is at most

$$\begin{aligned}\sum_{(i,j)\in S_U} w(i, j) &\leq \sum_{(i,j)\in S_U} (4 - d(i, j)) \\ &= 4|S_U| - (2|S_U| - 2) \\ &= 2|S_U| + 2. \end{aligned} \tag{4}$$

On the other hand, Rule 2 (IPS) implies that for every $(i, j) \in S_P$, $w(i, j) = |A(i, j)|$. Note that for each side square $|A(i, j)| = 3$ and for each corner square $|A(i, j)| = 2$. Since there are at most $2n - 2$ painted squares in the outer perimeter, and at most four of these are corner squares, we have that the total number of the walls in H is at least

$$\begin{aligned}\sum_{(i,j)\in S_P} w(i, j) &= \sum_{(i,j)\in S_P} |A(i, j)| \\ &\geq 4|S_P| - (2n - 2) - 4 \\ &= 4|S_P| - 2n - 2. \end{aligned} \tag{5}$$

Clearly we have

$$\sum_{(i,j)\in S_P} w(i, j) = \sum_{(i,j)\in S_U} w(i, j), \tag{6}$$

and hence Eqs. (4)–(6) imply that

$$4|S_P| - 2n - 2 \leq 2|S_U| + 2. \tag{7}$$

Since $|S_P| + |S_U| = n^2$, Eq. (7) implies that

$$\begin{aligned} 4(n^2 - |S_U|) - 2n - 2 &\leq 2|S_U| + 2, \\ 4n^2 - 2n - 4 &\leq 6|S_U|, \\ \frac{2n^2 - n - 2}{3} &\leq |S_U|. \end{aligned}$$

Thus Eq. (2) holds.

3.2 Upper Bound

In this section, we prove the following theorem.

Theorem 2. *For every integer n, $n \geq 2$, there is an instance $H \in \mathcal{H}_n$ such that*

$$\alpha(H) \leq 2\left\lceil \frac{n}{3} \right\rceil + 1. \tag{8}$$

We prove this theorem by constructing a desired instance H that satisfies Eq. (8). In the case where $2 \leq n \leq 6$, we can construct H as described in **Fig. 7**. It is easy to verify that every instance in Fig. 7 satisfies Eq. (8).

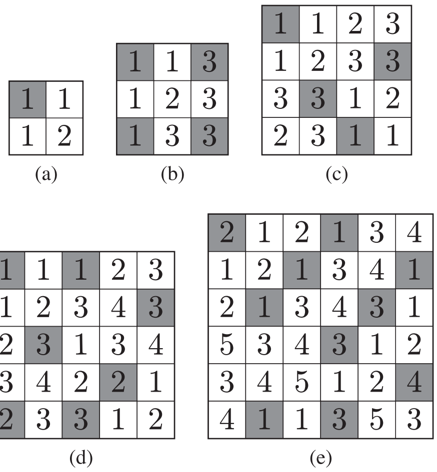


Fig. 7 Instances H that satisfy Eq. (8) for $2 \leq n \leq 6$. The gray squares display the unique solutions. (a) $\alpha(H) = 2$. (b) $\alpha(H) = 3$. (c) $\alpha(H) = 3$. (d) $\alpha(H) = 4$. (e) $\alpha(H) = 5$.

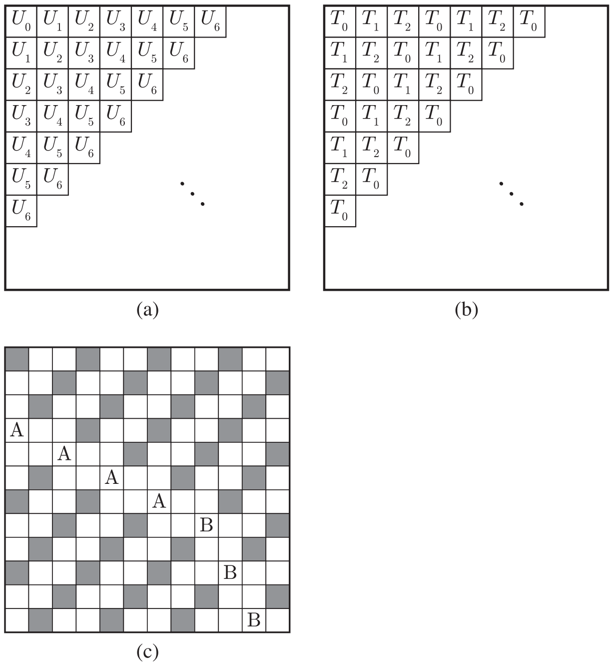


Fig. 8 (a) The subsets U_a of $[n]^2$, $0 \leq a \leq 2n - 2$. (b) The subsets T_0, T_1, and T_2 of $[n]^2$. (c) Solution S for H. Squares with a letter "A" belong to X_A and ones with a letter "B" belong to X_B.

Consider below the case where $n \geq 7$. To simplify our proof, we give a proof only for the case where $n \equiv 0 \pmod 3$. We can easily extend our proof to the other cases.

We first give a partition $S = (S_P, S_U)$ of $[n]^2$, and show that S is a feasible partition i.e., S satisfies Rules 2 (IPS) and 3 (CUS). Then, we construct a desired instance H for which S is the unique solution.

We focus on the sum of indices i and j of a row and a column. For every $0 \leq a \leq 2n - 2$, let

$$U_a = \{(i, j) \mid i + j = a\}.$$

(See **Fig. 8** (a).) We then define the following three sets T_0, T_1 and T_2 as follows:

$$T_k = \{(i, j) \mid i + j \equiv k \pmod 3\}$$

where $k \in \{0, 1, 2\}$. (See Fig. 8 (b).) Moreover, we define sets X_A and X_B. Figure 8 (c) illustrates the squares in X_A and X_B. The set X_A contains the square $(3, 0)$ and the squares (i, j) at two horizontal and one vertical step where $i + j \le n$. The set X_B contains the square $(n/3 + 3, 2n/3)$ and the squares (i, j) at one horizontal and two vertical step where $i + j \le 2n - 3$. More formally, the sets X_A and X_B are defined as follows:

$$X_A = \left\{ \left(\frac{a}{3}+2, \frac{2a}{3}-2 \right) \in T_0 \,\middle|\, 3 \le a \le n,\ a \equiv 0 \pmod 3 \right\} \qquad (9)$$

and

$$X_B = \left\{ \left(\frac{2a-n}{3} + 1, \frac{a+n}{3} - 1 \right) \in T_0 \mid n+3 \le a \le 2n-3, a \equiv 0 \pmod 3 \right\}. \qquad (10)$$

We define

$$X = X_A \cup X_B.$$

The desired partition $S = (S_P, S_U)$ is defined as

$$S_P = T_0 \setminus X \text{ and } S_U = T_1 \cup T_2 \cup X.$$

Then S gives the paint displayed in Fig. 8 (c).

The partition S is a feasible partition as in the following proposition:

Proposition 1. *$S = (S_P, S_U)$ is a feasible partition.*

Proof. We first show that S satisfies Rule 2 (IPS). A square (i, j) touches at most the following four squares: $(i-1, j)$, $(i+1, j)$, $(i, j-1)$, and $(i, j+1)$. Since $S_P \subseteq T_0$, we have $i + j \equiv 0 \pmod 3$ for every $(i, j) \in S_P$. We have $(i-1)+j \equiv 2 \pmod 3$, $(i+1)+j \equiv 1 \pmod 3$, $i+(j-1) \equiv 2 \pmod 3$, and $i+(j+1) \equiv 1 \pmod 3$, and hence $(i-1, j) \in T_2 \subseteq S_U$, $(i+1, j) \in T_1 \subseteq S_U$, $(i, j-1) \in T_2 \subseteq S_U$, and $(i, j+1) \in T_1 \subseteq S_U$. Thus S satisfies Rule 2 (IPS).

We then show that S satisfies Rule 3 (CUS). Let a be an arbitrary integer such that $3 \le a \le 2n - 3$ and $a \equiv 0 \pmod 3$. Since all the squares in U_{a-2} and U_{a-1} are in S_U, all the squares in $U_{a-2} \cup U_{a-1}$ are connected. Similarly, since all the squares in U_{a+1} and U_{a+2} are in S_U, all the squares in $U_{a+1} \cup U_{a+2}$ are connected. Now we show that $U_{a-2} \cup U_{a-1}$ and $U_{a+1} \cup U_{a+2}$ are connected via an element in X. Consider the following two cases.

Case 1: $3 \le a \le n$.

Note that $(a/3 + 2, 2a/3 - 2)$ is contained in $X_A \subseteq X \subseteq S_U$, and touches both $(a/3 + 1, 2a/3 - 2) \in U_{a-1}$ and $(a/3 + 2, 2a/3 - 1) \in U_{a+1}$. Thus all the squares in

$$U_{a-2} \cup U_{a-1} \cup \left\{ \left(\frac{a}{3} + 2, \frac{2a}{3} - 2 \right) \right\} \cup U_{a+1} \cup U_{a+2}$$

are connected.

Case 2: $n + 3 \le a \le 2n - 3$.

Note that $(2a/3 - n/3 + 1, a/3 + n/3 - 1)$ is contained in $X_B \subseteq X \subseteq S_U$, and touches both $(2a/3 - n/3, a/3 + n/3 - 1) \in U_{a-1}$ and $(2a/3 - n/3 + 1, a/3 + n/3) \in U_{a+1}$. Thus all the squares in

$$U_{a-2} \cup U_{a-1} \cup \left\{ \left(\frac{2a}{3} - \frac{n}{3} + 1, \frac{a}{3} + \frac{n}{3} - 1 \right) \right\} \cup U_{a+1} \cup U_{a+2}$$

are connected.

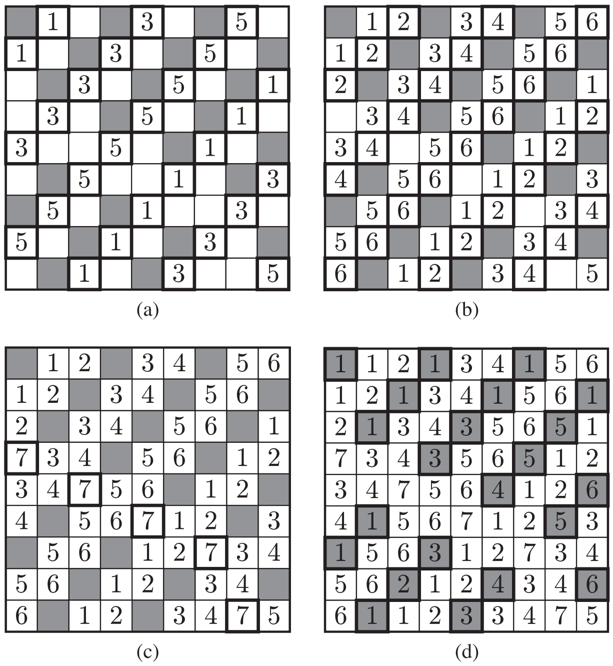


Fig. 9 Construction of the desired instance H. (a), (b), (c), and (d) correspond to Steps 1, 2, 3, and 4, respectively.

Thus we complete the proof. □

Now we construct H satisfying Eq. (8) for which S is the unique solution by the following four steps:

[Step 1] For every $(i, j) \in T_1$, we set

$$H_{i,j} = \begin{cases} 2 \cdot \left\lfloor \frac{i+j}{3} \right\rfloor + 1 & \text{if } \left\lfloor \frac{i+j}{3} \right\rfloor < \frac{n}{3}; \\ 2 \cdot \left(\left\lfloor \frac{i+j}{3} \right\rfloor - \frac{n}{3} \right) + 1 & \text{otherwise.} \end{cases} \qquad (11)$$

See **Fig. 9** (a).

[Step 2] For every $(i, j) \in T_2$, we set

$$H_{i,j} = \begin{cases} 2 \cdot \left\lfloor \frac{i+j}{3} \right\rfloor + 2 & \text{if } \left\lfloor \frac{i+j}{3} \right\rfloor < \frac{n}{3}; \\ 2 \cdot \left(\left\lfloor \frac{i+j}{3} \right\rfloor - \frac{n}{3} \right) + 2 & \text{otherwise.} \end{cases}$$

See Fig. 9 (b).

[Step 3] For every $(i, j) \in X$, we set

$$H_{i,j} = \frac{2n}{3} + 1.$$

See Fig. 9 (c).

[Step 4] For every $(i, j) \in S_P$, we set

$$H_{i,j} = \begin{cases} 1 & \text{if } 0 \le i \le 1; \\ H_{1,j} & \text{if } i \ge 2,\ \ j \equiv 0 \pmod 3); \\ H_{0,j} & \text{if } i \ge 2,\ \ j \equiv 1 \pmod 3); \\ H_{0,j} & \text{if } i \ge 2,\ \ j \equiv 2 \pmod 3). \end{cases} \qquad (12)$$

In other words, we set

$$H_{i,j} = \begin{cases} 1 & \text{if } 0 \le i \le 1; \\ \left\lfloor \frac{2 \cdot j}{3} \right\rfloor + 1 & \text{if } i \ge 2. \end{cases}$$

See Fig. 9 (d).

Clearly H satisfies Eq. (8). Now we prove that S is the unique solution for H by the following Propositions 2 and 3. In Proposition 2, we prove that S is a solution for H, that is, S satisfies Rule 1 (NRL). In Proposition 3, we prove that H admits a unique solution.

Proposition 2. *$S = (S_P, S_U)$ satisfies Rule 1 (NRL) for H.*

Proof. It suffices to show that if $(i, j) \in S_U$, then $H_{i,j} \neq H_{i,j'}$ for every $(i, j') \in S_U \setminus \{(i, j)\}$, and $H_{i,j} \neq H_{i',j}$ for every $(i', j) \in S_U \setminus \{(i, j)\}$.

Suppose Step 1 gives integers such that $H_{i,j_1} = H_{i,j_2}$ for some $i, j_1, j_2 \in [n]$ where $j_1 < j_2$. Since Eq. (11) implies that $H_{i,j}$ is monotonically increasing for $\lfloor (i+j)/3 \rfloor < n/3$ and for $\lfloor (i+j)/3 \rfloor \geq n/3$, we have $\lfloor (i+j_1)/3 \rfloor < n/3$ and $\lfloor (i+j_2)/3 \rfloor \geq n/3$. Thus

$$2 \cdot \left\lfloor \frac{i+j_1}{3} \right\rfloor + 1 = 2 \cdot \left(\left\lfloor \frac{i+j_2}{3} \right\rfloor - \frac{n}{3} \right) + 1.$$

Since $(i, j_1) \in T_1$ and $(i, j_2) \in T_1$, we have $(i + j_1) \equiv 1 \pmod 3$ and $(i + j_2) \equiv 1 \pmod 3$, and hence

$$\frac{i+j_1-1}{3} = \frac{i+j_2-1}{3} - \frac{n}{3}.$$

Therefore,

$$j_2 - j_1 \geq n.$$

This contradicts the fact that $j_1, j_2 \in [n]$. Similarly, we can prove that Steps 1 and 2 give integers so that no integer appears twice in a row or column.

Since we have never set $2n/3 + 1$ on any square in Steps 1 and 2 and X includes at most one square in each row and in each column, Step 3 gives integers so that no integer appears twice in any row and in any column. Thus, we complete the proof. □

Proposition 3. *S is the unique solution for H.*

Proof. Let $\hat{S} = (\hat{S}_P, \hat{S}_U)$ be an arbitrary solution for H. Below we show that $S_P \subseteq \hat{S}_P$. Then we prove that $S_U \subseteq \hat{S}_U$.

Proof of $S_P \subseteq \hat{S}_P$.

Consider first the square $(0, 0) \in S_P$. Since $H_{0,0} = H_{0,1} = H_{1,0}$, Rule 1 (NRL) implies that either "$(0, 0) \in \hat{S}_P$" or "$(0, 1) \in \hat{S}_P$ and $(1, 0) \in \hat{S}_P$." Thus, by Rule 3 (CUS), we have $(0, 0) \in \hat{S}_P$.

Consider next the squares in the 0th and 1st rows, that is, $(0, j) \in S_P$ and $(1, j) \in S_P$. Since $(0, 0) \in \hat{S}_P$, Rule 2 (IPS) implies that we have $(0, 1) \in \hat{S}_U$ and $(1, 0) \in \hat{S}_U$. Since $H_{0,1} = H_{1,0} = 1$, Rule 1 (NRL) and Eq. (12) imply that for each of $(0, j) \in S_P$ and $(1, j) \in S_P$, we have

$$(0, j) \in \hat{S}_P \text{ and } (1, j) \in \hat{S}_P, \tag{13}$$

respectively.

Lastly, we consider the squares $(i, j) \in S_P$ such that $2 \leq i \leq n-1$ and $j \in [n]$. We deal with the following three cases.

Case 1: $j \equiv 0 \pmod 3$.

In this case, since $(0, j) \in \hat{S}_P$ by Eq. (13), Rule 2 (IPS) implies that $(1, j) \in \hat{S}_U$. Thus, Rule 1 (NRL) and Eq. (12) imply that $(i, j) \in \hat{S}_P$.

Case 2: $j \equiv 1 \pmod 3$.

In this case, $j - 1 \equiv 0 \pmod 3$. Thus we have $(0, j-1) \in \hat{S}_P$ by Eq. (13), and hence Rule 2 (IPS) implies that $(0, j) \in \hat{S}_U$. Therefore, Rule 1 (NRL) and Eq. (12) imply that $(i, j) \in \hat{S}_P$.

Case 3: $j \equiv 2 \pmod 3$.

In this case, since $(1, j) \in \hat{S}_P$ by Eq. (13), Rule 2 (IPS) implies that $(0, j) \in \hat{S}_U$. Thus, Rule 1 (NRL) and Eq. (12) imply that $(i, j) \in \hat{S}_P$.

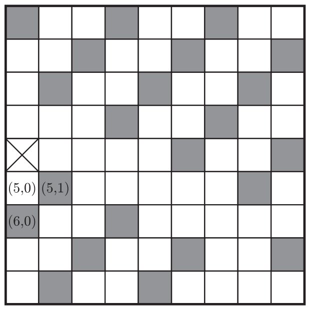


Fig. 10 Painting the crossed square $(4, 0)$ violates Rule 3 (CUS), because $(5, 0)$ is isolated.

1	1	2	1	3	4	1	5	6
1	2	1	3	4	1	5	6	1
2	1	3	4	3	5	6	5	1
7	3	4	3	5	6	5	1	2
3	4	7	5	6	4	1	2	6
4	1	5	6	7	1	2	5	3
1	5	6	3	1	2	7	3	4
5	6	2	1	2	4	3	4	6
6	1	1	2	3	3	4	7	5

Fig. 11 The instance H and its solution where $n = 9$.

Proof of $S_U \subseteq \hat{S}_U$.

Let $Z = S_U \setminus \hat{S}_U$. We prove that $Z = \emptyset$, that is, every square in S_U is not paintable.

Clearly $(4, 0) \notin Z$; otherwise the three squares $(4, 0)$, $(5, 1)$, and $(6, 0)$ are in $\hat{S}_P$, and hence $(5, 0) \in \hat{S}_U$ is isolated. (See **Fig. 10**.) Similarly, $(n-1, n-3) \notin Z$. Now we show that, for every $(i, j) \in S_U \setminus \{(4, 0), (n-1, n-3)\}$, $(i, j) \notin Z$. Recall that $S_U = T_1 \cup T_2 \cup X$.

Case 1: $(i, j) \in T_1 \setminus \{(4, 0)\}$.

Since $i + j \equiv 1 \pmod 3$, $(i-1, j) \in T_0$ or $(i, j-1) \in T_0$ is in $\hat{S}_P$. If $(i, j) \in Z$, we violate Rule 2 (IPS), and hence $(i, j) \notin Z$.

Case 2: $(i, j) \in T_2 \setminus \{(n-1, n-3)\}$.

Since $i + j \equiv 2 \pmod 3$, $(i+1, j) \in T_0$ or $(i, j+1) \in T_0$ is in $\hat{S}_P$. If $(i, j) \in Z$, we violate Rule 2 (IPS), and hence $(i, j) \notin Z$.

Case 3: $(i, j) \in X$.

Since $S_P = T_0 \setminus X$, Eqs. (9) and (10) imply that all the squares in U_{i+j} except (i, j) are in S_P. Thus, if $(i, j) \in Z$, the squares in U_{i+j-1} and the squares in U_{i+j+1} are disconnected. Hence, $(i, j) \notin Z$.

By the above discussion, we have $Z = \emptyset$. □

Figure 11 illustrates the example of the desired instance H and solution S where $n = 9$.

4. Maximum Hitori Number

In this section, we prove that $\lceil (4n^2 - 4n + 11)/5 \rceil \leq \bar{h}(n) \leq (4n^2 + 2n - 2)/5$. In Sections 4.1 and 4.2, we give the upper bound and the lower bound, respectively.

4.1 Upper Bound

In this section, we prove the following theorem:

Theorem 3. *For every integer n, $n \geq 2$, every instance $H \in \mathcal{H}_n$ satisfies*

$$\alpha(H) \leq \frac{4n^2 + 2n - 2}{5}. \tag{14}$$

For the case where $n = 2$, we have $(4n^2 + 2n - 2)/5 = 3.6$ and Eq. (14) clearly holds. Therefore, in the rest of this subsection, we consider only the case where $n \geq 3$. First, we prove the following lemma:

Lemma 1. *For every integer n, $n \geq 3$, every instance $H \in \mathcal{H}_n$ and its unique solution $S = (S_P, S_U)$ satisfy*

$$\alpha(H) \leq |S_U|. \tag{15}$$

Proof. For the sake of contradiction, assume that $\alpha(H) > |S_U|$. Since the number of different integers in H is more than the number of unpainted squares, there is an integer k such that $(i, j) \in S_P$ for every (i, j) satisfying $H_{i,j} = k$. Let (i_k, j_k) be such a square satisfying $H_{i_k,j_k} = k$.

Now we show that (i_k, j_k) is decolorable, that is, $S' = (S_P \setminus \{(i_k, j_k)\}, S_U \cup \{(i_k, j_k)\})$ is also a solution of H. This contradicts that S is the unique solution of H. Since S is a solution of H, we show only that decoloring (i_k, j_k) does not violate any rules.

Rule 1 (NRL):

Since all other squares labeled with k are painted, k is never repeated in S'.

Rule 2 (IPS):

Since decoloring (i_k, j_k) does not create a new pair of adjacent painted squares, there is no pair of adjacent painted squares in S'.

Rule 3 (CUS):

Since decoloring (i_k, j_k) does not break the connectivity of unpainted squares, all unpainted squares in S' are connected.

Therefore, there are at least two solutions and this contradicts that S is the unique solution of H. □

To prove Theorem 3, it suffices to verify

$$|S_U| - (2n - 2) \leq 4|S_P|. \tag{16}$$

Then, $|S_P| + |S_U| = n^2$ and Eq. (16) imply

$$|S_U| \leq \frac{4n^2 + 2n - 2}{5}. \tag{17}$$

Then from Eq. (17) and Lemma 1, we can easily have Eq. (14).

In the rest of the proof, we show that Eq. (16) holds. Before we verify Eq. (16), we define some terms. For a rectilinear grid, we say that two squares are *corner-adjacent* if these squares share exactly one intersection point of the grid. For a feasible partition $S = (S_P, S_U)$, we say that two squares in S_P are *corner-connected* if we can reach from one square to the other by tracing corner-adjacent squares each of which is in S_P. We call a maximal set of corner-connected squares a *corner-connected component*. Observe that S_P can be uniquely partitioned into corner-connected components. Let K be a corner-connected component in S_P. We say that K is *touched* if K includes at least one square in the outer perimeter. We say that K is a *small component* if K is not touched and $|K| \leq 2$, otherwise we say that K is a *large component*. We often say that K is small (large) if K is a small (large) component, respectively. Let S_{PS} and S_{PL} be a partition of S_P such that each square in S_{PS} is contained in a small component, and each square in S_{PL} is contained in a large component. In the rest of the proof, we say that a square is (corner-)adjacent to K if it is (corner-)adjacent to at least one of the squares in K.

Now we focus on some unpainted squares. Roughly, we want to claim that an unpainted square is paintable if it is corner-adjacent to a small component (see **Fig. 12** (a)), and an unpainted square may not be paintable if it is corner-adjacent to a large component (see Fig. 12 (b)). More formally, we prove the following lemmas:

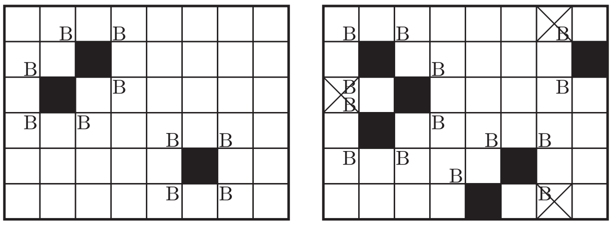


Fig. 12 (a) Two small components. (b) Three large components. The mark "B" indicates a square that is corner-adjacent to a corner-connected component. The crossed squares are not paintable.

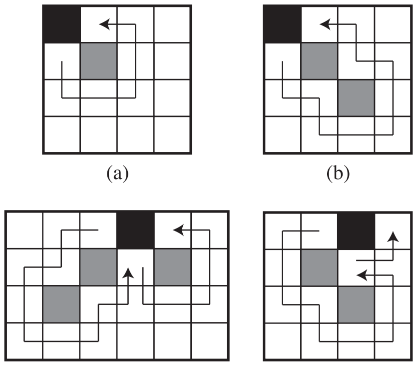


Fig. 13 The proof of Lemma 2.

Lemma 2. *Let $S = (S_P, S_U)$ be a feasible partition. Let $(i, j) \in S_U$ be a square in the outer perimeter. If (i, j) is not adjacent to any square in S_P and is not corner-adjacent to any square in S_{PL}, then (i, j) is always paintable.*

Proof. Without loss of generality, we assume $i = 0$.

(i) $(i, j) = (0, 0)$ or $(i, j) = (0, n - 1)$.

We give a proof only for $(i, j) = (0, 0)$. Since $(0, 0)$ is not adjacent to a square in S_P, $(1, 0)$ and $(0, 1)$ are in S_U. Thus, painting $(0, 0)$ does not violate Rule 2 (IPS).

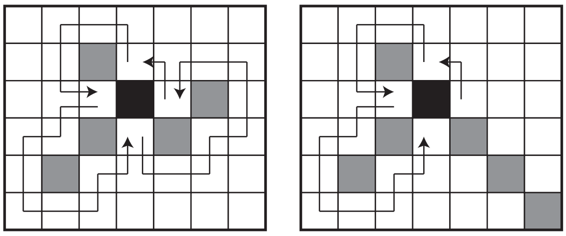
Next, we show that painting $(0, 0)$ does not violate Rule 3 (CUS), that is, $(1, 0)$ and $(0, 1)$ are still connected after painting $(0, 0)$.

If $(1, 1)$ is in S_U, clearly $(0, 0)$ is paintable because $(1, 0)$ and $(0, 1)$ are connected via $(1, 1)$.

Now we consider the case $(1, 1)$ is in S_P. Since $(0, 0)$ does not corner-adjacent to any square in S_{PL}, $(1, 1)$ is in the small component $\{(1, 1)\}$ or $\{(1, 1), (2, 2)\}$. Note that the corner-connected components $\{(1, 1), (0, 2)\}$ and $\{(1, 1), (2, 0)\}$ are touched and thus these are large components. If $(1, 1)$ is in the small component $\{(1, 1)\}$, $(1, 0)$ and $(0, 1)$ are connected with the path $(2, 0)$, $(2, 1)$, $(2, 2)$, $(1, 2)$, $(0, 2)$. (See **Fig. 13** (a).) If $(1, 1)$ is in the small component $\{(1, 1), (2, 2)\}$, $(1, 0)$ and $(0, 1)$ are connected with the path $(2, 0)$, $(2, 1)$, $(3, 1)$, $(3, 2)$, $(3, 3)$, $(2, 3)$, $(1, 3)$, $(1, 2)$, $(0, 2)$. (See Fig. 13 (b).)

(ii) $(i, j) = (0, j)$, $1 \leq j \leq n - 2$.

Since $(0, j)$ is not adjacent to a square in S_P, $(0, j - 1)$, $(1, j)$,



(a) (b)

Fig. 14 Painting (2, 3) does not violate Rule 3 (CUS).

and $(0, j+1)$ are in S_U. Thus, painting $(0, j)$ does not violate Rule 2 (IPS). In the rest of proof, we show that painting $(0, j)$ does not violate Rule 3 (CUS), that is, $(0, j-1)$, $(1, j)$, and $(0, j+1)$ are still connected after painting $(0, j)$. As in the case (i), we can show this. (See Figs. 13 (c) and (d).) □

Lemma 3. *Let $S = (S_P, S_U)$ be a feasible partition. Let $(i, j) \in S_U$ be a square not in the outer perimeter. If (i, j) is not adjacent to a square in S_P and is corner-adjacent to at most one square in S_{PL}, then (i, j) is always paintable.*

Proof. Since (i, j) is not adjacent to a square in S_P, $(i+1, j)$, $(i-1, j)$, $(i, j+1)$, and $(i, j-1)$ are in S_U. Thus, painting (i, j) does not violate Rule 2 (IPS).

Now, we show that painting (i, j) does not violate Rule 3 (CUS), that is, $(i+1, j), (i-1, j), (i, j+1)$, and $(i, j-1)$ are still connected after painting (i, j).

We consider the following two cases.

(i) (i, j) is corner-adjacent to no square in S_{PL}.

In this case, (i, j) may be corner-adjacent to only the squares in S_{PS}. In the same way as the proof of Lemma 2, we can show that $(i+1, j), (i-1, j), (i, j+1)$, and $(i, j-1)$ are connected after painting (i, j). (See **Fig. 14** (a).)

(ii) (i, j) is corner-adjacent to one square in S_{PL}.

Without loss of generality, we assume that $(i+1, j+1)$ is a square in S_{PL}. In this case, the large component may be touched and we cannot trace around the large component. However, since $(i, j+1)$ and $(i-1, j)$, $(i-1, j)$ and $(i, j-1)$, $(i, j-1)$, and $(i+1, j)$ are connected, $(i, j+1)$ and $(i, j+1)$ are always connected after painting (i, j). (See Fig. 14 (b).) □

These lemmas imply that if a square in S_U is not paintable, at least one of the following conditions holds:

Condition 1: The square in S_U is adjacent to a square in S_P.

Condition 2: The square in S_U is in the outer perimeter and corner-adjacent to at least one square in a large component.

Condition 3: The square in S_U is not in the outer perimeter and corner-adjacent to at least two squares in large components.

Let $S = (S_P, S_U)$ be a feasible partition. For S and $(i, j) \in S_U$, we define *protected number* $p(S, (i, j))$ as

$$p(S, (i, j)) = a(S, (i, j)) + 0.5 \cdot b(S, (i, j)), \tag{18}$$

where $a(S, (i, j))$ is the number of corner-connected components to which (i, j) is adjacent, and $b(S, (i, j))$ is the number of squares in S_{PL} to which (i, j) is corner-adjacent. Note that if (i, j) is in the outer perimeter and $p(S, (i, j)) < 0.5$ then (i, j) is paintable, and if (i, j) is not in the outer perimeter and $p(S, (i, j)) < 1$ then (i, j) is paintable.

Let K be a corner-connected component of S_P. For K, let A_K be the set of the squares in K together with the ones that are adjacent to K.

Now we define *protecting number* $p'(S, K)$ for each corner-connected component K as

$$p'(S, K) = \begin{cases} a'(S, K) + 0.5 \sum_{(i,j)\in K} b'(S, (i, j)) \\ \qquad\qquad \text{if } K \text{ is large,} \\ a'(S, K) \quad \text{if } K \text{ is small,} \end{cases} \tag{19}$$

where $a'(S, K)$ is the number of unpainted squares adjacent to K, and $b'(S, (i, j))$ is the number of unpainted squares corner-adjacent to (i, j).

Let $K_1, K_2, \ldots, K_m$ be the corner-connected components of S_P. We will verify

$$\sum_{(i,j)\in S_U} p(S, (i, j)) = \sum_{i=1}^{m} p'(S, K_i), \tag{20}$$

$$\sum_{i=1}^{m} p'(S, K_i) \le 4|S_P|, \tag{21}$$

and

$$\sum_{(i,j)\in S_U} p(S, (i, j)) \ge |S_U| - (2n-2). \tag{22}$$

From Eqs. (20), (21), and (22), we can easily verify Eq. (16) as required.

Proof of Eq. (20).

To verify that Eq. (20), we show that the following two equations hold:

$$\sum_{(i,j)\in S_U} a(S, (i, j)) = \sum_{i=1}^{m} a'(S, K_i) \tag{23}$$

and

$$\sum_{(i,j)\in S_U} b(S, (i, j)) = \sum_{(i,j)\in S_{PL}} b'(S, (i, j)). \tag{24}$$

Remember that $a(S, (i, j))$ is the number of corner-connected components to which (i, j) is adjacent and $a'(S, K)$ is the number of unpainted squares adjacent to K. Consider the graph $G = (V, E)$ defined as follows:

- $V = V_c \cup V_U$ where $V_c \cap V_U = \emptyset$, $|V_c| = m$, $|V_U| = |S_U|$ and each vertex in V_c corresponds to each corner-connected component and each vertex in V_U corresponds to each unpainted square.
- $E = \{(v, u) \mid v \in V_c, u \in V_U\}$, such that a square in the corner-connected component corresponding to v is adjacent to the square corresponding to u.

Then both $\sum_{(i,j)\in S_U} a(S, (i, j))$ and $\sum_{i=1}^{m} a'(S, K_i)$ are equal to the number of edges in G. Thus we have Eq. (23).

Remember that $b(S, (i, j))$ is the number of squares in S_{PL} and are corner-adjacent to (i, j), and $b'(S, (i, j))$ is the number of unpainted squares that are corner-adjacent to (i, j). Consider the graph $G = (V, E)$ defined as follows:

- $V = V_P \cup V_U$ where $V_P \cap V_U = \emptyset$, $|V_P|$ is the total number of squares in S_{PL}, $|V_U| = |S_U|$ and each vertex in V_P corresponds to each square in S_{PL} and each vertex in V_U corresponds to each unpainted square.
- $E = \{(v, u) \mid v \in V_P, u \in V_U,$

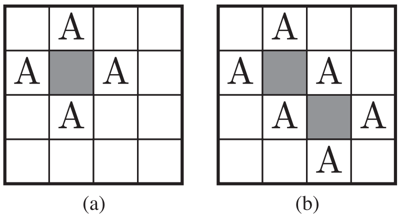


Fig. 15 Figure (a) shows $p'(S, K) = 4$ for $|K| = 1$, and Figure (b) shows $p'(S, K) = 6$ for $|K| = 2$. The mark "A" indicates a square that is adjacent to the corner-connected components.

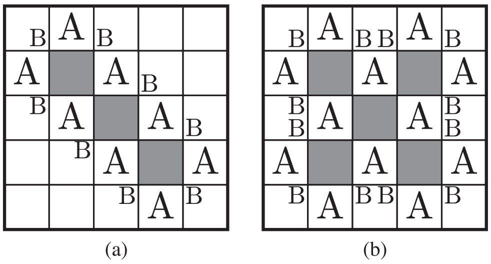


Fig. 16 Figure (a) shows $p'(S, K) = 12$ for $|K| = 3$, and Fig. (b) shows $p'(S, K) = 18$ for $|K| = 5$. The mark "A" indicates a square that is adjacent to the corner-connected component. The mark "B" indicates a square that is corner-adjacent to a corner-connected component.

the squares corresponding to v and u are corner-adjacent}. Thus both $\sum_{(i,j)\in S_U} b(S, (i, j))$ and $\sum_{(i,j)\in S_{PL}} b'(S, (i, j))$ are equal to the number of edges in G. Thus we have Eq. (24).

Proof of Eq. (21).

We verify that

$$p'(S, K) \leq 4|K| \tag{25}$$

holds for each corner-connected component K.

(i) K is small.

In this case,

$$p'(S, K) = a'(S, K). \tag{26}$$

Since K is not touched, $p'(S, K) = 4$ if $|K| = 1$ and $p'(S, K) = 6$ if $|K| = 2$. (See **Fig. 15**.) In both of the cases, Eq. (21) holds.

(ii) K is large.

Recall that

$$p'(S, K) = a'(S, K) + 0.5 \sum_{(i,j)\in K} b'(S, (i, j)). \tag{27}$$

First, we consider the case where $|K| \geq 3$ and give an upper bound of $a'(S, K)$. Since K is a corner-connected component, each square in K is corner-adjacent to another square in K. Besides, two corner-adjacent squares in K are adjacent to exactly two common unpainted squares. Since there are exactly $|K| - 1$ pairs of corner-adjacented squares, we have $a'(S, K) \leq 4|K| - 2(|K| - 1) = 2|K| + 2$. Next we give an upper bound of $\sum_{(i,j)\in K} b'(S, (i, j))$. Since there are exactly $|K| - 1$ pairs of corner-adjacented squares in K, we have $\sum_{(i,j)\in K} b'(S, (i, j)) \leq 4|K| - 2(|K| - 1) = 2|K| + 2$. Therefore, if $|K| \geq 3$, $p'(S, K) \leq 2|K| + 2 + 0.5(2|K| + 2) = 3|K| + 3 \leq 4|K|$ and Eq. (21) holds. (See **Fig. 16**.)

Now we consider the case $|K| \leq 2$. Since K is large, K is touched.

When $|K| = 1$, the only square in K is in the outer perimeter.

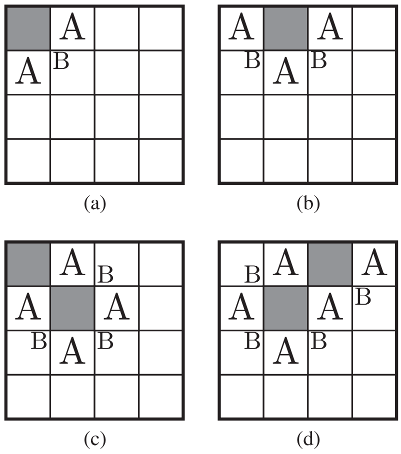


Fig. 17 Figure (a) shows $p'(S, K) = 2.5 \leq 4|K|$ for $|K| = 1$, Fig. (b) shows $p'(S, K) = 4 \leq 4|K|$ for $|K| = 1$, Fig. (c) shows $p'(S, K) = 5.5 \leq 4|K|$ for $|K| = 2$, and Fig. (d) shows $p'(S, K) = 7 \leq 4|K|$ for $|K| = 2$.

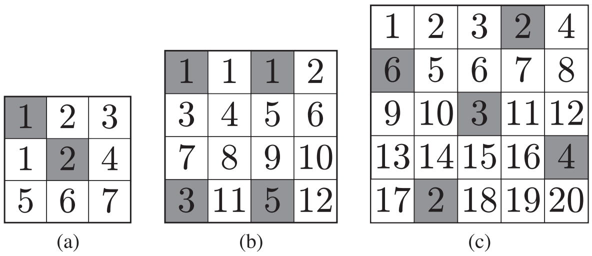


Fig. 18 Instances H that satisfy Eq. (29) for $3 \leq n \leq 5$. The gray squares display the unique solutions. (a) $\alpha(H) = 7$. (b) $\alpha(H) = 12$. (c) $\alpha(H) = 20$.

If the square is at a corner, there are two adjacent squares and one corner-adjacent square. Therefore, $p'(S, K) \leq 2.5 \leq 4|K|$. (See **Fig. 17** (a).) If the square is not at a corner, there are three adjacent squares and two corner-adjacent squares. Therefore, $p'(S, K) \leq 4 \leq 4|K|$. (See Fig. 17 (b).)

When $|K| = 2$, one of the squares in K is in the outer perimeter. In any case, $a'(S, K) \leq 5$ and $\sum_{(i,j)\in K} b'(S, (i, j)) \leq 4$. Therefore, $p'(S, K) \leq 7 \leq 4|K|$. (See Fig. 17 (c) and (d).)

Proof of Eq. (22).

If $S = (S_P, S_U)$ is the unique solution, every square (i, j) in S_U should be not paintable. Thus, we have

$$p(S, (i, j)) \geq \begin{cases} 0.5 & \text{if } (i, j) \text{ is in the outer perimeter,} \\ 1 & \text{otherwise.} \end{cases} \tag{28}$$

Since there are $4n - 4$ squares in the outer perimeter, even if all the squares in the outer perimeter are in S_U, Eq. (22) holds.

4.2 Lower Bound

In this section, we prove the following theorem.

Theorem 4. *For every integer n, $n \geq 3$, there is an instance $H \in \mathcal{H}_n$ such that*

$$\alpha(H) \geq \left\lceil \frac{4n^2 - 4n + 11}{5} \right\rceil. \tag{29}$$

We prove this theorem by constructing a desired instance H that satisfies Eq. (29). In the case where $3 \leq n \leq 5$, we can construct H as described in **Fig. 18**. It is easy to verify that every instance in Fig. 18 satisfies Eq. (29).

Consider below the case where $n \geq 6$. We show that we can

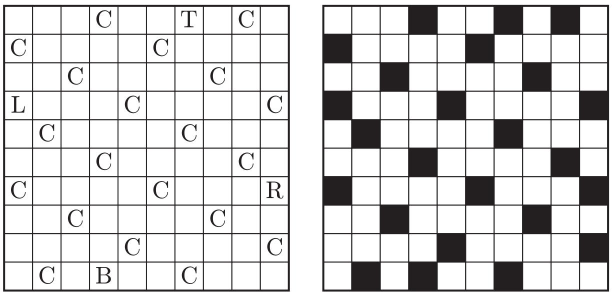


Fig. 19 Partition of $[n]^2$. (a) The squares with C (T, L, R, and B) are in T_c (T_t, T_l, T_r, and T_b, respectively). (b) Desired partition S.

construct the desired instance H such that

$$\alpha(H) = \begin{cases} \frac{4n^2-4n+20}{5} & \text{if } n \equiv 0 \pmod 5; \\ \frac{4n^2-4n+15}{5} & \text{if } n \equiv 1 \pmod 5; \\ \frac{4n^2-4n+12}{5} & \text{if } n \equiv 2 \pmod 5; \\ \frac{4n^2-4n+11}{5} & \text{if } n \equiv 3 \pmod 5; \\ \frac{4n^2-4n+12}{5} & \text{if } n \equiv 4 \pmod 5. \end{cases} \tag{30}$$

Then we can easily have Eq. (29).

To simplify our proof, we give a proof only for the case where $n \equiv 0 \pmod 5$. We can easily extend our proof to the other cases.

We first give a partition $S = (S_P, S_U)$ of $[n]^2$, and show that S satisfies Rules 2 (IPS) and 3 (CUS). Then, we construct the desired instance H for which S is the unique solution of H.

To define S, we define five sets of squares. Let

$$T_c = \{(i, j) \in [n]^2 \mid i + 2j \equiv 1 \pmod 5\}, \tag{31}$$

$$T_t = \{(0, 6), (0, 11), \ldots, (0, n-4)\},$$

$$T_l = \{(3, 0), (8, 0), \ldots, (n-7, 0)\},$$

$$T_r = \{(6, n-1), (11, n-1), \ldots, (n-4, n-1)\} \text{ and}$$

$$T_b = \{(n-1, 3), (n-1, 8), \ldots, (n-1, n-7)\}.$$

See **Fig. 19** (a). Note that these sets are pairwise disjoint. Let

$$T_o = T_t \cup T_l \cup T_r \cup T_b.$$

The desired partition $S = (S_P, S_U)$ is defined as

$$S_P = T_c \cup T_o \text{ and} \tag{32}$$

$$S_U = [n]^2 \setminus S_P. \tag{33}$$

Then S gives the paint displayed in Fig. 19 (b).

The partition S satisfies Rules 2 (IPS) and 3 (CUS) as in the following proposition:

Proposition 4. *$S = (S_P, S_U)$ satisfies Rules 2 (IPS) and 3 (CUS).*

Proof. We first show that S satisfies Rule 2 (IPS). Note that $(i, j), (i', j') \in [n]^2$ are adjacent if and only if

$$|i - i'| + |j - j'| = 1.$$

Let $(i, j), (i', j') \in T_c$. Then we have

$$|i - i'| + |j - j'| \geq 3.$$

Thus, any two squares in T_c are not adjacent. Let $(i, j), (i', j') \in T_o$. Then we have

$$|i - i'| + |j - j'| \geq 5.$$

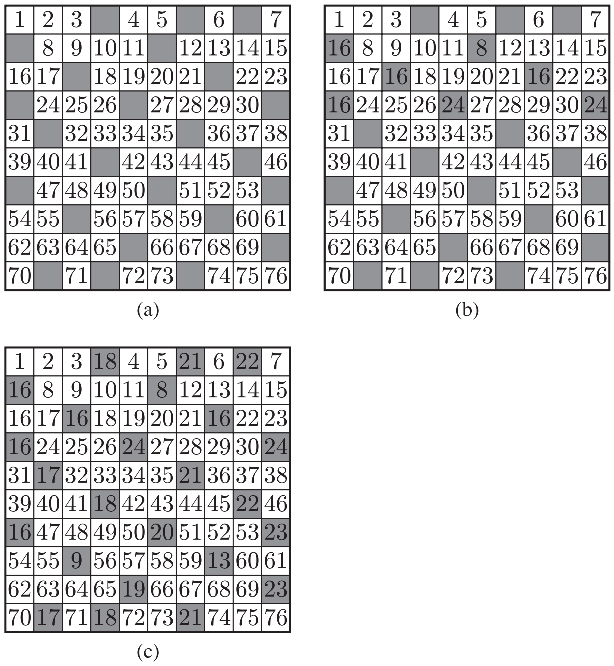


Fig. 20 Construction of the desired instance H. (a), (b) and (c) correspond to Steps 1, 2 and 3, respectively.

Now we show that each square in T_t is not adjacent to each square in T_c. For each $(0, 5j+1) \in T_t$, $(0, 5j) \notin T_c$, $(1, 5j+1) \notin T_c$, and $(0, 5j+2) \notin T_c$.

Similarly we can show that each square in T_l, T_r, and T_b are not adjacent to each square in T_c. Thus S satisfies Rule 2 (IPS).

We then show that S satisfies Rule 3 (CUS). We can easily check that the four corner squares are connected to each other. For any pair of the other unpainted squares, we need a corner-connected component of at least three painted squares to disconnect them. However, no such corner-connected component exists. Thus, Rule 3 (CUS) holds and we complete the proof. □

Now we construct H satisfying Eq. (30) for which S is the unique solution by the following three steps.

[Step 1] We set distinct numbers to the squares in S_U. See **Fig. 20** (a).

[Step 2] We set a number for every $(i, j) \in S_P$ satisfying $i = 1, 2, 3$ as follows:

$$H_{i,j} = \begin{cases} H_{1,1} & \text{if } i = 1 \text{ and } j > 0; \\ H_{3,1} & \text{if } i = 3 \text{ and } j > 0; \\ H_{2,0} & \text{otherwise.} \end{cases} \tag{34}$$

See Fig. 20 (b).

[Step 3] We set a number for every $(i, j) \in S_P$ satisfying $i = 0$ or $4 \leq i \leq n-1$ as follows:

$$H_{i,j} = \begin{cases} H_{1,j} & \text{if } j \equiv 2 \pmod 5; \\ H_{2,j} & \text{otherwise.} \end{cases} \tag{35}$$

See Fig. 20 (c).

Since each square in S_U has a distinct number, S satisfies Rule 1 (NRL). Thus, S is a solution for H.

Now we prove that S is the unique solution for H by the following proposition:

Proposition 5. *S is the unique solution for H.*

Proof. Let $\hat{S} = (\hat{S}_P, \hat{S}_U)$ be an arbitrary solution for H. As in

Proposition 3, below we show that $S_P \subseteq \hat{S}_P$. Then we prove that $S_U \subseteq \hat{S}_U$.

Proof of $S_P \subseteq \hat{S}_P$.

Consider the squares $(1,0),(3,0) \in S_P$. Since $H_{1,0} = H_{2,0} = H_{3,0}$, Rule 1 (NRL) implies that at least two of $(1,0)$, $(2,0)$, and $(3,0)$ must be painted. Thus, by Rule 2 (IPS), we have $(1,0)$, $(3,0) \in \hat{S}_P$.

Consider next the squares in the 1st row, that is, $(1, j) \in S_P$. Since $(1,0) \in \hat{S}_P$, Rule 2 (IPS) implies that we have $(1,1) \in \hat{S}_U$.

By Eq. (34), for every $(i, j) \in S_P$ in the 1st row except $(1,0)$, we have set

$$H_{i,j} = H_{1,1}.$$

Thus, for every $(i, j) \in S_P$ in the 1st row except $(1,0)$, we have

$$(i, j) \in \hat{S}_P. \tag{36}$$

Similarly, we can show that every $(i, j) \in S_P$ in the 2nd and 3rd rows are in $\hat{S}_P$.

Lastly, we consider the squares $(i, j) \in S_P$ such that $i = 0$ or $i \geq 4$. We deal with the following five cases.

Case 1: $j \equiv 0 \pmod 5$.

By Eq. (31), we have $(1, j) \in T_c \subseteq S_P$. By Eq. (36), $(1, j) \in \hat{S}_P$. Then, Rule 2 (IPS) implies that $(2, j) \in \hat{S}_U$. By Eq. (35), $H_{i,j} = H_{2,j}$. Thus, by Rule 1 (NRL), $(i, j) \in \hat{S}_P$.

Case 2: $j \equiv 1 \pmod 5$.

By Eq. (31), we have $(2, j+1) \in T_c \subseteq S_P$. As in the case 1, we have $(2, j) \in \hat{S}_U$. By Eq. (35), $H_{i,j} = H_{2,j}$. Thus, by Rule 1 (NRL), $(i, j) \in \hat{S}_P$.

Case 3: $j \equiv 2 \pmod 5$.

By Eq. (31), we have $(2, j) \in T_c \subseteq S_P$. As in the case 1, we have $(1, j) \in \hat{S}_U$. By Eq. (35), $H_{i,j} = H_{1,j}$. Thus, by Rule 1 (NRL), $(i, j) \in \hat{S}_P$.

Case 4: $j \equiv 3 \pmod 5$.

By Eq. (31), we have $(2, j-1) \in T_c \subseteq S_P$. As in the case 1, we have $(2, j) \in \hat{S}_U$. By Eq. (35), $H_{i,j} = H_{2,j}$. Thus, by Rule 1 (NRL), $(i, j) \in \hat{S}_P$.

Case 5: $j \equiv 4 \pmod 5$.

By Eq. (31), we have $(3, j) \in T_c \subseteq S_P$. As in the case 1, we have $(2, j) \in \hat{S}_U$. By Eq. (35), $H_{i,j} = H_{2,j}$. Thus, by Rule 1 (NRL), $(i, j) \in \hat{S}_P$.

Proof of $S_U \subseteq \hat{S}_U$.

Let $Z = S_U \setminus \hat{S}_U$. We now prove that $Z = \emptyset$. Clearly $(0,1) \notin Z$; otherwise $(0,0) \in \hat{S}_U$ is isolated. Similarly, $(1, n-1)$, $(n-2, 0)$, $(n-1, n-2) \notin Z$.

Since each (i, j) is adjacent to at least one square in S_P, for every $(i, j) \in S_U \setminus \{(0,1),(1,n-1),(n-2,0),(n-1,n-2)\}$, $(i, j) \notin Z$. Thus, we have $Z = \emptyset$.

Finally, we show that H satisfies Eq. (30). The step 1 of our construction implies that all squares in S_U have distinct numbers. Thus, we have

$$\alpha(H) = |S_U|.$$

Now we count squares in S_U. By Eq. (32), we have

$$S_U = [n]^2 \setminus (T_c \cup T_t \cup T_l \cup T_r \cup T_b).$$

Then we have

$$\begin{aligned} |S_U| &= \left|[n]^2 \setminus (T_c \cup T_t \cup T_l \cup T_r \cup T_b)\right| \\ &= n^2 - \left(\frac{n^2}{5} + \left(\frac{n}{5} - 1\right) + \left(\frac{n}{5} - 1\right) + \left(\frac{n}{5} - 1\right) + \left(\frac{n}{5} - 1\right)\right) \\ &= \frac{4n^2 - 4n + 20}{5}. \end{aligned}$$

□

5. Conclusions

In this paper, we investigate new combinatorial characteristics of Hitori, called hitori number $h(n)$ and maximum hitori number $\bar{h}(n)$. We prove that $\lceil (2n-1)/3 \rceil \leq h(n) \leq 2\lceil n/3 \rceil + 1$ for every integer n, $n \geq 2$, and $\lceil (4n^2 - 4n + 11)/5 \rceil \leq \bar{h}(n) \leq (4n^2 + 2n - 2)/5$ for every integer n, $n \geq 3$.

In other words, there is no instance of n-hitori using more than $(4n^2 + 2n - 2)/5$ different integers, while there is an instance of n-hitori using $\lceil (4n^2 - 4n + 11)/5 \rceil$ different integers. For 100-hitori, we have $7{,}924 \leq \bar{h}(100) \leq 8{,}000$. These results improved the results mentioned in Ref. [6], $6{,}731 \leq \bar{h}(100) \leq 8{,}911$.

Although our upper and lower bounds of hitori number $h(n)$ are close, there is a large gap between the bounds of maximum hitori number $\bar{h}(n)$. We think that tightening the gap on $\bar{h}(n)$ is an interesting challenge.

5.1 Instances with a Moderate Number of Different Integers

Since our results imply that we can construct an instance H with $\alpha(H) = 2n/3 + 1$ and H' with $\alpha(H') = (4n^2 - 4n + 11)/5$, readers may have the following natural question: Does there exist an instance H such that $\alpha(H) = i$ for each $2\lceil n/3 \rceil + 1 \leq i \leq \lceil (4n^2 - 4n + 11)/5 \rceil$? We can give a positive answer to this question by constructing desired instances.

For an instance H, the square (i, j) is called *free* if H remains an instance even if we change the integer in (i, j) from $H_{i,j}$ to $\alpha(H) + 1$. A free square (i, j) is called *multi* if there exists another square $(i', j') \neq (i, j)$ such that $H_{i,j} = H_{i',j'}$, otherwise, the free square is called *solo*.

Figure 21 (a) is the instance H of 7-hitori obtained by the construction we give in Section 3.2. Since $(1,1)$ in H is a multi free square, we can obtain another instance H' by changing $(1,1)$ from 2 to $\alpha(H) + 1 = 7$. (See Fig. 21 (b).) Obviously, $\alpha(H') = \alpha(H) + 1$. In the same way, we can make an instance H' from a given instance H so that $\alpha(H') = \alpha(H) + 1$ as long as there exists at least one multi free square. (See Fig. 21 (c).)

Let (i^*, j^*) be a free square. Let x be an integer such that $x \neq H_{i,j^*}$ and $x \neq H_{i^*,j}$ for each $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$. Then H clearly remains an instance even if we change the integer in (i^*, j^*) from H_{i^*,j^*} to x.

Figure 22 (a) is the instance H of 7-hitori obtained by the construction we give in Section 4.2. Since $(6,6)$ in H is a solo free square, and $3 \neq H_{i,6}$ and $3 \neq H_{6,j}$ for each $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$, the instance H' obtained by changing $(6,6)$ from 36 to 3 is also an instance. (See Fig. 22 (b).) Since the number 3 is used in H, $\alpha(H') = \alpha(H) - 1$. In the same way, we can make an instance H' from a given instance H so that $\alpha(H') = \alpha(H) - 1$ as long as there exists at least one solo free square. (See Fig. 22 (c).)

Note that changing the integer in a multi/solo free square may affect many other multi/solo free squares and decrease the num-

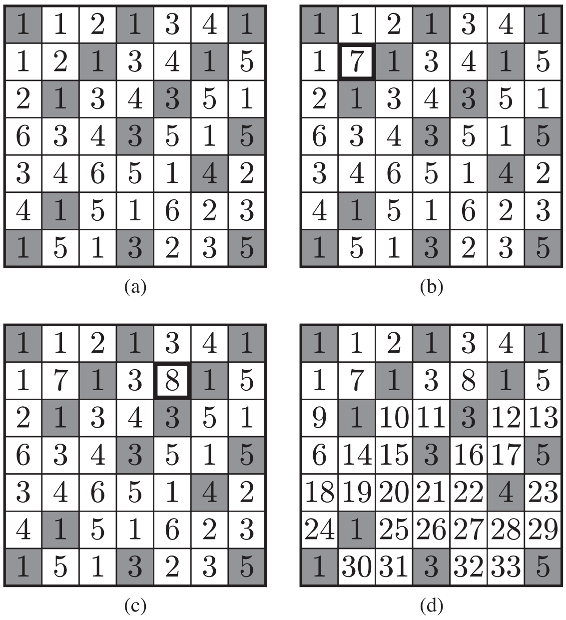


Fig. 21 Four instances of 7-hitori. (a) $\alpha(H) = 6$. (b) $\alpha(H) = 7$. (c) $\alpha(H) = 8$. (d) $\alpha(H) = 33$.

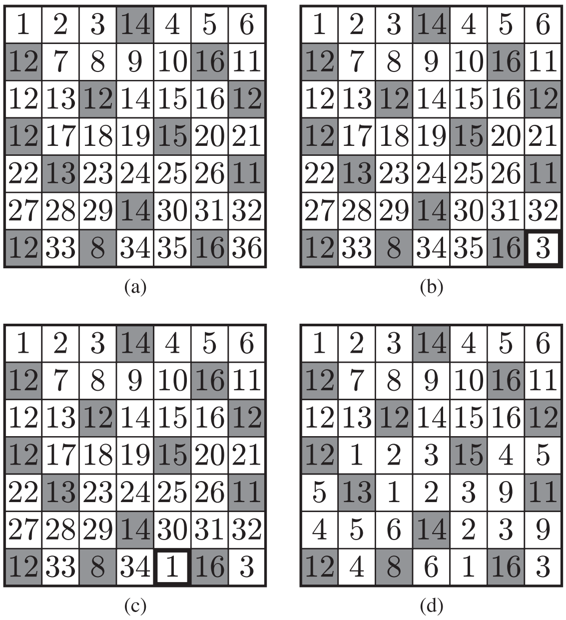


Fig. 22 Four instances of 7-hitori. (a) $\alpha(H) = 36$. (b) $\alpha(H) = 35$. (c) $\alpha(H) = 34$. (d) $\alpha(H) = 16$.

ber of such squares by more than 1. For example, $(0, 2)$ is a solo free square in Fig. 22 (a) but not in Fig. 22 (b).

We can see that the instance H constructed in Section 3.2 has many multi free squares. Furthermore, as hinted in Fig. 21 (d), there is a way for applying the local change to H for increasing α at least $n^2/2$ times. Similarly, Fig. 22 (d) hints that we can apply the other local change for decreasing α to the instance constructed in Section 4.2 at least $n^2/2$ times. Therefore, there exists an instance H such that $\alpha(H) = i$ for each $2\lceil n/3 \rceil + 1 \leq i \leq \lceil (4n^2 - 4n + 11)/5 \rceil$.

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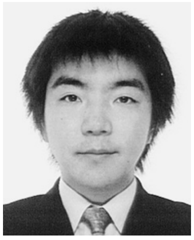

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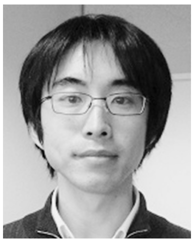

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