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$\gamma_k(n) = \max\{\lfloor n/(2k+1)\rfloor, 1\}$ for Maximal Outerplanar Graphs with n mod $(2k+1) \le 6$

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Abstract: Let $G = (V, E)$ be an undirected graph with a set V of nodes and a set E of edges, $|V| = n$. A node v is said to *distance-k dominate* a node w if w is reachable from v by a path consisting of at most k edges. A set $D \subseteq V$ is said a *distance-k dominating set* if every node can be distance-k dominated by some $v \in D$. The size of a *minimum* distance-k dominating set, denoted by $\gamma_k(G)$, is called the *distance-k domination number* of G. The value $\gamma_k(n)$ is defined by $\gamma_k(n) = \max\{\gamma_k(G) : G \text{ has } n \text{ nodes}\}$. This paper considers $\gamma_k(n)$ for *maximal outerplanar graphs*. There is a conjecture $\gamma_k(n) = \max\{\lfloor n/(2k+1)\rfloor, 1\}$, which was proved for $k = 1, 2$. This paper gives a unified and simpler proof for $k = 1, 2, 3$. In fact, a *stronger* result is shown that for all $n > 2k$ and $r = n \bmod (2k+1) \le 6$, there exist at least $2k + 1 - r$ distinct distance-k dominating sets of size at most $\lfloor n/(2k+1)\rfloor$, which can be found in linear time.

Keywords: distance domination, maximal outerplanar graph, linear-time algorithm

1. Introduction

Let $G = (V, E)$ be an undirected graph with a set V of nodes and a set E of edges, where $|V| = n$. A node v is said to *distance-k dominate* a node w if w is reachable from v by a path consisting of at most k edges. A set $D \subseteq V$ is said a *distance-k dominating set* if every node can be distance-k dominated by some node $v \in D$. The size of a minimum distance-k dominating set, denoted by $\gamma_k(G)$, is called the *distance-k domination number* of G. Let

$$\gamma_k(n) = \max\{\gamma_k(G) : G \text{ is a graph of } n \text{ nodes}\}.$$

In particular, $\gamma_1(\cdot)$ is the well-known *domination number*.

Domination is one of the fundamental topics in graph theory, see Refs. [1], [5], [6], [10], [11], [12]. This paper considers $\gamma_k(n)$ for *maximal outerplanar graphs* (MOG). A graph is said *outerplanar* if it can be drawn in the plane without crossing and the nodes belong to the unbounded outer face. It is *maximal* if adding an extra edge breaks this property. It is known that a graph is outerplanar if and only if it does not contain K_4 or $K_{2,3}$ as a minor (Ref. [3]), and a MOG is a *visibility graph*, i.e., a *triangulation* graph of a simple polygon of n nodes (Ref. [4]). See illustrations in **Fig. 1**.

In general, it is not trivial to determine $\gamma_k(G)$ even for a MOG. Nevertheless, since the outer boundary C of a MOG is a Hamilton cycle in G, we see $\gamma_k(G) \le \gamma_k(C) = \left\lceil \frac{n}{2k+1} \right\rceil$. Hence $\gamma_k(G) = 1$ if $n \le 2k$. Thus in the following we only consider for $n > 2k$.

The above argument shows $\gamma_k(n) \le \left\lceil \frac{n}{2k+1} \right\rceil$. But in general it is not tight. Instead there is a conjecture $\gamma_k(n) = \left\lfloor \frac{n}{2k+1} \right\rfloor$, proved for $k = 1, 2$ (Refs. [1], [10]). In this paper, we give a unified and simpler proof for $k = 1, 2, 3$. In fact, we show a stronger result that for all $r = n \bmod (2k+1) \le 6$ (hence for all $k \le 3$ and $n > 2k$), there exist at least $2k + 1 - r$ *distinct* distance-k dominating sets of size at most $\left\lfloor \frac{n}{2k+1} \right\rfloor$, which can be found in linear time.

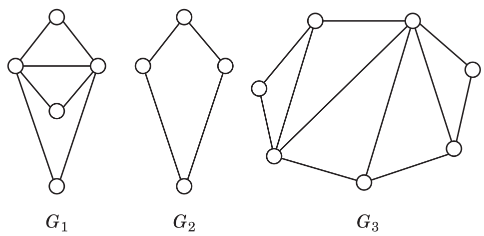

Fig. 1 An illustration of some graphs. Graph G_1 is planar but not outerplanar since it has a $K_{2,3}$ minor. On the other hand, G_2 is outerplanar but not maximal. G_3 is a maximal outerplanar graph and it is a triangulation of the outer polygon.

Related works Campos and Wakabayashi [2] showed $\gamma_1(n) = \lfloor (n+t)/4 \rfloor$, where t is the number of degree-2 nodes ($t \ge 2$). This result was independently proved by Tokunaga [11] using a coloring-based and simpler proof.

2. Preliminaries

Let $P = u - w - v$ denote a path with nodes u, w, v and edges (u, w), (w, v). A *triangle ear* (simply an *ear* in the following) with respect to a graph $G = (V, E)$ is such a path $P = u - w - v$ that $w \notin V$, $u, v \in V$, and $(u, v) \in E$ (see an illustration in **Fig. 2**). We use $G + P$ to denote the graph obtained by adding P to G, and similarly $G + P_1 + \cdots + P_i = (G + P_1 + \cdots + P_{i-1}) + P_i$ for $i \ge 2$. In this paper, we prove the following theorem.

Theorem 1 For any $k \ge 1$, $p \ge 1$, $0 \le r \le \min\{6, 2k\}$ and $n = p(2k+1) + r$, $\gamma_k(G) \le p = \left\lfloor \frac{n}{2k+1} \right\rfloor$ for any graph $G = C + P_1 + \cdots + P_r$, where C is a simple cycle of $p(2k+1) = n - r$ nodes, P_i are ears with respect to $C + P_1 + \cdots + P_{i-1}$, $i \ge 2$. Moreover, at least $2k + 1 - r$ distinct distance-k dominating set of G consisting of at most p nodes of C can be found in $O(n)$ time.

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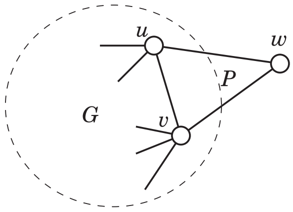

Fig. 2 An illustration of an ear $P = u - w - v$ with respect to G.

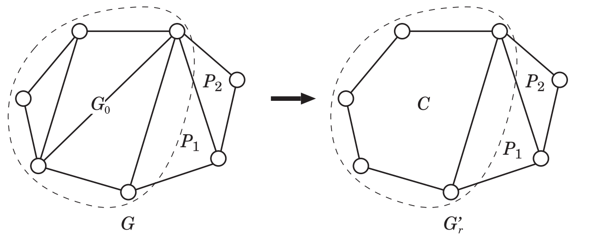

Fig. 3 An illustration for Corollary 1: the $k = 2$ case for graph G_3 in Fig. 1.

Corollary 1 $\gamma_k(G) \le \left\lfloor \frac{n}{2k+1} \right\rfloor$ for a MOG G of $n \ge 2k+1$ nodes and $k = 1, 2, 3$.

Proof. Let $p = \left\lfloor \frac{n}{2k+1} \right\rfloor \ge 1$ and $r = n - p(2k+1)$. We have $0 \le r \le 2k \le 6$ since $k \le 3$.

It is well-known (and easy to see) that any MOG with four or more nodes must have an ear $P = u - w - v$ on the outer boundary, where w is of degree two. Removing w we get a MOG with one fewer nodes. Repeating this procedure we can get an *ear decomposition* $G = G_0 + P_1 + \cdots + P_r$, where G_0 is a MOG of $p(2k+1) = n - r$ nodes and P_i are ears on the outer boundary of $G_{i-1} = G_0 + \cdots + P_{i-1}$, $i = 1, \ldots, r$.

Let C be the outer boundary of G_0. Clearly C is a Hamilton cycle of G_0. Thus P_1 is an ear with respect to C too, and graph $G'_1 = C + P_1$ has the same outer boundary as G_1. Repeating the argument, we see P_i is an ear with respect to graph $G'_{i-1} = C + \cdots + P_{i-1}$ too, and $G'_i = G'_{i-1} + P_i$ has the same outer boundary as G_i, $i \ge 2$. See an illustration in **Fig. 3**.

By Theorem 1, we have $\gamma_k(G'_r) \le p$. Since graph $G = G_r$ has the same node set as G'_r but with a superset of edges, $\gamma_k(G) \le \gamma_k(G'_r) \le p = \left\lfloor \frac{n}{2k+1} \right\rfloor$. □

Since the tight example in Ref. [10] for $k = 1$ also serves as a tight example for any $k \ge 2$, we have the next corollary.

Corollary 2 $\gamma_k(n) = \left\lfloor \frac{n}{2k+1} \right\rfloor$ for MOGs of $n \ge 2k+1$ nodes and $k = 1, 2, 3$. □

Corollary 3 For a MOG with $n \ge 2k+1$ nodes, at least $2k+1-r$ distinct distance-k dominating set of size at most $\left\lfloor \frac{n}{2k+1} \right\rfloor$ can be found in $O(n)$ time if $k \le 3$, where $r = n \bmod (2k+1)$.

Proof. An ear decomposition $G = G_0 + P_1 + \cdots + P_r$ can be found by repeatedly finding and removing degree-2 nodes. For that purpose, we store the graph by an adjacency list and calculate the degrees in $O(n)$ time (notice that the number of edges is $2n - 3$). We store the nodes using a bucket by their degrees. This can be done in $O(n)$ time. Finding a node with (residual) degree two takes $O(1)$ time. Then we set its degree to zero and for *all its neighbors in the adjacency list*, subtract their degrees by one unless it is zero (notice that we do not change the adjacency list). Then we update the bucket and continue. It is easy to see that the total time for updating the bucket is $O(n)$ as there are $O(n)$ edges.

On the other hand, determining the Hamilton cycle C for G_0 requires $O(n)$ time (Ref. [8]). Finding $2k+1-r$ distinct distance-k dominating set for G'_r, which is also a distance-k dominating set for G, requires $O(n)$ time by Theorem 1. Thus the total running time is $O(n)$. □

Remark We remark that Theorem 1 can be applied to non-MOGs. For example, it can be applied to graph G_1 in Fig. 1, which is even not outerplanar.

3. Proof for Theorem 1

In this section, we prove Theorem 1. Let $\mathrm{dist}_G(u, v)$ denote the distance between nodes u and v in a graph G, i.e., the minimum number of edges required to connect u and v in G. Given a set D of nodes, let $\mathrm{dist}_G(D, v)$ denote the distance between D and a node v, i.e.,

$$\mathrm{dist}_G(D, v) = \min_{u \in D} \{\mathrm{dist}(u, v)\}.$$

Thus D is a distance-k dominating set for graph G if and only if $\mathrm{dist}_G(D, v) \le k$ for all nodes v. The next lemma is obvious by the definition of ear.

Lemma 1 Let $G = (V, E)$ be an undirected graph, $D \subseteq V$ be a set of nodes, and P be an ear to G. Then $\mathrm{dist}_{G+P}(D, v) = \mathrm{dist}_G(D, v)$ for all $v \in V$. □

3.1 Case $r = 0$ (i.e., $G = C$ is a Cycle of $p(2k+1)$ Nodes)

Let the $p(2k+1)$ nodes of cycle C be $v_1, v_2, \ldots, v_{p(2k+1)}$ such that v_{i+1} is a neighbor of v_i. Obviously $\gamma_k(C) = p$. In fact, there are exactly $2k+1$ size-p distinct distance-k dominating sets on C:

$$S_i = \left\{v_i, v_{2k+1+i}, \ldots, v_{(p-1)(2k+1)+i}\right\}, \ i = 1, 2, \ldots, 2k+1.$$

Finding and outputting all of them requires $O(n)$ time, proving this case.

3.2 Case $r \ge 1$: a Glance of the Proof

Consider sets S_i defined in Section 3.1. First notice that for any *adjacent* nodes u and u' on C, their distances to an S_i can have exactly one of the next two relations:

- $\mathrm{dist}_C(S_i, u) = \mathrm{dist}_C(S_i, u') = k$,
- $\mathrm{dist}_C(S_i, u) - \mathrm{dist}_C(S_i, u') = \pm 1$.

Moreover, knowing u, u', $\mathrm{dist}_C(S_i, u)$ and $\mathrm{dist}_C(S_i, u')$ (of valid values), we can *uniquely* determine S_i in $O(1)$ time.

Observe that adding an ear P to C may make some S_i infeasible (i.e., cannot distance-k dominate $C + P$), but there can be *exactly one* of them. We conjecture that adding r ears can make at most r sets S_i infeasible, and thus Theorem 1 holds for all r. So far we can prove it for $r \le 6$, as shown in the following subsections.

3.3 Case $r = 1$ (One Ear)

Suppose that an ear $P_1 = u_1 - w_1 - u'_1$ is added to C and node w_1 is *not* distance-k dominated by some S_i (see an illustration in **Fig. 4**). Clearly this can happen only if $\mathrm{dist}_{C+P_1}(S_i, u_1) = \mathrm{dist}_{C+P_1}(S_i, u'_1) = k$, thus $\mathrm{dist}_C(S_i, u_1) = \mathrm{dist}_C(S_i, u'_1) = k$ by Lemma 1, hence S_i is unique and can be determined in $O(1)$ time as shown in Section 3.2. This proved for case $r = 1$.

3.4 Case $r = 2$ (Two Ears)

Let S_i be the unique set that cannot distance-k dominate $C + P_1$.

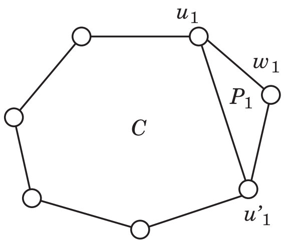

Fig. 4 Illustration for Case $r = 1$.

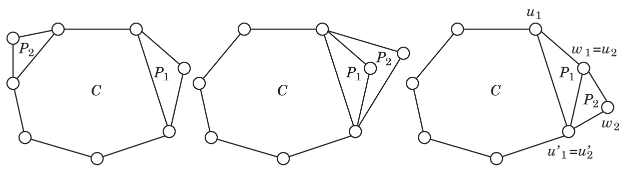

Fig. 5 Illustration for Case $r = 2$: In the first two sub-cases, P_2 is an ear to C too (notice that the graph may not be outerplanar), hence the argument for Case $r = 1$ can be applied again; In the last sub-case, P_2 is an ear with respect to P_1.

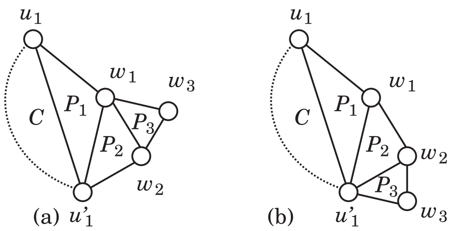

Fig. 6 Illustration for case $r = 3$: We only need to consider these two sub-cases.

Suppose an ear $P_2 = u_2 - w_2 - u'_2$ is added to $C + P_1$ (see **Fig. 5**). Clearly if P_2 is an ear with respect to C too, then we are done, because, by applying the argument for $r = 1$, P_2 can make at most one more S_h infeasible to $C + P_1 + P_2$. Thus we only need to consider when P_2 is not an ear with respect to C but to P_1. Without loss of generality, assume $u_2 = w_1$ and $u'_2 = u'_1$.

Now suppose that w_2 is *not* distance-k dominated by some set S_h that distance-k dominates $C + P_1$. This can happen only if $\mathrm{dist}_{C+P_1+P_2}(S_h, u_2) = \mathrm{dist}_{C+P_1+P_2}(S_h, u'_2) = k$. Hence $\mathrm{dist}_C(S_h, u'_1) = k$ and $\mathrm{dist}_C(S_h, u_1) = k - 1$ by easy calculation. Therefore S_h can be *uniquely* determined as shown in Section 3.2. Thus we have at least $2k - 1$ sets S_j ($j \notin \{i, h\}$) that distance-k dominates both w_1 and w_2, hence $C + P_1 + P_2$.

Determining the sub-case and, if necessary, S_h, requires $O(1)$ time, hence the running time for case $r = 2$ is $O(n)$ too. For $k = 1$, since $r = n \bmod (2k + 1) \leq 2$, the proof finishes here.

3.5 Case $r = 3$ (Three Ears, Thus $k \geq 2$)

So far we showed that adding two ears P_1 and P_2 can make at most two of the sets S_j infeasible to $C + P_1 + P_2$. Now we show that adding a third ear $P_3 = u_3 - w_3 - u'_3$ to $C + P_1 + P_2$ can make at most one more S_j infeasible.

It is easy to see that if P_2 is an ear to C, or P_2 is an ear to P_1 but P_3 is an ear to C or P_1, then we can apply the same argument for case $r = 2$. In fact, by Lemma 1, we only need to consider sub-cases in which every P_i is an ear to P_{i-1}. For $r = 3$, they are illustrated by sub-cases (a) and (b) in **Fig. 6**.

Assume that w_3 is *not* distance-k dominated by some S_ℓ that distance-k dominates $C + P_1 + P_2$. Sub-case (a) can happen only if $\mathrm{dist}_C(S_\ell, u'_1) = k - 1$ and $\mathrm{dist}_C(S_\ell, u_1) = k$. Hence we can determine the unique S_ℓ. For sub-case (b), it can happen only if $\mathrm{dist}_{C+P_1+P_2+P_3}(S_\ell, u'_1) = \mathrm{dist}_{C+P_1+P_2+P_3}(S_\ell, w_2) = k$, implying $\mathrm{dist}_{C+P_1}(S_\ell, w_1) = k - 1$, hence $\mathrm{dist}_C(S_\ell, u_1) = k - 2$. This is *impossible* since $\mathrm{dist}_C(S_\ell, u'_1) = k$, showing that such an S_ℓ does not exist in sub-case (b). Determining the sub-case and S_ℓ requires $O(1)$ time, proving Case $r = 3$.

3.6 Case $r = 4$ (Four Ears)

We showed that there can be at most three of S_j infeasible to $C + P_1 + P_2 + P_3$. Now we want to show that adding a fourth ear $P_4 = u_4 - w_4 - u'_4$ can make at most *one* more infeasible. As a conclusion, this is a *false* proposition. Nevertheless, we show how to overcome this difficulty by careful argument. Again, we only need to consider the sub-cases in which every P_i is an ear of P_{i-1}, as shown in **Fig. 7**.

Assume that w_4 is not distance-k dominated by some S_q that distance-k dominates $C + P_1 + P_2 + P_3$. This can happen only if $\mathrm{dist}_{C+P_1+\cdots+P_4}(S_q, u_4) = \mathrm{dist}_{C+P_1+\cdots+P_4}(S_q, u'_4) = k$. Then we can calculate feasible labels $\mathrm{dist}_C(S_q, u_1)$ and $\mathrm{dist}_C(S_q, u'_1)$. Easy calculation shows that it is impossible for sub-cases (a-1) and (b-1). For sub-case (a-2), the unique feasible distance labeling is $\mathrm{dist}_C(S_q, u_1) = k - 2$ and $\mathrm{dist}_C(S_q, u'_1) = k - 1$. For sub-case (b-2), however, there are *two* feasible labelings:

- $\mathrm{dist}_C(S_q, u_1) = k - 2$ and $\mathrm{dist}_C(S_q, u'_1) = k - 1$,
- $\mathrm{dist}_C(S_q, u_1) = k$ and $\mathrm{dist}_C(S_q, u'_1) = k - 1$.

Nevertheless, recall that sub-case (b-2) is derived from sub-case (b) (see Section 3.5), for which *all sets S_i feasible to $C + P_1 + P_2$ are feasible to $C + P_1 + P_2 + P_3$* too. Therefore the total number of sets S_j infeasible to $C + P_1 + \cdots + P_4$ for sub-case (b-2) is still no more than $2 + 0 + 2 = 4$. On the other hand, since it is clear that the total running time is $O(n)$, we finished proving Case $r = 4$. For $k = 2$, since $r = n \bmod (2k + 1) \leq 4$, the proof for $\gamma_2(G) \leq \max\{1, \lfloor n/5 \rfloor\}$ finishes here. □

3.7 Cases $r = 5, 6$ (Hence $k \geq 3$)

Again, we only need to consider the sub-cases in which every P_i is an ear of P_{i-1}. For each sub-case, define

k_i = the number of S_j that cannot distance-k dominate $C + P_1 + \cdots + P_i$.

All we want to show (see Section 3.2) is that for all r, $k_r \leq r$. So far we have shown it for $r \leq 4$. Now we prove it for $r = 5, 6$. See **Figs. 8**, **9**, in which we start from $r = 4$ and have marked k_r for each sub-case. The detail of the distance labels are omitted since it is much technical and not interesting.

4. Conclusion and Future Work

This paper showed $\gamma_k(n) = \max\left\{\left\lfloor \frac{n}{2k+1} \right\rfloor, 1\right\}$ for $k \leq 3$ and MOGs with a linear-time construction algorithm. In fact, letting $r = n \bmod (2k+1)$, it shows a stronger result that at least $2k+1-r$ distinct distance-k dominating sets of size at most $\left\lfloor \frac{n}{2k+1} \right\rfloor$ can be found in linear time for all $n \geq 2k + 1$ and $r \leq 6$. Currently we are working on a simple proof for $n \bmod (2k + 1) \geq 7$ cases (or to develop a counterexample), and trying to improve the results for guarding numbers and vertex cover numbers as considered in Ref. [1].

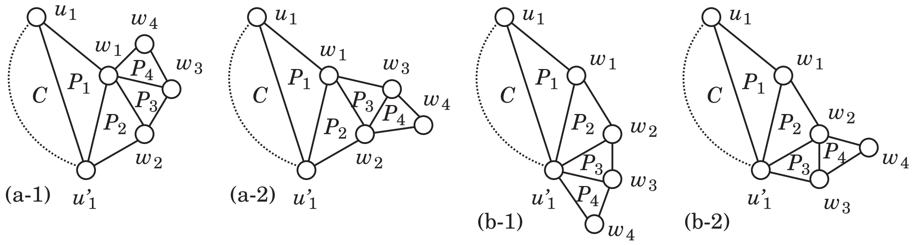

Fig. 7 Sub-cases for $r = 4$. (a-∗) and (b-∗) are derived from sub-cases (a) and (b) in Fig. 6, respectively. In all sub-cases, P_i is an ear of P_{i-1} for all i.

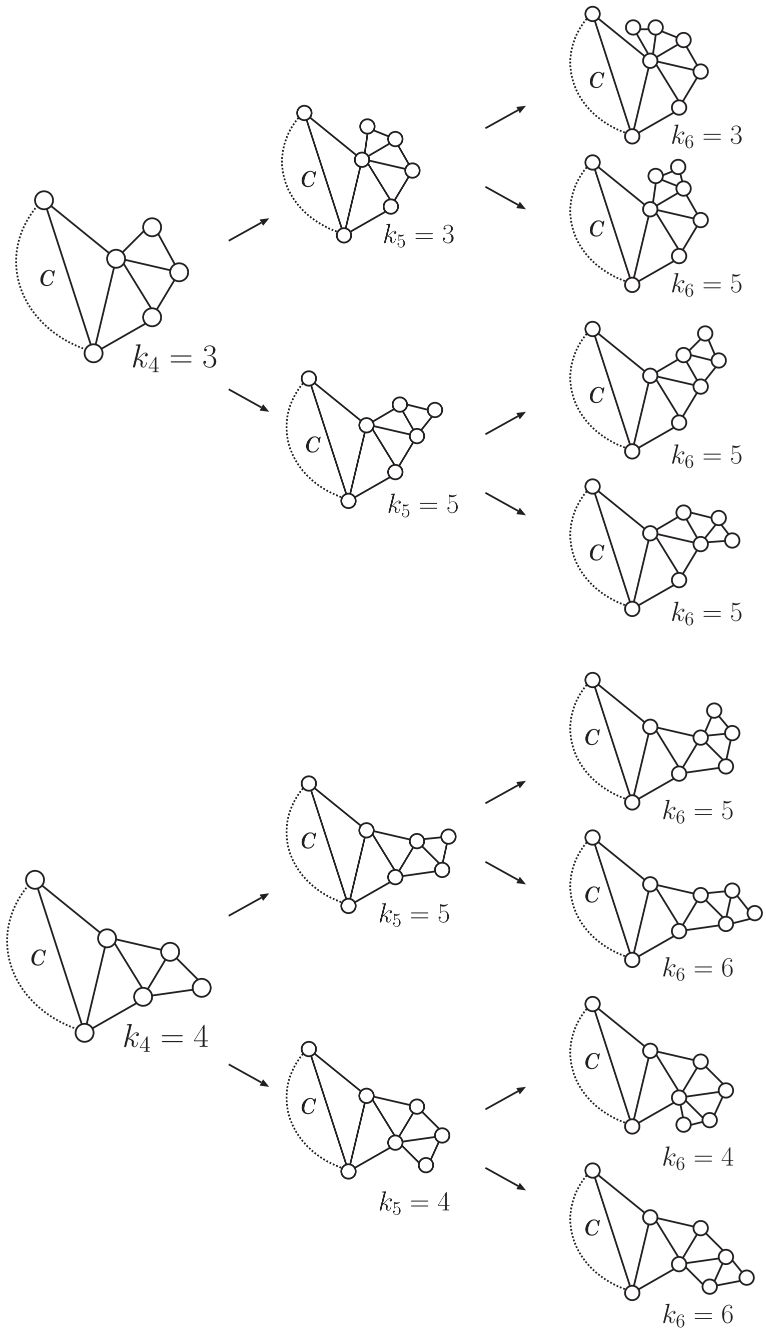

Fig. 8 Sub-cases for $r = 5, 6$ (part 1 of 2).

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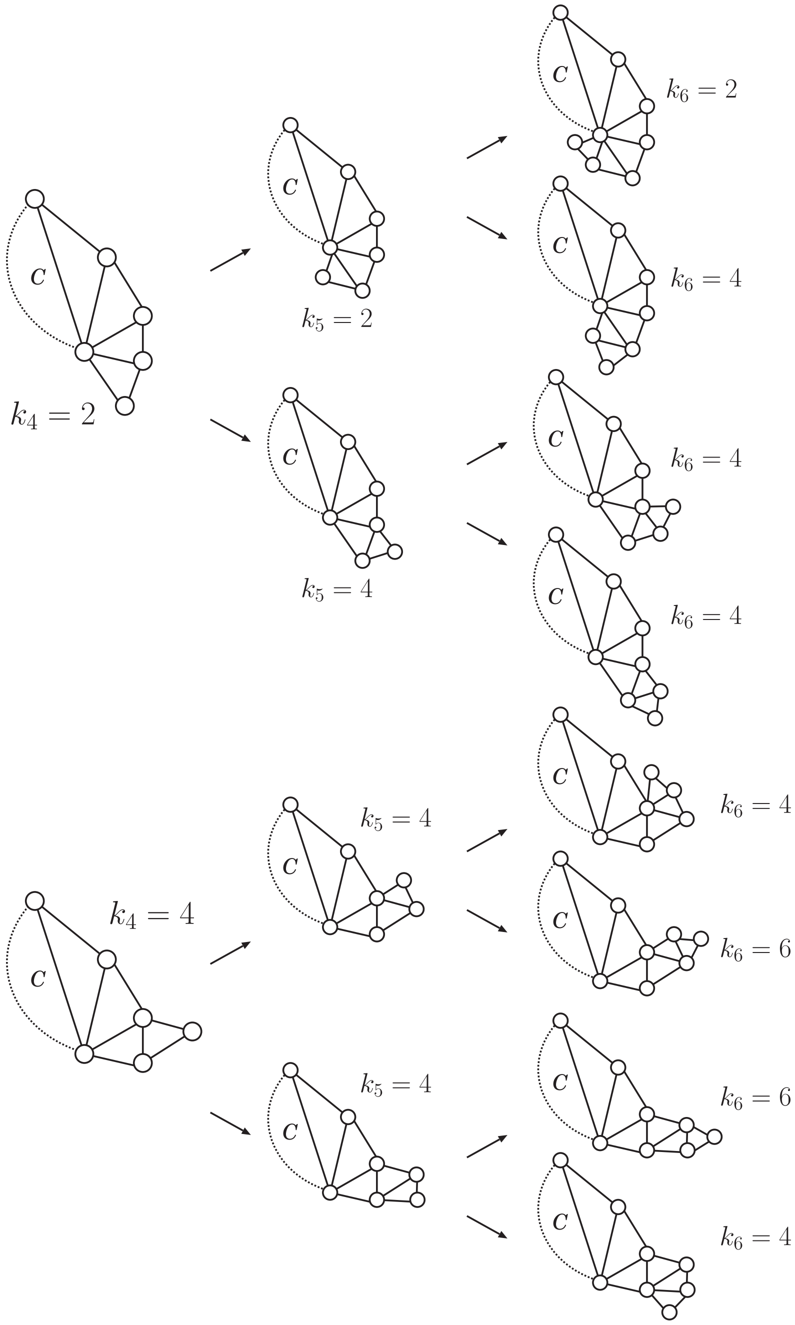

Fig. 9 Sub-cases for $r = 5, 6$ (part 2 of 2).

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