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On the Metric Dimension of Biregular Graph

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Abstract: The metric dimension of a connected graph G is the minimum number of vertices in a subset W of $V(G)$ such that all other vertices are uniquely determined by its vector distance to the vertices in W. In this paper, we consider a connected graph G where every vertex of G has relatively same probability to resolve some distinct vertices in G, namely a (μ, σ)-regular graph. We give tight lower and upper bounds on the metric dimension of a connected (μ, σ)-regular graphs of order $n \geq 2$ where $1 \leq \mu \leq n-1$ and $\sigma = n-1$.

Keywords: (μ, σ)-regular graph, basis, metric dimension, resolving set

1. Introduction

Throughout this paper, all graphs are finite, simple, and connected. The *vertex set* and the *edge set* of graph G are denoted by $V(G)$ and $E(G)$, respectively. The distance between two distinct vertices $u, v \in V(G)$, denoted by $d_G(u, v)$, is the length of a shortest $u - v$ path in G. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(G)$. For $v \in V(G)$, a *representation* of v with respect to W is defined as k-tuple $r(v \mid W) = (d_G(v, w_1), d_G(v, w_2), \ldots, d_G(v, w_k))$. A set W of vertices *resolves* a graph G if every two distinct vertices $x, y \in V(G)$ satisfy $r(x \mid W) \neq r(y \mid W)$. The resolving set of G with minimum cardinality is called a *basis* of G, and we call its cardinality as the *metric dimension* of G, denoted by $\beta(G)$.

The resolving set problem was introduced independently by Harary and Melter [12], and by Slater [18]. Slater considered the minimum resolving set of a graph as the location of the placement of a minimum number of sonar/loran detecting devices in a network. So, the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. This topic is also applied to various areas, including coin weighing problem [17], drug discovery [6], robot navigation [13], network discovery and verification [2], connected joins in graphs [17], and strategies for mastermind game [7].

In general, finding the metric dimension of a graph is a difficult problem. There is no effective algorithm that can be used to determine the metric dimension of any graph. Garey and Johnson [11] showed that determining the metric dimension of a graph is an NP-complete problem. Diaz et al. [8] also stated that determining the metric dimension of a graph is NP-hard even for bounded-degree planar graphs. Epstein et al. [9] extended the hardness of metric dimension for split graphs, bipartite graphs, co-bipartite graphs, and line graphs of bipartite graphs. They showed that the metric dimension can be computed efficiently for cographs, k-edge-augmented trees, and wheels. Kratica et al. [14] have determined the metric dimension of hypercube graphs of order at most 131072 and Hamming graph of order at most 4913, by using genetic algorithm. Meanwhile Fernau et al. [10] have proven that a minimum resolving set of a chain graph can be constructed in linear time.

The metric dimension for some certain classes of graphs is known. Chartrand et al. [5] have shown that path graphs and complete graphs are the only graph of order n with metric dimension 1 and $n-1$, respectively. They also studied the metric dimension of cycles and trees. The metric dimension of random graph has been studied in [3]. Zejnilović et al. [20] applied this topic to graphs with missing edges. Some results on the metric dimension of a graph obtained from a graph operation, can be seen in Refs. [4], [15], [16], [19].

In this paper, we consider a regular graph. The graph G is called μ-regular if every vertex in G is adjacent to μ other vertices. Thus, every vertex of G has the same probability to resolve some distinct vertices of G. Chartrand et al. [5] have initiated the research of metric dimension for regular graphs. They provided the metric dimension of 2-regular and $(n-1)$-regular graphs of order n. The resolving set of regular bipartite graphs has been investigated in Ref. [1]. Now, we consider a biregular graph. For integers $\mu, \sigma \geq 1$, a graph G is called a (μ, σ)-regular graph if every vertex of G is adjacent to μ or σ other vertices in G. In case of $\mu = \sigma$, we have a μ-regular graph (or σ-regular graph). In this paper, we determine the metric dimension of a connected (μ, σ)-regular graphs of order $n \geq 2$ where $1 \leq \mu \leq n-1$ and $\sigma = n-1$.

2. The Main Results

In this section, we define G as a $(\mu, n-1)$-regular graph of order $n \geq 2$ for $\mu \in \{1, 2, \ldots, n-1\}$. For a vertex $v \in V(G)$, we recall the *degree* of v in G, denoted by $d_G(v)$, is the number of adjacent vertices to v in G.

Definition 1 Let G be a $(\mu, n-1)$-regular graph of order $n \geq 2$

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for $\mu \in \{1, 2, \ldots, n-1\}$. Let $A, B \subseteq V(G)$ such that:

- $A = \{v \mid d_G(v) = n-1, v \in V(G)\}$
- $B = \{v \mid d_G(v) = \mu, v \in V(G)\}$

Note that, since G is a $(\mu, n-1)$-regular graph of order n, then A is non-empty. We can say that for $(n-1)$-regular graph G, $A = V(G)$ and $B = \emptyset$. Chartrand et al. [5] have proven that the metric dimension of $(n-1)$-regular graph is $n-1$. Now, we assume that $B \neq \emptyset$. By Definition 1, an induced subgraph of G by B may be disconnected.

Definition 2 Let G be a $(\mu, n-1)$-regular graph of order $n \geq 2$ for $\mu \in \{1, 2, \ldots, n-1\}$. For $m \geq 1$, let $\{B_1, B_2, \ldots, B_m\}$ be the partition of B such that for $1 \leq i \leq m$, $G[B_i]$ is a maximal connected induced subgraph of G by B_i.

In four propositions below, we give some properties of $(\mu, n-1)$-regular graph of order n.

Proposition 3 Let G be a $(\mu, n-1)$-regular graph of order n for $\mu \in \{1, 2, \ldots, n-1\}$. Then $diam(G) \leq 2$.

Proof. Let u and v be two non-adjacent vertices of G. Note that, there exists $a \in A$ such that $au, av \in E(G)$. Therefore, $d_G(u, v) \leq 2$. ■

Proposition 4 For $1 \leq \mu \leq n-1$, the number of vertices of A is at most μ. In particular, $|A| \in \{1, 2, \ldots, \mu\}$.

Proof. For $v \in B$, let $T_A(v) = \{x \in A \mid xv \in E(G)\}$ and $T_B(v) = \{x \in B \mid xv \in E(G)\}$. It is clear that $|T_A(v)| + |T_B(v)| = \mu$. Since $|T_B(v)|$ is non-negative integer, we have that $|T_A(v)| \leq \mu$, and since every vertex in A is adjacent to all other vertices in G, we have $|A| \leq \mu$. ■

Proposition 5 For $m \geq 2$, $i, j \in \{1, 2, \ldots, m\}$, and $i \neq j$, if x and y are two distinct vertices in B_i and $z \in B_j$, then $d_G(x, z) = d_G(y, z) = 2$.

Proof. By definition, it is clear that $xz, yz \notin E(G)$. Let a be a vertex in A. By definition of A, we have that $d_G(x, z) = d_G(x, a) + d_G(a, z) = 2 = d_G(y, a) + d_G(a, z) = d_G(y, z)$. ■

Proposition 6 For $m \geq 2$, if there exists $i \in \{1, 2, \ldots, m\}$ such that $|B_i| = 1$, then every $j \in \{1, 2, \ldots, m\} \setminus \{i\}$ satisfies $|B_j| = 1$.

Proof. Let $v \in B_i$. So, v is only adjacent to all vertices in A. Suppose that there exists $j \in \{1, 2, \ldots, m\} \setminus \{i\}$ such that $|B_j| \geq 2$. Let x and y be two distinct vertices in B_j. So, x and y are also adjacent to all vertices in A. If $xy \in E(G)$, then $d_G(x) \neq d_G(v)$, a contradiction. So, we assume that $xy \notin E(G)$. It follows that $G[B_j]$ is a null graph, which is a graph without edges. Thus, we have a contradiction with $G[B_j]$ is a maximal connected induced subgraph of G. ■

Let W be a resolving set of G. In Lemma 7 below, we show that at most one vertex of A does not belong to W. Moreover, we also prove that we can always find a resolving set W of G such that one vertex of A does not belong to W, which can be seen in Lemma 8.

Lemma 7 Let W be a resolving set of G. Then $|A \setminus W| \leq 1$.

Proof. Suppose that $|A \setminus W| \geq 2$. Let u and v be two distinct vertices in A which are not in W. Since u and v are adjacent to every vertex in $V(G) \setminus \{u, v\}$, then we have that $r(u \mid W) = r(v \mid W)$, a contradiction. ■

Lemma 8 There exists a resolving set W of G such that $|A \setminus W| = 1$.

Proof. By considering Lemma 7, let S be a resolving set of G where $A \subseteq S$. Let $a \in A$. We define a set $S' = S \setminus \{a\}$. Note that a is adjacent to every vertex in S'. If every vertex $z \in V(G) \setminus S$ satisfies $r(z \mid S) \neq (1, 1, \ldots, 1)$, then S' is a resolving set of G. Otherwise, let x be a vertex in $V(G) \setminus S$ which satisfies $r(x \mid S) = (1, 1, \ldots, 1)$. It follows that $r(x \mid S') = (1, 1, \ldots, 1)$. Then, we define $S'' = S' \cup \{x\}$. Since the representation of all vertices of G are different, we obtain that S'' is also a resolving set of G. ■

By considering Proposition 6, we use the following definition.

Definition 9 Let $\mathcal{F}$ and $\mathcal{G}$ are collections of all $(\mu, n-1)$-regular graphs of order $n \geq 2$ where for $1 \leq i \leq m$, $|B_i| = 1$ and $|B_i| \geq 2$, respectively.

Note that, for an integer $n \geq 2$, it may be exists $\mu \in \{1, 2, \ldots, n-1\}$ such that for any construction of a graph G, either $G \in \mathcal{F}$ or $G \in \mathcal{G}$. For an example, it is easy to see that if G is a $(2, n-1)$-regular graph of order an even $n \geq 2$, then $G \in \mathcal{F}$.

In the following theorem, we give an exact value of the metric dimension of a $(\mu, n-1)$-regular graph which is an element of $\mathcal{F}$.

Theorem 10 Let $G \in \mathcal{F}$. Then

$$\beta(G) = |A| + m - 2.$$

Proof. Let $|A| = t$. Note that $|B| = m$ and G is isomorphic to $(m+1)$-complete partite graph where one part has t vertices and the cardinality of other m parts is one for each. In the other hand, $G \cong K_{m,1,1,\ldots,1}$. In Ref. [15], it has been proven that $\beta(K_{m,1,1,\ldots,1}) = t + m - 2 = |A| + m - 2$. ■

Now, we assume that $G \in \mathcal{G}$. Note that, by definition of $(\mu, n-1)$-regular graph and Proposition 5, every vertex $z \in V(G) \setminus B_i$ satisfies $d_G(u, z) = d_G(v, z)$ where u and v are distinct vertices in B_i. Therefore, $G[B_i]$ must be resolved by itself.

Lemma 11 Let $G \in \mathcal{G}$. Let W be a resolving set of G. Then for $m \geq 1$ and $i \in \{1, 2, \ldots, m\}$, B_i contributes at least $\beta(G[B_i])$ vertices in W.

Proof. Let W be a resolving set of G. Suppose that there exists $i \in \{1, 2, \ldots, m\}$ such that B_i contributes at most $\beta(G[B_i]) - 1$ vertices in W. Let $W_i = W \cap B_i$. Since $|W_i| < \beta(G[B_i])$, then it is clear that there exist two different vertices $u, v \in B_i$ such that $r(u \mid W_i) = r(v \mid W_i)$. By considering definition of G and Proposition 5, we obtain that for $z \in V(G) \setminus B_i$, $d_G(u, z) = d_G(v, z)$. It follows that if $W' = W \setminus W_i$, then $r(u \mid W') = r(v \mid W')$, which implies $r(u \mid W) = r(v \mid W)$, a contradiction. ■

The direct consequence of Lemmas 8 and 11 above is Corollary 12 below.

Corollary 12 Let $G \in \mathcal{G}$. Then

$$\beta(G) \geq \left(\sum_{i=1}^{m} \beta(G[B_i]) \right) + |A| - 1.$$

We recall that the *joint graph* of H_1 and H_2, denoted by $H_1 + H_2$, is a graph with $V(H_1 + H_2) = V(H_1) \cup V(H_2)$ with $V(H_1) \cap V(H_2) = \emptyset$ and $E(H_1 + H_2) = E(H_1) \cup E(H_2) \cup \{xy \mid x \in V(H_1), y \in V(H_2)\}$. Now, let us consider a subgraph of G. Let $a \in A$. For $i \in \{1, 2, \ldots, m\}$, we have that $G[B_i \cup \{a\}]$ is isomorphic to a graph $G[B_i] + K_1$. Lemma 13 below, proved in Ref. [16], is a useful property to determine $\beta(G)$.

Lemma 13 [16] Let Q be a connected graph. There exists a basis S of $Q + K_1$ such that $S \subseteq V(Q)$.

The direct consequence of Lemma 13 above is Corollary 14 below.

Corollary 14 Let Q be a connected graph. Let S be a basis of $Q + K_1$ satisfying Lemma 13. For $v \in V(Q + K_1)$, $r(v \mid S) = (1, 1, \ldots, 1)$ if and only if v is a vertex from K_1.

For $m \geq 1$ and $1 \leq i \leq m$, let S_i be a basis of $B_i + K_1$ satisfying Lemma 13. For a vertex $a \in A$, we define $S(a) = A \setminus \{a\}$. Note that $|S(a)| = |A| - 1$. Then we define a vertex set $W = S(a) \cup \bigcup_{1 \leq i \leq m} S_i$. In most cases, W resolves $V(G)$. In Lemma 15 below, we give a condition for W such that W does not resolve $V(G)$.

Lemma 15 Let $G \in \mathcal{G}$. For $m \geq 1$ and $1 \leq i \leq m$, let S_i be a basis of $G[B_i] + K_1$ satisfying Lemma 13. For $a \in A$, let $S(a) = A \setminus \{a\}$. Let $W = S(a) \cup \bigcup_{1 \leq i \leq m} S_i$. For $x, y \in V(G)$, $r(x \mid W) = r(y \mid W)$ if and only if $x \in B_i$, $y \in B_j$, $r(x \mid S_i) = (2, 2, \ldots, 2)$, $r(y \mid S_j) = (2, 2, \ldots, 2)$, and $i \neq j$.

Proof. ($\Leftarrow$) Let $z_1, z_2 \in V(G)$. By the definition of G, z_1 is adjacent to x and y if and only if $z_1 \in A$. Now we assume that $z_1, z_2 \notin A$. By considering Propositions 3 and 5, if $z_1 \notin B_i$ and $z_2 \notin B_j$ then we have $d_G(x, z_1) = 2 = d_G(y, z_2)$. So, $r(x \mid S) = (2, 2, \ldots, 2) = r(y \mid S)$ for $S \in \{S_1, S_2, \ldots, S_m\}$ and $r(x \mid S(a)) = (1, 1, \ldots, 1) = r(y \mid S(a))$, which implies $r(x \mid W) = r(y \mid W)$.

($\Rightarrow$) Since $|A \setminus W| = 1$, by considering Lemma 8 and Corollary 14, both $x, y \notin A$. For $i, j \in \{1, 2, \ldots, m\}$, let $x \in B_i$ and $y \in B_j$. Note that, i and j must be different since S_i is a basis of $G[B_i] + K_1$ which implies every two distinct vertices in $G[B_i] + K_1$ has distinct representation with respect to S_i.

Now, let $w \in W$. If $w \in A$, then it is clear that $d_G(x, w) = d_G(y, w) = 1$. Now, we assume that $w \in B$. If $w \notin B_i \cup B_j$, then $d_G(x, w) = d_G(y, w) = 2$. Otherwise, we have either $w \in B_i$ or $w \in B_j$. Let $w \in B_i$. If $xw \in E(G)$, then $d_G(x, w) = 1 \neq 2 = d_G(y, w)$. Therefore, w must be non-adjacent to x. It implies that $r(x \mid S_i) = (2, 2, \ldots, 2)$. By a similar argument, we also have $r(y \mid S_j) = (2, 2, \ldots, 2)$. ■

So, if the condition in Lemma 15 occurs, then we need to add some vertices in $W = S(a) \cup \bigcup_{1 \leq i \leq m} S_i$ such that a new set is a resolving set of G.

Lemma 16 Let $G \in \mathcal{G}$. Then

$$\beta(G) \leq \left(\sum_{i=1}^{m} \beta(G[B_i] + K_1) \right) + |A| + m - 2.$$

Proof. For an $m \geq 1$ and $1 \leq i \leq m$, let S_i be a basis of $G[B_i] + K_1$ satisfying Lemma 13. For $a \in A$, let $S(a) = A \setminus \{a\}$. Let $S = S(a) \cup \bigcup_{1 \leq i \leq m} S_i$. We distinguish two cases.

(1) S does not satisfy the condition in Lemma 15

Then choose $W = S$. Since S_i is a basis of $G[B_i] + K_1$, S_i resolves B_i, which implies that W resolves $V(G)$. Therefore,

$$\beta(G) \leq \sum_{i=1}^{m} \beta(G[B_i] + K_1) < \left(\sum_{i=1}^{m} \beta(G[B_i] + K_1) \right) + |A| + m - 2.$$

(2) S satisfies the condition in Lemma 15

For $i \in \{1, 2, \ldots, m\}$, let $Q_i = \{x \in B_i \mid r(x \mid S_i) = (2, 2, \ldots, 2)\}$. We also define $Q = \bigcup_{1 \leq i \leq m} Q_i$. For a vertex $y \in Q$, let $Q(y) = Q \setminus \{y\}$. Now, we define $W = S \cup Q(y)$. Since S_i resolves B_i for $1 \leq i \leq m$, $S(a)$ resolves A, and $Q(y)$ resolves Q, then W is a resolving set of G. Note that $|Q| \leq m$ which implies $|Q(y)| \leq m - 1$. Therefore,

$$\beta(G) \leq |W| \leq \left(\sum_{i=1}^{m} \beta(G[B_i] + K_1) \right) + |A| + m - 2. \quad ■$$

Combining the results in Corollary 12 and Lemma 16, we obtain the following bounds of metric dimension of $(\mu, n-1)$-regular graphs.

Theorem 17 Let $G \in \mathcal{G}$. Then

$$\left(\sum_{i=1}^{m} \beta(G[B_i]) \right) + |A| - 1 \leq \beta(G) \leq \left(\sum_{i=1}^{m} \beta(G[B_i] + K_1) \right) + |A| + m - 2.$$

In Theorems 18 and 19 below, we give an existence of a $(\mu, n-1)$-regular graph $G \in \mathcal{G}$, such that $\beta(G)$ is equal to either lower bound or upper bound of Theorem 17, respectively.

Theorem 18 There exists a graph $G \in \mathcal{G}$, such that

$$\beta(G) = \left(\sum_{i=1}^{m} \beta(G[B_i]) \right) + |A| - 1.$$

Proof. For $\mu \geq 2$ and $2 \leq i \leq m$, let $G[B_i]$ be isomorphic to complete graph K_t of order $t \geq 2$, and $|A| = r = \mu - t + 1$. Note that $n = (m-1)t + \mu + 1$. We will show that the metric dimension of this $(\mu, n-1)$-regular graph G is equal to the lower bound of Theorem 17. Also by Theorem 17, we only need to prove that $\beta(G) \leq \left(\sum_{i=1}^{m} \beta(G[B_i]) \right) + |A| - 1$.

Let $A = \{a_1, a_2, \ldots, a_r\}$ and $B_i = \{b_{i,1}, b_{i,2}, \ldots b_{i,t}\}$. Chartrand et al. [5] have shown that $\beta(K_t) = t - 1$. Let $W_i = B_i \setminus \{b_{i,t}\}$ for $1 \leq i \leq m$, and $W_A = A \setminus \{a_r\}$. Now, we define $W = W_A \cup W_1 \cup W_2 \cup \ldots \cup W_m$. Let $x, y \in V(G) \setminus W$. We assume that $x \in B_i$. Then there exists $j \neq i$ such that a vertex $z \in W \cap B_j$ satisfies $yz \in E(G)$ but $xy \notin E(G)$. It follows that $r(x \mid W) \neq r(y \mid W)$. ■

Theorem 19 There exists a graph $G \in \mathcal{G}$, such that

$$\beta(G) = \left(\sum_{i=1}^{m} \beta(G[B_i] + K_1) \right) + |A| + m - 2.$$

Proof. For $\mu \geq 3$ and $1 \leq i \leq m$, let $G[B_i]$ be isomorphic to cycle graph C_8 of order 8, and $|A| = r = \mu - 2$. Note that $n = 8m + \mu - 2$. We will show that the metric dimension of this $(\mu, n-1)$-regular graph G is equal to upper bound of Theorem 17. Also by Theorem 17, we only need to prove that $\beta(G) \geq \left(\sum_{i=1}^{m} \beta(G[B_i] + K_1) \right) + |A| + m - 2$. Caceres et al. [4] have proven that $\beta(C_8 + K_1) = 3$. Therefore, we have to show that $\beta(G) \geq 4m + |A| - 2$.

Suppose that $\beta(G) \leq 4m + |A| - 3$ and S is a basis of G. By Lemma 8, there are at most $4m - 2$ vertices of B in S. For $i \in \{1, 2, \ldots, m\}$, let $S_i = S \cap B_i$. If there exists $i \in \{1, 2, \ldots, m\}$ such that $|S_i| < \beta(C_8 + K_1) = 3$, then it is clear that there exist two different vertices x and y in B_i with the same representation with respect to S_i, which implies $r(x \mid S) = r(y \mid S)$. So, for $1 \leq i \leq m$, we have $|S_i| \geq 3$. Since $|S \cap B| \leq 4m - 2$, there exist $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$ such that $|S_i| = 3 = |S_j|$. Note that, for $k \in \{i, j\}$, there exists $x_k \in B_k$ such that $r(x_k \mid S_k) = (2, 2, 2)$.

Since $x_k z \in E(G)$ where $z \in A$ and $x_k z \notin E(G)$ where $z \in B \setminus B_k$, by considering Proposition 3, we obtain that $r(x_i \mid S) = r(x_j \mid S)$, a contradiction. ■

In Theorem 20 below, we give an example of a $(\mu, n-1)$-regular graph $G \in \mathcal{G}$, such that G does not satisfy conditions in Lemma 15 and having metric dimension $\left(\sum_{i=1}^{m} \beta(G[B_i] + K_1)\right) + |A| - 1$.

Theorem 20 There exists a graph $G \in \mathcal{G}$, such that

$$\beta(G) = \left(\sum_{i=1}^{m} \beta(G[B_i] + K_1)\right) + |A| - 1$$

Proof. For $\mu \geq 3$ and $1 \leq i \leq m$, let $G[B_i]$ be isomorphic to cycle graph C_7 of order 7, and $|A| = r = \mu - 2$. Note that $n = 7m + \mu - 2$. We will show that the metric dimension of this $(\mu, n-1)$-regular graph G is equal to $\left(\sum_{i=1}^{m} \beta(G[B_i] + K_1)\right) + |A| - 1$. Caceres et al. [4] have proven that $\beta(C_7 + K_1) = 3$. Therefore, we will show that $\beta(G) = 3m + |A| - 1$.

Let $b_1, b_2, \ldots, b_7$ be seven vertices in C_7 where $E(C_7) = \{b_i b_{i+1}, b_7 b_1 \mid 1 \leq i \leq 6\}$. Let Λ be a basis of $C_7 + K_1$ satisfying Lemma 13 such that for every vertex $x \in V(C_7 + K_1)$, $r(x \mid \Lambda) \neq (2, 2, 2)$. Now, for $1 \leq i \leq m$, let $A = \{a_1, a_2, \ldots, a_r\}$ and $B_i = \{b_{i,1}, b_{i,2}, \ldots b_{i,6}\}$.

For the upper bound, we define $W_A = A \setminus \{a_r\}$, $W_i = \{b_{i,j} \mid b_j \in \Lambda\}$ for $1 \leq i \leq m$, and $W = W_A \cup \bigcup_{i=1}^{m} W_i$. Since W_A resolves A, W_i resolves B_i, and W does not satisfy the condition in Lemma 15, then we obtain that W is a resolving set of G. Therefore,

$$\beta(G) \leq |W| = 3m + |A| - 1.$$

Now, suppose that $\beta(G) \leq 3m + |A| - 2$ and S is a basis of G. By Lemma 8, there are at most $3m - 1$ vertices of B in S. For $i \in \{1, 2, \ldots, m\}$, let $S_i = S \cap B_i$. So, there exists $i \in \{1, 2, \ldots, m\}$ such that $|S_i| < \beta(C_7 + K_1) = 3$. Thus, it is clear that there exist two different vertices x and y in B_i with the same representation with respect to S_i, which implies $r(x \mid S) = r(y \mid S)$, a contradiction. Therefore, we obtain that

$$\beta(G) \geq |W| = 3m + |A| - 1. \quad ■$$

We also give an existence of a $(\mu, n-1)$-regular graph $G \in \mathcal{G}$ such that $\beta(G) = k$ where $k \in \{\alpha + 1, \alpha + 2, \ldots, \gamma - 1\}$ with $\alpha = \left(\sum_{i=1}^{m} \beta(G[B_i])\right) + |A| - 1$ and $\gamma = \left(\sum_{i=1}^{m} \beta(G[B_i] + K_1)\right) + |A| - 1$.

Theorem 21 There exists a graph $G \in \mathcal{G}$, such that $\beta(G) = k$ where $k \in \{\alpha+1, \alpha+2, \ldots, \gamma-1\}$ with $\alpha = \left(\sum_{i=1}^{m} \beta(G[B_i])\right) + |A| - 1$ and $\gamma = \left(\sum_{i=1}^{m} \beta(G[B_i] + K_1)\right) + |A| - 1$.

Proof. For $\mu \geq 3$, $1 \leq i \leq t$, and $t + 1 \leq j \leq m$, let $G[B_i]$ and $G[B_j]$ be isomorphic to cycle graph C_3 of order 3, and C_7 of order 7, respectively. Let $|A| = r = \mu - 2$. Note that $n = 3t + 7(m - t) + \mu - 2$. Chartrand et al. [5] have shown that for $s \geq 3$, $\beta(C_s) = 2$. Meanwhile, Caceres et al. [4] have proven that $\beta(C_3 + K_1) = 3 = \beta(C_7 + K_1)$. We will show that $\beta(G) = 2t + 3(m - t) + |A| - 1 = 3m - t + |A| - 1$. Note that,

$$\begin{aligned}
\left(\sum_{i=1}^{m} \beta(G[B_i])\right) + |A| - 1 &= 2m + |A| - 1 \\
&< 3m - t + |A| - 1 \\
&= \beta(G) \\
&< 3m + |A| - 1 \\
&= \left(\sum_{i=1}^{m} \beta(G[B_i] + K_1)\right) + |A| - 1.
\end{aligned}$$

Let $V(C_3) = \{b_1, b_2, b_3\}$ and $E(C_3) = \{b_i b_{i+1}, b_3 b_1 \mid 1 \leq i \leq 2\}$. Let $V(C_7) = \{c_1, c_2, \ldots, c_7\}$ and $E(C_7) = \{c_i c_{i+1}, c_7 c_1 \mid 1 \leq i \leq 6\}$. Let Γ be a basis of C_3, and Λ be a basis of $C_7 + K_1$ satisfying Lemma 13 such that for every vertex $x \in V(C_7 + K_1)$, $r(x \mid \Lambda) \neq (2, 2, 2)$. Now, for $1 \leq i \leq t$, and $t + 1 \leq j \leq m$, let $A = \{a_1, a_2, \ldots, a_r\}$, $B_i = \{b_{i,1}, b_{i,2}, b_{i,3}\}$, and $B_j = \{c_{j,1}, c_{j,2}, \ldots, c_{j,7}\}$.

For the upper bound, we define $W_A = A \setminus \{a_r\}$, $W_i = \{b_{i,l} \mid b_l \in \Gamma\}$ for $1 \leq i \leq t$, $W_j = \{c_{j,l} \mid b_l \in \Lambda\}$ for $t + 1 \leq j \leq m$, and $W = W_A \cup \bigcup_{i=1}^{t} W_i \bigcup_{j=t+1}^{m} W_j$. Since W_A resolves A, W_i resolves B_i, W_j resolves B_j, and W does not satisfy the condition in Lemma 15, we obtain that W is a resolving set of G. Therefore,

$$\beta(G) \leq |W| = 3m - t + |A| - 1.$$

Now, suppose that $\beta(G) \leq 3m - t + |A| - 2$ and S is a basis of G. By Lemma 8, there are at most $3m - t - 1$ vertices of B in S. For $i \in \{1, 2, \ldots, m\}$, let $S_i = S \cap B_i$. So, there exists $i \in \{1, 2, \ldots, t\}$ or $j \in \{t + 1, t + 2, \ldots, m\}$ such that $|S_i| < \beta(C_3) = 2$ or $|S_j| < \beta(C_6 + K_1) = 3$. If $|S_i| < \beta(C_3) = 2$, it is clear that there exist two different vertices x_1 and x_2 in B_i with the same representation with respect to S_i, which implies $r(x_1 \mid S) = r(x_2 \mid S)$, a contradiction. By the same argument, if $|S_j| < \beta(C_6 + K_1) = 3$, there exist two different vertices y_1 and y_2 in B_j such that $r(x_1 \mid S) = r(x_2 \mid S)$, a contradiction. Therefore, we obtain that

$$\beta(G) \geq |W| = 3m - t + |A| - 1. \quad ■$$

3. Conclusion

Let G be a $(\mu, n-1)$-regular graph of order n for $\mu \in \{1, 2, \ldots, n-1\}$. Let $A, B \subseteq V(G)$ where $A = \{v \mid d_G(v) = n-1, v \in V(G)\}$ and $B = \{v \mid d_G(v) = \mu, v \in V(G)\}$. For $m \geq 1$, let $B_1, B_2, \ldots, B_m$ be partitions of B such that for $1 \leq i \leq m$, $G[B_i]$ is a maximal connected induced subgraph of G.

In this paper, we provide an exact value of the metric dimension of a connected $(\mu, n-1)$-regular graphs G of order $n \geq 2$ where for $1 \leq i \leq m$, $|B_i| = 1$. We also give tight lower and upper bound of $\beta(G)$ where for $1 \leq i \leq m$, $|V(G[B_i])| \geq 2$. We also show an existence of a connected $(\mu, n-1)$-regular graph G of order $n \geq 2$ where for $1 \leq i \leq m$, $|V(G[B_i])| \geq 2$, such that $\beta(G)$ is not equal to those either lower or upper bound above.

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References

[1] Bača, M., Baskoro, E.T., Salman, A.N.M., Saputro, S.W. and Suprijanto, D.: The metric dimension of regular bipartite graphs, *Bull. Math. Soc. Sci. Math. Roumanie*, Vol.54(102), No.1, pp.15–28 (2011).

[2] Beuliova, Z., Eberhard, F., Erlebach, T., Hall, A., Hoffman, M., Mihalak, M. and Ram, I.: Network discovery and verification, *IEEE J. Sel. Areas Commun.*, Vol.24, pp.2168–2181 (2006).

[3] Bollobás, B., Mitsche, D. and Prałat, P.: Metric dimension for random graphs, *Electron. J. Combin.*, Vol.20, P1 (2013).

[4] Caceres, J., Hernando, C., Mora, M., Puertas, M.L., Pelayo, I.M., Seara, C. and Wood, D.R.: On the metric dimension of some families of graphs, *Electronic Notes in Discrete Math.*, Vol.22, pp.129–133 (2005).

[5] Chartrand, G., Eroh, L., Johnson, M.A. and Oellermann, O.R.: Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, Vol.105, pp.99–113 (2000).

[6] Chartrand, G. and Zhang, P.: The theory and application of resolvability in graphs: A survey, *Congr. Numer.*, Vol.160, pp.47–68 (2003).

[7] Chavatal, V.: Mastermind, *Combinatorica*, Vol.3, pp.325–329 (1983).

[8] Diaz, J., Pottonen, O., Serna, M. and van Leeuwen, E.J.: On the complexity of metric dimension, *Proc. ESA 2012*, LNCS, Vol.7501, pp.419–430, Springer (2012).

[9] Epstein, L., Levin, A. and Woeginger, G.J.: The (weighted) metric dimension of graphs: Hard and easy cases, *Proc. WG 2012*, LNCS, Vol.7551, pp.114–125, Springer (2012).

[10] Fernau, H., Heggernes, P., van't Hof, P., Meister, D. and Saei, R.: Computing the metric dimension for chain graphs, *Inf. Process. Lett.*, Vol.115, pp.671–676 (2015).

[11] Garey, M.R. and Johnson, D.S.: *Computers and Intractability: A Guide to the Theory of NP-completeness*, W.H. Freeman, California (1979).

[12] Harary, F. and Melter, R.A.: On the metric dimension of a graph, *Ars Combin.*, Vol.2, pp.191–195 (1976).

[13] Khuller, S., Raghavachari, B. and Rosenfield, A.: Landmarks in graphs, *Discrete Appl. Math.*, Vol.70, pp.217–229 (1996).

[14] Kratica, J., Kovačević-Vujčić, V. and Čangalović, M.: Computing the metric dimension of graphs by genetic algorithms, *Comput. Optim. Appl.*, Vol.44, pp.343–361 (2009).

[15] Saputro, S.W., Baskoro, E.T., Salman, A.N.M. and Suprijanto, D.: The metric dimension of a complete *n*-partite graph and its Cartesian product with a path, *J. Combin. Math. Combin. Comput.*, Vol.71, pp.283–293 (2009).

[16] Saputro, S.W., Simanjuntak, R., Uttunggadewa, S., Assiyatun, H., Baskoro, E.T., Salman, A.N.M. and Bača, M.: The metric dimension of the lexicographic product of graphs, *Discrete Math.*, Vol.313, pp.1045–1051 (2013).

[17] Sebo, A. and Tannier, E.: On metric generators of graphs, *Math. Oper. Res.*, Vol.29, pp.383–393 (2004).

[18] Slater, P.J.: Leaves of Trees, *Congr. Numer.*, Vol.14, pp.549–559 (1975).

[19] Yero, I.G., Kuziak, D. and Rodriguez-Velázquez, J.A.: On the metric dimension of corona product graphs, *Comput. Math. Appl.*, Vol.61, pp.2793–2798 (2011).

[20] Zejnilovic, S., Mitsche, D., Gomes, J. and Sinopoli, B.: Extending the metric dimension to graphs with missing edges, *Theor. Comput. Sci.*, Vol.609, pp.384–394 (2016).

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