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On the Fixed Degree Tree Graph

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Abstract: A 2-switch on a simple graph G consists of deleting two edges $\{u, v\}$ and $\{x, y\}$ of G and adding the edges $\{u, x\}$ and $\{v, y\}$, provided the resulting graph is a simple graph. It is well known that if two graphs G and H have the same set of vertices and the same degree sequence, then H can be obtained from G by a finite sequence of 2-switches. While the 2-switch transformation preserves the degree sequence other conditions like connectivity may be lost. We study the restricted case where 2-switches are applied to trees to obtain trees.

Keywords: tree graph, fixed degree, 2-switch

1. Introduction

The *tree graph* of a connected graph G is the graph $T(G)$ whose vertices are the spanning trees of G, and two trees P and Q are adjacent if P can be obtained from Q by deleting an edge p of P and adding an another edge q of Q. It is easy to prove that $T(G)$ is always connected and Cummins [4] proved that if G has a cycle, then $T(G)$ is hamiltonian.

Some variations of the tree graph have been studied, like the *adjacency tree graph* studied by Zhang and Chen [11] and by Heinrich and Liu [8], the *leaf exchange tree graph* studied by Broersma and Li [3] and by Harary, Mokken and Plantholt [6]; and the *tree graph defined by a set of cycles* studied by Li, Neumann-Lara and Rivera-Campo [9].

Let $n \geq 2$ be an integer and consider the complete graph K_n with vertices $v_1, v_2, \ldots, v_n$. Let $\sigma = (d_1, d_2, \ldots, d_n)$ be the degree sequence of a spanning tree T of K_n. We define the *fixed degree tree graph of* K_n, with respect to σ, as the graph $T_\sigma(K_n)$ whose vertices are the spanning trees of K_n with degree sequence σ; that is the spanning trees S of K_n such that $\deg_S(v_i) = d_i$ for $i = 1, 2, \ldots, n$. Two spanning trees P and Q of K_n are adjacent in $T_\sigma(K_n)$ if there are non-adjacent edges p and r of P and non-adjacent edges q and s of Q, such that Q can be obtained from P by deleting p and r and adding q and s. An example appears in **Fig. 1**.

This transformation of graphs is known as a 2−switch. Havel [7] and Hakimi [5] (see also Berge [2]) proved that if two simple graphs G and H with vertex set V are such that $d_G(v) = d_H(v)$ for each $v \in V$, then H can be obtained from G by a finite sequence of 2-switches. Bereg and Ito [1] gave a formula for the minimum number of 2-switches needed to obtain H from G.

A graph H obtained from a tree T by a 2-switch may not be a tree. In this article we present some results related to the connectivity and traversability of the graphs $T_\sigma(K_n)$, in which all vertices are trees. For a connected graph G, the *distance* $d(u, v)$ between two vertices u and v of G is the length of a shortest path joining u and v in G. The *diameter*, *diam*(G), of a connected graph G is the maximum distance among the vertices of G.

2. Preliminary Results

We say that a sequence of integers $\sigma = (d_1, d_2, \ldots, d_n)$ is an *arboreal sequence* of order n if there exists a tree T with n vertices $v_1, v_2, \ldots v_n$ such that $d_T(v_i) = d_i$ for $i = 1, 2, \ldots, n$.

We need the following well known results.

Theorem 1. *A sequence $\sigma = (d_1, d_2, \ldots, d_n)$ of integers is an arboreal sequence if and only if*
$1 \leq d_i \leq n - 1$ *for* $i = 1, 2, \ldots, n$, *and*
$d_1 + d_2 + \ldots + d_n = 2(n - 1)$.

Theorem 2. [10] *Let $\sigma = (d_1, d_2, \ldots, d_n)$ be an arboreal sequence of order n. The number of spanning trees of K_n with degree sequence σ is*

$$\frac{(n-2)!}{(d_1-1)!(d_2-1)!\ldots(d_n-1)!}.$$

Theorem 3. *Let G be a graph with maximum degree Δ and for $i = 1, 2, \ldots, \Delta$ let n_i be the number vertices of G with degree i. Then*

$$\sum_{\{u,v\} \in E(G)} (d_G(u) + d_G(v)) = \sum_{i=1}^{\Delta} i^2 n_i.$$

Theorem 2 gives the order of $T_\sigma(K_n)$. The degree of a vertex in $T_\sigma(K_n)$ corresponding to a tree P is given by the number of pairs of non-adjacent edges in P. Then by a counting argument we have the following theorem.

Theorem 4. *For every arboreal sequence σ, the graph $T_\sigma(K_n)$ is a $\left(\binom{n}{2} - \frac{1}{2}\sum_{i=1}^{\Delta} i^2 n_i\right)$-regular graph where n_i is the number of vertices of degree i and Δ is the largest integer in σ.*

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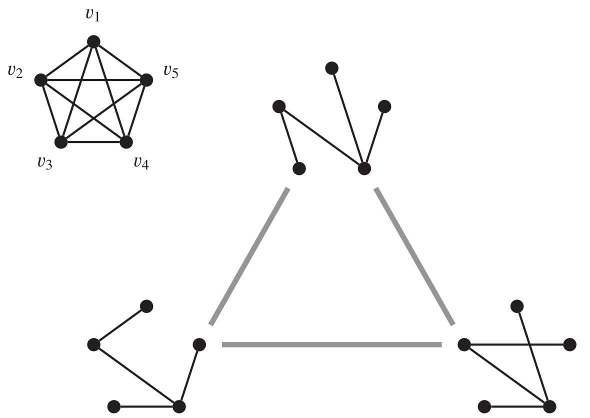

Fig. 1 $T_\sigma(K_5)$ with $\sigma = (1, 2, 1, 3, 1)$.

Proof. Let σ be an arboreal sequence, let P be a vertex of $T_\sigma(K_n)$ and let $e = \{u, v\}$ be and edge of P. Since the number of adjacent edges to e is $d_P(u) + d_P(v) - 1$, the number of non-adjacent edges to e is $n - d_P(u) - d_P(v)$. Adding over all edges of P and using Theorem 3 we obtain:

$$\sum_{\{u,v\}\in E(P)} \frac{n - d_P(u) - d_P(v)}{2} = \frac{n(n-1)}{2} - \sum_{\{u,v\}\in E(P)} \frac{d_P(u) + d_P(v)}{2}$$
$$= \binom{n}{2} - \frac{1}{2}\sum_{i=1}^{\Delta} i^2 n_i.$$

□

3. Main Results

Let $\sigma = (d_1, d_2, \ldots, d_n)$ be an arboreal sequence. For any vertex v of K_n we denote by $\sigma(v)$ the integer d_i, where i is such that $v = v_i$. Let v be a vertex in K_n such that $\sigma(v) = 1$. For each vertex u with $\sigma(u) > 1$, let $H_v(u)$ be the subgraph of $T_\sigma(K_n)$ induced by those spanning trees of K_n with degree sequence σ in which v is adjacent to u.

Lemma 5. *Let σ be an arboreal sequence of order $n \geq 4$. Let v be a vertex of K_n such that $\sigma(v) = 1$. For each vertex u of K_n with $\sigma(u) > 1$ the graph $H_v(u)$ is isomorphic to $T_{\lambda_u}(K_n - v)$, where λ_u is the arboreal sequence of order $n - 1$ given by $\lambda_u(u) = \sigma(u) - 1$ and $\lambda_u(w) = \sigma(w)$ for each vertex w with $w \in V(K_n) - \{u, v\}$.*

Proof. Let $\Theta : V(H_v(u)) \to V(T_{\lambda_u}(K_n - v))$ be given by $\Theta(P) = P - v$. Since $\{v, u\}$ is a terminal edge of P and $d_P(v) = 1$, then $P - v$ is a spanning tree of $K_n - v$; it is clear that Θ is a bijective function. If two trees P and Q are adjacent in $H_v(u)$, then there exist edges p and r in P different from $e = \{v, u\}$ and edges q and s in Q, also different from e, such that $Q = (P - \{p, r\}) + \{q, s\}$. Clearly $\Theta(Q) = Q - v = ((P - v) - \{p, r\}) + \{q, s\} = (\Theta(P) - \{p, r\}) + \{q, s\}$. Therefore $\Theta(P)$ and $\Theta(Q)$ are adjacent in $T_{\lambda_u}(K_n - v)$. Analogously if $\Theta(P)$ and $\Theta(Q)$ are adjacent in $T_{\lambda_u}(K_n - v)$, then P and Q are adjacent in $H_v(u)$. □

Lemma 6. *Let σ be an arboreal sequence and let Q be a spanning tree of K_n with degree sequence σ. Let v be a vertex of K_n such that $\sigma(v) = 1$. For each vertex u not adjacent to v in Q with $\sigma(u) > 1$, there exists a spanning tree P of K_n, also with degree sequence σ, containing the edge $\{v, u\}$, and such that P is adjacent to Q in $T_\sigma(K_n)$.*

Proof. Let u be a vertex not adjacent to v in Q and let x be the vertex adjacent to v in Q. Since $\sigma(u) > 1$, there is a vertex y adjacent to u in Q not lying in the vu path of Q. Let $P = (Q - \{\{v, x\}, \{u, y\}\}) + \{\{v, u\}, \{x, y\}\}$. Clearly $\{v, u\}$ is an edge of P, and Q is adjacent to P in $T_\sigma(K_n)$. □

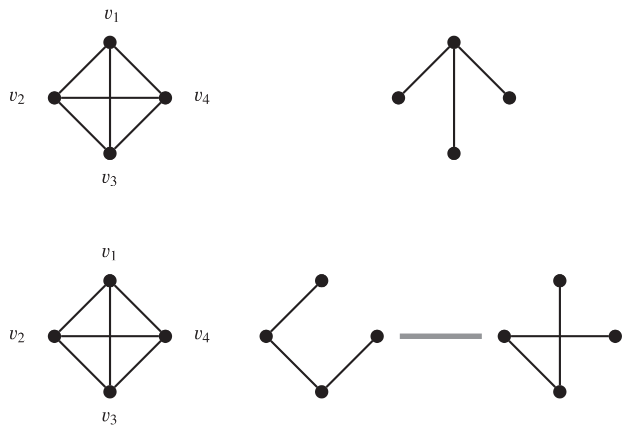

Fig. 2 $T_{(3,1,1,1)}(K_4)$ and $T_{(1,2,2,1)}(K_4)$.

Theorem 7. *Let $n \geq 4$ be an integer. For every arboreal sequence σ, $diam(T_\sigma(K_n)) \leq n - 3$. In particular, $T_\sigma(K_n)$ is connected.*

Proof. The result holds for $n = 4$, see **Fig. 2**. We proceed by induction assuming that for an integer $m \geq 4$, $diam(T_\lambda(K_m)) \leq m-3$ for every arboreal sequence λ . We prove that $diam(T_\sigma(K_{m+1})) \leq m - 2$ for any arboreal sequence σ.

Let v be a vertex of K_{m+1} for which $\sigma(v) = 1$ and let P and Q be vertices of $T_\sigma(K_{m+1})$. If there is a vertex u of K_{m+1} with $\sigma(u) > 1$ such that both P and Q are vertices of $H_v(u)$, then $d(P, Q) \leq diam(H_v(u)) = diam(T_\sigma(K_{m+1}) - v) \leq m - 3$ by Lemma 5 and by the induction hypothesis, where λ is the arboreal sequence of order m given by $\lambda(u) = \sigma(u) - 1$ and $\lambda(w) = \sigma(w)$ for $w \in V(K_m) - \{u, v\}$.

If P is a vertex of $H_v(u)$ and Q is a vertex of $H_v(w)$ with $u \neq w$, then by Lemma 6 there is a vertex R of $H_v(u)$ which is adjacent to Q in $T_\sigma(K_{m+1})$. In this case $d(P, Q) \leq d(P, R) + 1 \leq diam(H_v(u))+1 = diam(T_\sigma(K_{m+1})-v)+1 \leq (m-3)+1 = m-2$. □

Theorem 8. *Let $n \geq 4$ be an integer and σ an arboreal sequence. For each tree in $T_\sigma(K_n)$, there exists a hamiltonian path in $T_\sigma(K_n)$ that starts in P.*

Proof. The result holds for $n = 4$, see Fig. 2. We proceed by induction assuming that for an integer $m \geq 4$ and for every arboreal sequence λ and every spanning tree Q of K_m with degree sequence λ, the graph $T_\lambda(K_m)$ contains a hamiltonian path starting in Q. We prove the result for $T_\sigma(K_{m+1})$.

As in the proof of the previous theorem consider a vertex v of K_{m+1} for which $\sigma(v) = 1$ and let $u_1, u_2, \ldots, u_r$ be the vertices of K_{m+1} with $\sigma(u_i) > 1$. For $i = 1, 2, \ldots, r$ let λ_i be the arboreal sequence of order m given by $\lambda_i(u_i) = \sigma(u_i) - 1$ and $\lambda_i(w) = \sigma(w)$ for $v \neq w \neq u_i$.

Let P be a vertex of $T_\sigma(K_{m+1})$. Without loss of generality let us suppose P is a vertex of $H_v(u_1)$. By Lemma 5 the graph $H_v(u_1)$ is isomorphic to $T_{\lambda_1}(K_{m+1} - v)$ and by the induction hypothesis $T_{\lambda_1}(K_{m+1} - v)$ contains a hamiltonian path that starts in $P - v$; this implies that $H_v(u_1)$ contains a hamiltonian path T_1 that starts in P. Let Q_1 denote the other end of T_1. By Lemma 6 there exists a vertex P_2 of $H_v(u_2)$ which is adjacent to Q_1 in $T_\sigma(K_{m+1})$. Again by Lemma 5 and by the induction hypothesis, there is a hamilto-

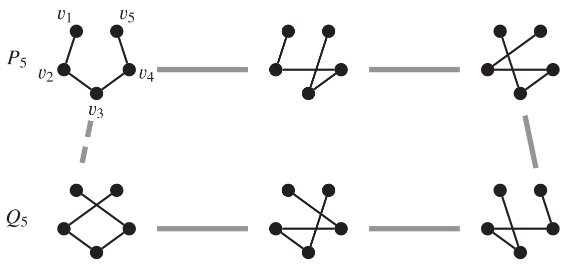

Fig. 3 Hamiltonian path joining P_5 and Q_5 in $T_\sigma(K_5)$.

nian path T_2 of $H_v(u_2)$ that starts in P_2 and ends at some vertex Q_2. Clearly this process can be continued to obtain a hamiltonian path of $T_\sigma(K_{m+1})$ that starts in P. □

Theorem 9. *If $n \geq 5$ and $\sigma_n = (1, 2, 2, \ldots, 2, 1)$, then $T_{\sigma_n}(K_n)$ is hamiltonian.*

Proof. Let $v_1, v_2, \ldots, v_n$ denote the vertices of K_n. We prove by induction that for each integer $n \geq 5$ and for each ordering $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$ of the vertices of K_n, the graph $T_{\sigma_n}(K_n)$ contains a hamiltonian path that starts in $P_n = (v_{i_1}, v_{i_2}, \ldots, v_{i_n})$ and ends in $Q_n = (v_{i_1}, v_{i_{n-1}}, v_{i_{n-2}}, \ldots, v_{i_2}, v_{i_n})$. The result follows since P_n and Q_n are adjacent in $T_{\sigma_n}(K_n)$.

We show the case $n = 5$ and the inductive step for the ordering $v_{i_k} = v_k$ for $k = 1, 2, \ldots, n$. All other orderings may be treated analogously. **Figure 3** shows that $T_{\sigma_5}(K_5)$ contains a hamiltonian path that starts in P_5 and ends in Q_5. We proceed by induction assuming that for certain integer $m \geq 5$ and for each spanning path $P = (v_{i_1}, v_{i_2}, \ldots, v_{i_m})$ of K_m the graph $T_{\sigma_m}(K_m)$ contains a hamiltonian path that starts in P and ends in $Q = (v_{i_1}, v_{i_{m-1}}, v_{i_{m-2}}, \ldots, v_{i_2}, v_{i_m})$ and consider the graph $T_{\sigma_{m+1}}(K_{m+1})$, where σ_{m+1} is the arboreal sequence $(1, 2, 2, \ldots, 2, 1)$ of order $m + 1$. Let

$$\begin{aligned} P^1_{m+1} &= (v_1, v_2, \ldots, v_{m+1}) = P_{m+1}, \\ Q^1_{m+1} &= (v_1, v_2, v_m, v_{m-1}, \ldots, v_3, v_{m+1}). \end{aligned}$$

For $i = 2, \ldots, m-2$, let

$$\begin{aligned} P^i_{m+1} &= (v_1, v_{i+1}, v_{i+2}, \ldots, v_m, v_2, v_3, \ldots, v_i, v_{m+1}), \\ Q^i_{m+1} &= (v_1, v_{i+1}, v_i, \ldots, v_2, v_m, v_{m-1}, \ldots, v_{i+2}, v_{m+1}), \end{aligned}$$

and let

$$\begin{aligned} P^{m-1}_{m+1} &= (v_1, v_m, v_2, v_3, \ldots, v_{m-1}, v_{m+1}), \\ Q^{m-1}_{m+1} &= (v_1, v_m, v_{m-1}, \ldots, v_2, v_{m+1}) = Q_{m+1}. \end{aligned}$$

For $i = 1, 2, \ldots, m-1$ let H_i be the subgraph of $T_{\sigma_{m+1}}(K_{m+1})$, induced by the spanning paths of K_{m+1} in which v_1 is adjacent to v_{i+1}. By Lemma 5, H_i is isomorphic to $T_{\lambda_{i+1}}(K_{m+1} - v_1)$, where λ_{i+1} is the arboreal sequence of order m given by $\lambda_{i+1}(v_{i+1}) = 1$ and $\lambda_{i+1}(v_j) = 2$ if $1 \neq j \neq i+1$. By the induction hypothesis $T_{\lambda_{i+1}}(K_{m+1} - v_1)$ contains a hamiltonian path that starts in $P^i_{m+1} - v_1$ and ends in $Q^i_{m+1} - v_1$. This implies that H_i contains a hamiltonian path R_i that starts in P^i_{m+1} and ends in Q^i_{m+1}.

Finally, observe that for $i = 1, 2, \ldots, m-2$, $P^{i+1}_{m+i} = Q^i_{m+1} - \{\{v_1, v_{i+1}\}, \{v_{i+2}, v_{m+1}\}\} + \{\{v_1, v_{i+2}\}, \{v_{i+1}, v_{m+1}\}\}$ which implies that Q^i_{m+1} and P^{i+1}_{m+i} are adjacent in $T_{\sigma_{m+1}}(K_{m+1})$. Therefore $R_1, R_2, \ldots, R_{m-1}$ can be joined to form a hamiltonian path in $T_{\sigma_{m+1}}(K_{m+1})$ that starts in $P_{m+1} = P^1_{m+1}$ and ends in $Q_{m+1} = Q^{m-1}_{m+1}$,

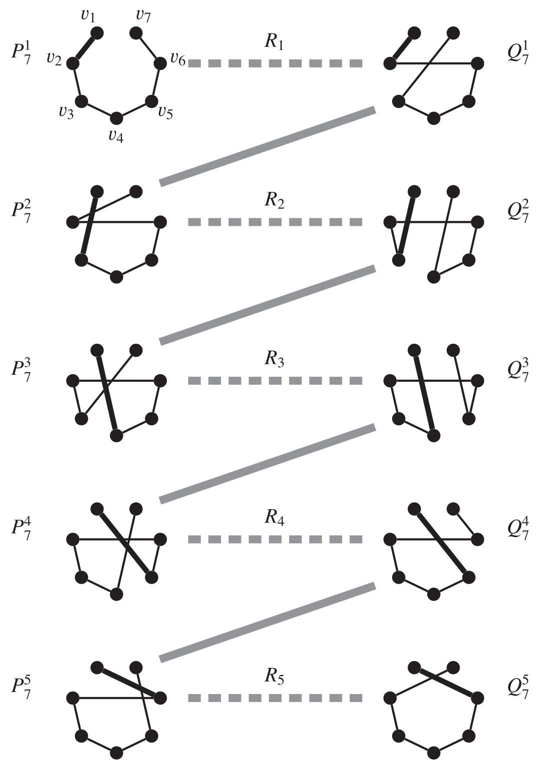

Fig. 4 Case $m + 1 = 7$ in Theorem 9.

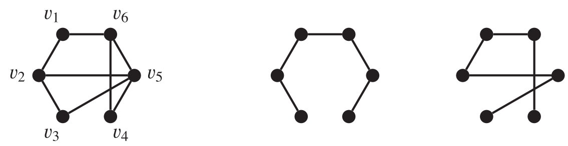

Fig. 5 $T_{(2,2,1,1,2,2)}(G)$ is disconected.

see **Fig. 4** for the case $m + 1 = 7$. □

The fixed degree tree graph may be defined for any connected graph G as follows: Let σ be the degree sequence of a spanning tree Q of G and let $T_\sigma(G)$ be the graph whose vertices are the spanning trees S of G such that $d_S(u) = d_Q(u)$ for each vertex u of G. As in the case $G = K_n$, two trees P and S are adjacent in $T_\sigma(G)$ if there are non-adjacent edges p and r of P and non-adjacent edges t and s of S, such that S can be obtained from P by deleting p and r and adding t and s.

A fixed degree tree graph $T_\sigma(G)$ of a connected graph may no longer be connected as shown in **Fig. 5**. For complete bipartite graphs we have the following results.

Let n and m be positive integers. A sequence σ of order $n + m$ is (n, m)-*arboreal* if there is an spanning tree T of $K_{n,m}$ that has σ as its degree sequence.

Let (X_m, Y_n) be the bipartition of the complete bipartite graph $K_{m,n}$. Let $X_m = \{x_1, x_2, \ldots, x_m\}$, $Y_n = \{y_1, y_2, \ldots, y_n\}$ and $\sigma = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$ be an (m, n)-arboreal sequence. For any vertex x of X_m, we denote by $\sigma(x)$ the integer a_i, where i is such that $x = x_i$ and we denote $\sigma(y)$ the integer b_i, where i is such that $y = y_i$ for any vertex y of Y_n. Let x be a vertex in X_m such that $\sigma(x) = 1$. For each vertex y with $\sigma(y) > 1$, let $H_x(y)$ be the subgraph of $T_\sigma(K_{m,n})$ induced by those spanning trees of $K_{m,n}$ with degree sequence σ in which x is adjacent to y.

Lemma 10. *Let σ be an (m, n)-arboreal sequence with $m \geq 3$*

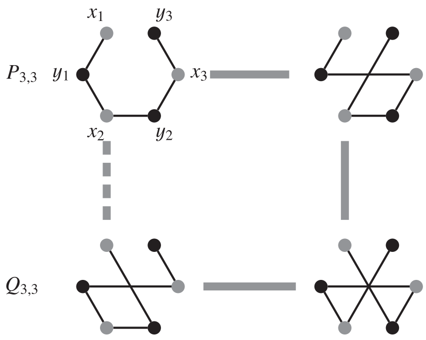

Fig. 6 The graph $T_{\sigma_{3,3}}(K_{3,3})$.

and $n \geq 3$, and let (X_m, Y_n) be the bipartition of the complete bipartite graph $K_{m,n}$. Let x be a vertex of X_m such that $\sigma(x) = 1$. For each vertex y of Y_n with $\sigma(y) > 1$ the graph $H_x(y)$ is isomorphic to $T_{\lambda_y}(K_{m,n} - x)$, where λ_y is the $(m-1, n)$-arboreal sequence given by $\lambda_y(y) = \sigma(y) - 1$, $\lambda_y(w) = \sigma(w)$ for each vertex w in Y_n with $w \neq y$ and $\lambda_y(v) = \sigma(v)$ for each vertex v in X_m with $v \neq x$.

Theorem 11. *Let n and m be positive integers. The graph $T_\sigma(K_{m,n})$ is connected for every (m, n)-arboreal sequence σ.*

The proofs are similar to those of Lemma 5 and Theorem 7, respectively, and are omitted here.

For $n \geq 3$, let $\sigma_{n,n}$ be the (n, n)-arboreal sequence given by $\sigma_{n,n}(x_1) = 1 = \sigma_{n,n}(y_n)$, $\sigma_{n,n}(x_i) = 2$ for $i = 2, 3, \ldots, n$ and $\sigma_{n,n}(y_j) = 2$ for $j = 1, 2, \ldots, n-1$; and let $\sigma_{n+1,n}$ be the $(n+1, n)$-arboreal sequence given by $\sigma_{n+1,n}(x_1) = 1 = \sigma_{n+1,n}(x_{n+1})$, $\sigma_{n+1,n}(x_i) = 2$ for $i = 2, 3, \ldots, n$ and $\sigma_{n+1,n}(y_j) = 2$ for $j = 1, 2, \ldots, n$.

Theorem 12. *Let $n \geq 3$ be an integer. The graphs $T_{\sigma_{n,n}}(K_{n,n})$ and $T_{\sigma_{n+1,n}}(K_{n+1,n})$ are hamiltonian.*

Proof. We prove that for any ordering $x_{i_1}, x_{i_2}, \ldots x_{i_n}$ of X_n and any ordering $y_{j_1}, y_{j_2}, \ldots, y_{j_n}$ of Y_n, the graph $T_{\sigma_{n,n}}(K_{n,n})$ contains a hamiltonian path that starts in $P_{n,n} = (x_{i_1}, y_{j_1}, x_{i_2}, y_{j_2}, \ldots, x_{i_n}, y_{j_n})$ and ends in $Q_{n,n} = (x_{i_1}, y_{j_{n-1}}, x_{j_{n-1}}, \ldots, x_{i_2}, y_{j_1}, x_{i_n}, y_{j_n})$ and that for any ordering $x_{i_1}, x_{i_2}, \ldots, x_{i_n}, x_{i_{n+1}}$ of X_{n+1} and any ordering $y_{j_1}, y_{j_2}, \ldots, y_{j_n}$ of Y_n, the graph $T_{\sigma_{n+1,n}}(K_{n+1,n})$ contains a hamiltonian path that starts in $R_{n+1,n} = (x_{i_1}, y_{j_1}, x_{i_2}, y_{j_2}, \ldots, x_{i_n}, y_{j_n}, x_{i_{n+1}})$ and ends in $S_{n+1,n} = (x_{i_1}, y_{j_n}, x_{i_n}, \ldots, x_{i_2}, y_{j_1}, x_{i_{n+1}})$. The results follows since $P_{n,n}$ and $Q_{n,n}$ are adjacent in $T_{\sigma_{n,n}}(K_{n,n})$, and since $R_{n+1,n}$ and $S_{n+1,n}$ are adjacent in $T_{\sigma_{n+1,n}}(K_{n+1,n})$.

We show the base of induction and the inductive steps for $T_{\sigma_{m+1,m}}(K_{m+1,m})$ and $T_{\sigma_{m+1,m+1}}(K_{m+1,m+1})$ for the ordering $x_{i_k} = x_k$, $y_{j_l} = y_l$ for all corresponding values of k and l. All other orderings may be treated in an analogous way.

Let p be the order of the complete bipartite graph $K_{n,n}$ or $K_{n+1,n}$. For $p = 6$, **Fig. 6** shows that $T_{\sigma_{3,3}}(K_{3,3})$ contains a path that starts in $P_{3,3}$ and ends in $Q_{3,3}$.

We proceed by induction assuming $p = t \geq 6$, that $T_{\sigma_{m,m}}(K_{m,m})$ contains a hamiltonian path between the vertices $P_{m,m}$ and $Q_{m,m}$ for $t = 2m$, and that $T_{\sigma_{m+1,m}}(K_{m+1,m})$ contains a hamiltonian path between the vertices $R_{m+1,m}$ and $S_{m+1,m}$ for $t = 2m + 1$. We then consider the case with $p = t + 1$ vertices.

For p odd, in $T_\sigma(K_{m+1,m})$, let

$$
\begin{aligned}
P^1_{m+1,m} &= (x_1, y_1, x_2, y_2, \ldots, x_{m+1}) = R_{m+1,m} \\
Q^1_{m+1,m} &= (x_1, y_1, x_m, y_{m-1}, x_{m-1}, \ldots, x_2, y_m, x_{m+1}).
\end{aligned}
$$

For $k = 2, \ldots, m-1$, let

$$
\begin{aligned}
P^k_{m+1,m} &= (x_1, y_k, x_{k+1}, y_{k+1} \ldots, x_m, y_1, x_2, \ldots, x_k, y_m, x_{m+1}) \\
Q^k_{m+1,m} &= (x_1, y_k, x_k, y_{k-1}, \ldots, y_1, x_m, y_{m-1}, , \ldots, x_{k+1}, y_m, x_{m+1})
\end{aligned}
$$

and let

$$
\begin{aligned}
P^m_{m+1,m} &= (x_1, y_m, x_m, y_1, x_2, y_2, \ldots, y_{m-1}, x_{m+1}) \\
Q^m_{m+1,m} &= (x_1, y_m, x_m, y_{m-1}, \ldots, y_1, x_{m+1}) = S_{m+1,m}.
\end{aligned}
$$

For $k = 1, 2, \ldots, m$ let H_k be the subgraph of $T_{\sigma_{m+1,m}}(K_{m+1,m})$, induced by the spanning paths of $K_{m+1,m}$ in which x_1 is adjacent to y_k. By Lemma 10, H_k is isomorphic to $T_{\sigma^k_{m,m}}(K_{m+1,m} - x_1)$ where $\sigma^k_{m,m}$ is the (m, m)-arboreal sequence given by $\sigma^k_{m,m}(y_k) = 1$, $\sigma^k_{m,m}(y_i) = 2$ if $i \neq k$, $\sigma^k_{m,m}(x_m) = 1$ and $\sigma^k_{m,m}(x_j) = 2$ if $1 \neq j \neq m$.

By the induction hypothesis, for $k = 1, 2, \ldots, m-1$, $T_{\sigma^k_{m,m}}(K_{m+1,m} - x_1)$ contains a hamiltonian path that starts in $P^k_{m+1,m} - x_1$ and ends in $Q^k_{m+1,m} - x_1$. This implies that H_k contains a hamiltonian path A_k that starts in $P^k_{m+1,m}$ and ends in $Q^k_{m+1,m}$.

Also by the induction hypothesis, $T_{\sigma^m_{m,m}}(K_{m+1,m} - x_1)$ contains a hamiltonian path that starts in $(x_{i_1}, y_{j_1}, x_{i_2}, y_{j_2}, \ldots, x_{i_m}, y_{j_m}) = (x_{m+1}, y_{m-1}, \ldots, y_2, x_2, y_1, x_m, y_m)$ and ends in $(x_{i_1}, y_{j_{m-1}}, x_{j_{m-1}}, \ldots, x_{i_2}, y_{j_1}, x_{i_m}, y_{j_m}) = (x_{m+1}, y_1, x_2, \ldots, x_m, y_m)$. As above, this implies that H_m contains a hamiltonian path A_m that starts in $(x_{m+1}, y_{m-1}, \ldots, y_2, x_2, y_1, x_m, y_m, x_1)$ and ends in $(x_{m+1}, y_1, x_2, \ldots, x_m, y_m, x_1)$.

Notice that $(x_{m+1}, y_{m-1}, \ldots, y_2, x_2, y_1, x_m, y_m, x_1)$ and $(x_{m+1}, y_1, x_2, \ldots, y_m, x_1)$ are, respectively, the paths $P^m_{m+1,m}$ and $Q^m_{m+1,m}$ traversed backwards. Therefore A_m is a hamiltonian path of H_m that starts in $P^m_{m+1,m}$ and ends in $Q^m_{m+1,m}$.

Observe that for $k = 2, 3, \ldots, m-1$, $P^k_{m,m+1} = Q^{k-1}_{m,m+1} - \{\{x_1, y_{k-1}\}, \{x_k, y_k\}\} + \{\{x_1, y_k\}, \{x_k, y_{k-1}\}\}$ and that $P^m_{m,m+1} = Q^{m-1}_{m+1,m} - \{\{x_1, y_{m-1}\}, \{x_{m+1}, y_m\}\} + \{\{x_1, y_m\}, \{x_{m+1}, y_{m-1}\}\}$, which implies that $P^k_{m+1,m}$ and $Q^{k-1}_{m+1,m}$ are adjacent in $T_{\sigma_{m+1,m}}(K_{m+1,m})$ for $k = 2, 3, \ldots, m$. Therefore $A_1, A_2, \ldots, A_m$ can be joined to form a hamiltonian path in $T_{\sigma_{m+1,m}}(K_{m+1,m})$ that starts in $R_{m+1,m} = P^1_{m+1,m}$ and ends in $S_{m+1,m} = Q^m_{m+1,m}$, see **Fig. 7** for the case $p = 9$.

For p even, in $T_\sigma(K_{m+1,m+1})$ let

$$
\begin{aligned}
R^1_{m+1,m+1} &= (x_1, y_1, x_2, y_2, \ldots, x_m, y_m) = P_{m+1,m+1}, \\
S^1_{m+1,m+1} &= (x_1, y_1, x_m, y_{m-1}, x_{m-1}, \ldots, x_2, y_m).
\end{aligned}
$$

And for $k = 2, \ldots, m$, let

$$
\begin{aligned}
R^k_{m+1,m+1} &= (x_1, y_k, x_{k+1}, y_{k+1}, \ldots, x_m, y_1 x_2, y_2, \ldots, x_k, y_m) \\
S^k_{m+1,m+1} &= (x_1, y_k, x_k, y_{k-1}, \ldots, y_{n-1}, x_m, y_{m-1}, \ldots, x_{k+1}, y_m) \\
&= Q_{m+1,m+1}.
\end{aligned}
$$

For $k = 1, 2, \ldots, m$ let H_k be the subgraph of $T_{\sigma_{m+1,m+1}}(K_{m+1,m+1})$, induced by the spanning paths of $K_{m+1,m+1}$ in which x_1 is adjacent to y_k. By Lemma 10, H_k is isomorphic to $T_{\sigma^k_{m+1,m}}(K_{m+1,m+1} - x_1)$ where $\sigma^k_{m+1,m}$ is the $(m+1, m)$-arboreal sequence given by $\sigma^k_{m+1,m}(y_k) = 1$, $\sigma^k_{m+1,m}(y_m) = 1$, $\sigma^k_{m+1,m}(y_i) = 2$ if $m \neq i \neq k$, and $\sigma^k_{m,m}(x_j) = 2$ if $j \neq 1$.

By the induction hypothesis, $T_{\sigma^k_{m+1,m}}(K_{m+1,m+1} - x_1)$ contains a hamiltonian path that starts in $R^k_{m+1,m+1} - x_1$ and ends in $S^k_{m+1,m+1} - x_1$ for $k = 1, 2, \ldots, m$. As above, this implies that H_k contains a hamiltonian path B_k that starts in $R^k_{m+1,m+1}$ and ends

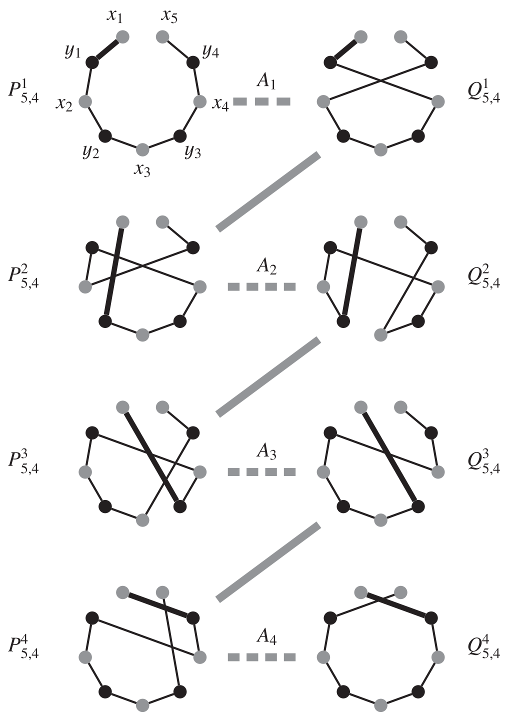

Fig. 7 Case $p = 9$ in Theorem 12.

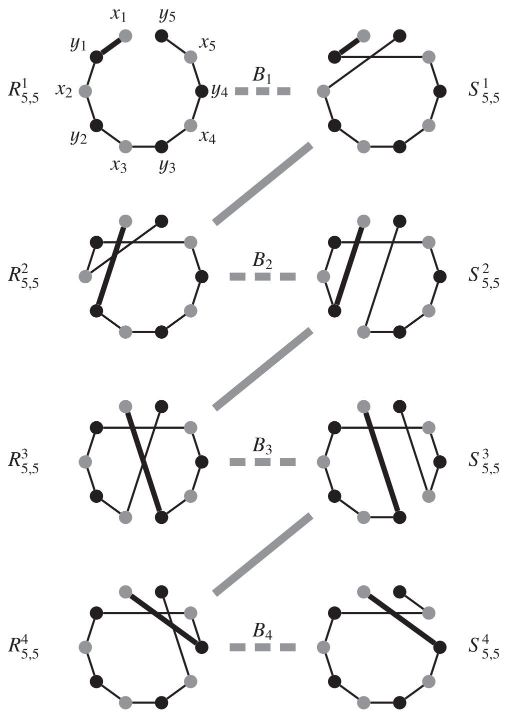

Fig. 8 Case $p = 10$ in Theorem 12.

in $S^k_{m+1,m+1}$.

Finally observe that for $k = 2, 3, \ldots, m$, $R^k_{m+1,m+1} = S^{k-1}_{m+1,m+1} - \{\{x_1, y_{k-1}\}, \{x_k, y_k\}\} + \{\{x_1, y_k\}, \{x_k, y_{k-1}\}\}$ which implies that $R^k_{m+1,m+1}$ and $S^{k-1}_{m+1,m+1}$ are adjacent in $T_{\sigma_{m+1,m+1}}(K_{m+1,m+1})$. Therefore $B_1, B_2, \ldots, B_m$ can be joined to form a hamiltonian path in $T_{\sigma_{m+1,m+1}}(K_{m+1,m+1})$ that starts in $P_{m+1,m+1} = R^1_{m+1,m+1}$ and ends in $Q_{m+1,m+1} = S^m_{m+1,m+1}$, see **Fig. 8** for the case $p = 10$. □

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