

Regular Paper

Searching with Advice: Robot Fence-Jumping

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Abstract: We study a new problem concerning search in the plane involving a robot and an immobile treasure, initially placed at distance 1 from each other. The length β of an arc (a fence) which is placed within the perimeter of the disk centered at the initial position of the robot, as well as the promise that the treasure is outside the fence, is given as part of the input. The goal is to devise movement trajectories so that the robot locates the treasure in minimum time. Notably, although the presence of the fence limits searching uncertainty, the location of the fence is unknown, and the worst case analysis is determined adversarially. Nevertheless, the robot has the ability to make cross-cuts by moving in the interior of the disk. In particular, the robot can attempt a number of chord-jump moves if it happens to be within the fence or if an endpoint of the fence is discovered. The optimal solution to our question can be obtained as a solution to a complicated optimization problem, which involves trigonometric functions, and trigonometric equations that do not admit closed form solutions. For the 1-Jump Algorithm, we fully describe the optimal trajectory, and provide an analysis of the associated cost as a function of β . Our analysis indicates that the optimal k-Jump Algorithm requires that the robot has enough memory and computation power to compute the optimal chord-jumps. Motivated by this, we give an abstract performance analysis for every k-Jump Algorithm. Subsequently, we present a highly efficient Halving Heuristic k-Jump Algorithm that can effectively approximate the optimal k-Jump Algorithm, with very limited memory and computation requirements. Further, we explore randomized algorithms and analyze their expected worst-case performance.

Keywords: disk, fence, optimization, robot, search, speed, treasure

1. Introduction

Geometric search is concerned with finding a target placed in a geometric region. It has been investigated in many areas of mathematics, theoretical computer science, and robotics. In each instance, one aims to provide search algorithms that optimize a certain cost, which may take into account a variety of important characteristics and features of the domain, computational abilities of the searcher, assumptions about the target, etc. In this paper, we introduce and study *fence-jumping search*, a new search problem involving a robot, an unknown stationary fence (barrier), and an unknown stationary target (or treasure) in the plane.

The location of the treasure is unknown to the robot. However, the robot has knowledge that at the start it is located at a distance of 1 (unit) away from the treasure. Equivalently, the treasure is stationed on the perimeter of a disk (within the known environment), which is centered at the start point of the robot. A *fence*, a given circular arc of length β , is placed on the perimeter of the disk, whose location is also unknown to the robot. Further, the robot has the knowledge that the treasure is located on the perimeter but not on the fence. Depending on its trajectory, the robot may move along the perimeter of the disk and occasionally, e.g., when it is within the fence, it may move along a chord, or as we say to *fence-jump*, so as to reduce the time necessary to

perform the search. We will analyze several fence-jumping algorithms that will allow us to reach the treasure in minimal time.

We study the fence-jumping search problem for one robot starting at the center of the disk and moving at a constant speed 1. We assume the treasure is stationary and placed by an adversary at the beginning of each round depending on the fence location. The adversary always positions the treasure on the perimeter, but outside the fence. The robot may move anywhere on the disk in an attempt to find this treasure; it is also able to use geometric knowledge so as to decrease the amount of time spent during the search. That is to say, since the robot knows that the treasure is not located on the fence, it could try to bypass it by “jumping over the fence.” The goal of this paper is to determine a trajectory so that the robot finds the treasure in optimal time.

1.1 Related Work

Search problems were first proposed sixty years ago when Beck [4] and Bellman [5] asked an important question tied to minimizing the optimal search path. Motivated from this, several search optimization problems have been studied that take into account several parameters including the use of a fixed [8], [16] or a mobile target [18], the tools searchers have access to, the number of searchers, the communication restrictions. Often, the essential part of the robot activity is the recognition and/or mapping of the terrain. In the case of a known structure, the main objective of the search is to minimize the time it takes to find the treasure. Search-

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ing for a motionless target has also been studied in the cow-path problem [4], lost in a forest problem [12], [15] and plane searching problem [2], [3].

In their well known paper [3], Baeza-Yates et al. study the worst-case time for search involving one robot and a treasure placed at an unknown location in the line. Useful surveys on search theory can also be found in Refs. [6] and [10].

Search by multiple robots with communication capabilities has been considered in Refs. [11], [14], while Refs. [8], [9] study the evacuation of k robots searching for an exit located on the perimeter of a disk. The problem of finding trajectories for obstacle avoidance in both known and unknown terrains has also been considered in several papers, including in Refs. [1], [7], [17].

1.2 Outline and Results of the Paper

As a main objective, our approach will have to design algorithms for finding the treasure in optimal time, while adapting to the fence structure located on the perimeter. This is leading us to propose algorithms that attempt to deliver the optimal shortcuts necessary to exit and/or avoid the fence structure.

An outline of our results is as follows. In Section 2, we introduce the basic concepts and analyze a simple search algorithm for finding the treasure without involving any jumps. In Section 3, we introduce and analyze the optimal 1-Jump search algorithm. In Section 4, we propose a generic description of k -Jump algorithms. In Section 5, we study a k -Jump algorithm based on a *halving* heuristic, which approximates the optimal jump without relying on solutions of trigonometric optimization problems. In Section 6, we contrast the choices and performance of the Halving k -Jump Algorithm with the choices and performance of the Optimal k -Jump Algorithm that was obtained using optimization software packages, for $k \leq 3$. In Section 7 we explore randomized algorithms. We conclude with Section 8.

2. Preliminary Observations

First we introduce the basic concepts and assumptions of our model. Initially, we assume that the robot is located in the center of the disk with a radius of 1, and a treasure is located on the perimeter of the disk at distance 1 from the robot. We define this treasure to be a point on the disk and, thus, does not take any space on the perimeter. The treasure location is always unknown to the robot until it moves directly over its point location. That is to say, the robot has no vision capabilities, in that it becomes aware of what each point on the disk is, i.e., a fence point, treasure or nothing special, only if the point is visited. The robot moves at the same speed throughout its search on the disk and the movement of the robot from the center always takes 1 unit of time. The robot has the computational power to numerically solve trigonometric equations through the use of deterministic processors which possess the required memory for these processes.

Recall that the goal of the robot is to optimize the length of its trajectory using various types of movements, i.e., the robot may walk on the fence or even jump over the fence moving along a chord (within the interior of the disk).

To begin, we provide a naive solution to our treasure finding problem, which we will then improve with a number of algo-

rithms. In what follows, we denote the length of the fence by β , given as part of the input. In any deterministic algorithm being considered, the robot will first move to an arbitrary point on the perimeter of the disk, thereafter referred to as the *basic landing point*, and then it will start moving/searching the disk counterclockwise. Algorithm 1 is described below.

Algorithm 1 0-Jump Algorithm

- 1: Walk to the perimeter of the disk
 - 2: Continue walking on the perimeter counterclockwise
 - 3: **if** while you are moving outside the fence you reach a fence endpoint **then**
 - 4: Jump along the corresponding chord of length
 - 5: $2 \sin(\beta/2)$
 - 6: **else**
 - 7: Walk on perimeter until you find treasure
-

Our work focuses on algorithms that perform well under worst case analysis. As such, the performance of any algorithm will be determined after an adversary decides on the location of both the basic landing point, the fence itself, and the treasure. For the sake of exposition, we now present the worst case termination time depending on the location of the basic landing point.

Lemma 1 The worst case termination time of Algorithm 1 is ^{*1}

$$\begin{cases} c_0^0 := 1 + 2\pi - \beta + 2 \sin(\beta/2) & \text{(Fig. 1 (a))} \\ c_1^0 := 1 + 2\pi & \text{(Fig. 1 (b))} \end{cases}$$

where the reference in the right column indicates the Figure which applies to the case.

Proof. Suppose that the basic landing point is outside the fence, as seen in **Fig. 1 (a)**, and say that the clockwise distance between the landing point and the fence is $x \in (0, 2\pi - \beta)$. It is straightforward that the adversary would place the treasure clockwise in between the landing point and the fence, at a clockwise distance $y \in (0, x)$ from the landing point. Then, for all $x \in (0, 2\pi - \beta)$ the cost of the algorithm would be

$$\sup_{y \in (0, x)} \{1 + 2\pi - \beta + 2 \sin(\beta/2) - y\} = 1 + 2\pi - \beta + 2 \sin(\beta/2).$$

In the other case, the landing point is within the fence, as illustrated in **Fig. 1 (b)**. Suppose that the clockwise distance between the landing point and the endpoint of the fence is $x \in (0, \beta)$. Also suppose that the clockwise distance between the same fence endpoint and the treasure is $y \in (0, 2\pi - \beta)$. Then, the robot will locate the treasure in time

$$\sup_{x, y} \{1 + 2\pi - x - y\} = 1 + 2\pi.$$

This proves Lemma 1. ■

It is intuitive that having the basic landing point outside the fence is a “favorable event” in that for all β , $c_0^0 \leq c_1^0$. This follows formally from the fact that the non-negative expression $\beta - 2 \sin(\beta/2)$ is increasing in $\beta > 0$. Hence, the worst-case performance of Algorithm 1 is $1 + 2\pi$.

^{*1} The usefulness of notation c_j^i for the cost of the algorithm will be transparent in the next sections

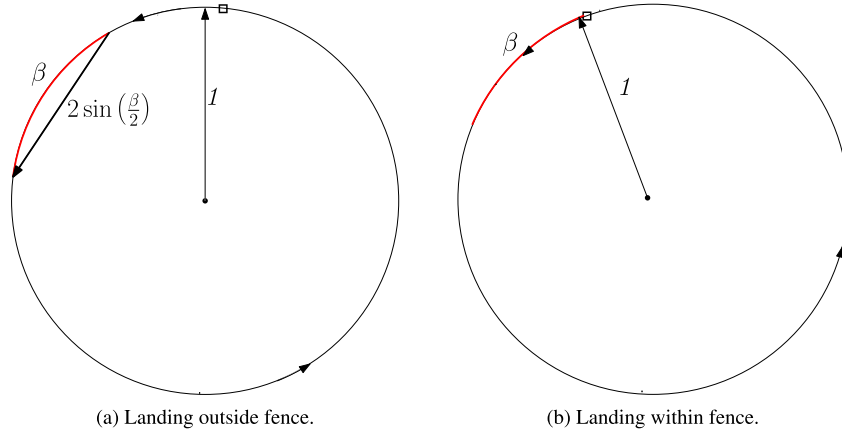


Fig. 1 Basic landing point.

Next, we focus on algorithms that can address the choice of the adversary placing (basic) landing points within the fence. In particular, we analyze optimally a special family of intuitive algorithms, called k -jump that perform as follows. First the robot reaches the perimeter of the disk at an arbitrary point. If that landing point is within the fence, the robot tries to shortcut along a carefully chosen chord, and continues for at most k jumps and as long as the landing points fall within the fence. If any landing point falls outside the fence, the robot needs to naturally apply a remedy phase, in which any unexplored critical part is first explored before the robot continues with its exploration from the latest landing point. Notice that these algorithms are completely determined by the length choice of the (at most) k chords, and this is what we determine optimally.

3. The Optimal 1-Jump Algorithm

In this section we analyze the optimal 1-Jump Algorithm, which also serves as a warm-up for the analysis of the generic k -Jump Algorithm. 1-Jump Algorithms are fully determined by the (unique) chord jump of corresponding arc-length α they make in case the basic landing point (of the robot) is within the fence.

It is worthwhile discussing the required specifications for the algorithm to be correct. First, we require the jump to be in “counterclockwise” direction, i.e., that $\alpha \leq \pi$ (this also breaks the symmetry for the adversarial placements of the fence and the treasure). Second, we further require that the chord jump does not pass over the area that could hold the treasure, landing back to the fence. For that, it is of importance that $\alpha \leq 2\pi - \beta$. To summarize, the 1-Jump Algorithm is fully determined by choosing α satisfying

$$0 < \alpha \leq \min\{\pi, 2\pi - \beta\}.$$

To resume, Algorithm 2 with parameter α runs similarly to Algorithm 1, except from the case that the last landing of Algorithm 1 (which happens to be the basic one) is within the fence. If that happens, Algorithm 2 makes a counterclockwise jump corresponding to arc length α . If the 1st-jump landing point is in the fence, then it runs Algorithm 1. Otherwise, the 1st-jump landing point is outside the fence and the robot applies the following *remedy phase*: move clockwise along the periphery of the disk until the endpoint of the fence is found, say at arc distance x , and

then return to the 1st-jump landing point along the corresponding chord of length $2 \sin(x/2)$, and continue executing Algorithm 1.

Algorithm 2 1-Jump

- 1: Walk to the perimeter of the disk
 - 2: **if** your landing point is inside the fence **then**
 - 3: make a ccw chord jump of arc length α
 - 4: Perform steps 3–7 of Algorithm 1
 - 5: **else**
 - 6: move cw along perimeter until endpoint of the fence
 - 7: return to jump landing point along corresponding chord
 - 8: continue searching on the perimeter
-

Lemma 2 Depending on the landing points, the cost of Algorithm 2 with parameter α is

$$\begin{cases} c_0^1 := 1 + 2\pi - \beta + 2 \sin(\beta/2) & \text{(Fig. 1 (a))} \\ c_1^1 := c_0^1 + 4 \sin(\alpha/2) - 2 \sin(\beta/2) & \text{(Fig. 2 (a))} \\ c_2^1 := 1 + 2\pi - (\alpha - 2 \sin(\alpha/2)) & \text{(Fig. 2 (b))} \end{cases} \quad (1)$$

with the understanding that c_0^1, c_1^1, c_2^1 are functions on β and α , and the reference in the right column indicates the figure which applies to the case.

Proof. Clearly, if the basic landing point is in the fence, then the cost of Algorithm 2 c_0^1 is equal to cost c_0^0 of Algorithm 1 (for the same case).

Suppose now that the basic landing point is in the fence. Algorithm 2 performs a counterclockwise chord jump of length $2 \sin(\alpha/2)$. We examine two more subcases. In the first subcase, the 1st-jump landing point is outside the fence as seen in **Fig. 2 (a)**, say at clockwise distance $x \in (0, \alpha)$ from the fence. Then the robot follows the remedy phase spending $x + 2 \sin(x/2)$ more time to come back to the same landing point. Clearly, the worst positioning of the treasure is to be arbitrarily clockwise close to the fence. That would make the robot search for an additional time of $2\pi - \beta - x$, for a total of

$$1 + x + 2 \sin(x/2) + 2\pi - \beta - x = 1 + 2 \sin(x/2) + 2\pi - \beta.$$

Since $x \leq \alpha \leq \min\{\pi, 2\pi - \beta\}$ and by the monotonicity of $\sin(x/2)$ we see, as promised, that the cost in that case is no more than

$$\sup_{0 < x < \alpha} \{1 + 2 \sin(\alpha/2) + 2 \sin(x/2) + 2\pi - \beta\}$$

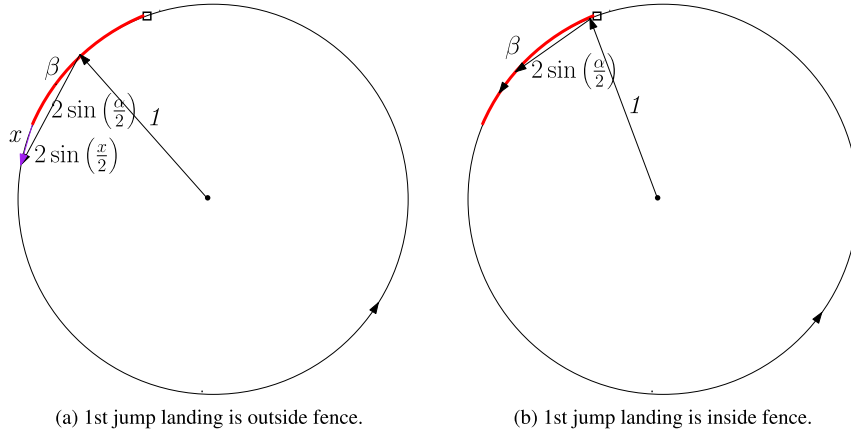


Fig. 2 1-Jump Algorithm basic landing point.

$$\begin{aligned}
 &= 1 + 4 \sin(\alpha/2) + 2\pi - \beta \\
 &= c_0^1 + 4 \sin(\alpha/2) - 2 \sin(\beta/2)
 \end{aligned}$$

In the second subcase, the 1st-jump landing point is in the fence and is illustrated in Fig. 2 (b). It is not difficult to see that the worst configuration in this case is when the robot's basic landing point is arbitrarily close to the clockwise endpoint of the fence, while the treasure is arbitrarily close to the same endpoint and outside the fence. Clearly, in that case the robot traverses the whole disk, saving only an arc of length α which is jumped over using the corresponding chord of length $2 \sin(\alpha/2)$. Overall, the cost in this case becomes $1 + 2\pi - \alpha + \sin(\alpha/2)$. This completes the proof of Lemma 2. ■

Critical to our analysis toward specifying the optimal choice of α is the solution to a specific equation that does not admit a closed form. Consider expression $\alpha + 2 \sin(\alpha/2)$ which is monotonically increasing. As such, for every $\beta \in \mathbb{R}$, the equation $\alpha + 2 \sin(\alpha/2) = \beta$ admits a unique solution in α . Motivated by this observation we write that “ α_β is the unique real number satisfying equation $\alpha_\beta + 2 \sin(\alpha_\beta/2) = \beta$.”

Moreover, since $\alpha + 2 \sin(\alpha/2)$ is increasing in the variable α , so is α_β in the variable β . We are now ready to define and analyze the optimal 1-Jump Algorithm.

Theorem 1 Let γ be the unique solution to equation $\pi = \gamma - \sin(\gamma/2)$ ($\gamma \approx 4.04196$). The optimal 1-Jump Algorithm chooses jump step corresponding to arc length $\alpha = \alpha_\beta$ if $\beta \leq \gamma$, $\alpha = 2\pi - \beta$ if $\beta > \gamma$ and terminates in time

$$1 + \begin{cases} 2\pi - \alpha_\beta + 2 \sin(\alpha_\beta/2) & \text{if } \beta \leq \gamma \\ \beta + 2 \sin(\beta/2) & \text{if } \beta > \gamma. \end{cases} \quad (2)$$

Proof. By Lemma 2, the optimal 1-Jump Algorithm is determined by choosing α that minimizes

$$\sup_{0 < \alpha < \min\{\pi, \pi - \beta\}} \{c_0^1, c_1^1(\alpha), c_2^1(\alpha)\},$$

where in the expression above, we make the dependence on α explicit. Again, it should be clear that having the basic landing point outside the fence is a “favorable event.” Intuitively, this is the only case that the robot makes full use of the fact that the treasure does not lie within the fence, jumping over it and using the corresponding chord. Effectively, this implies that for all β , α we

have $c_0^1 \leq \min\{c_1^1(\alpha), c_2^1(\alpha)\}$.

Next, for any $\beta \in (0, 2\pi)$ we need to choose α so as to minimize $\max\{c_1^1(\alpha), c_2^1(\alpha)\}$. To that end, note that c_1^1, c_2^1 exhibit different monotonicities with respect to α so that, if possible, the minimum will be attained when the two costs are equal. Equating the two costs gives that $\alpha + 2 \sin(\alpha/2) = \beta$. Recall that we have denoted the unique solution to the equation by α_β which is increasing in β . Since the jump step needs to stay no more than $\min\{\pi, 2\pi - \beta\}$, the choice $\alpha = \alpha_\beta$ (which is the best possible) is valid only when $\alpha_\beta \leq \min\{\pi, 2\pi - \beta\}$. Numerically we can compute $\alpha_\pi \approx 1.66$, which, due to the monotonicity of α_β , implies that the dominant constraint is that $\alpha_\beta \leq 2\pi - \beta$, and hence any restrictions will be imposed for $\beta > \pi$. Indeed, setting $\alpha_\beta = 2\pi - \beta$, and substituting in $\alpha_\beta + 2 \sin(\alpha_\beta/2) = \beta$, we obtain $2\pi - \beta + 2 \sin(\pi - \beta/2) = \beta$. The value of β that satisfies this equation is $\beta \approx 4.04196$.

To resume, as long as $\beta \leq \gamma$, the best choice for the jump is the solution to the equation $\alpha + 2 \sin(\alpha/2) = \beta$. When $\beta > \gamma$, the best jump step is equal to $2\pi - \beta$.

From the discussion above, the induced cost when $\beta \leq \gamma$ would be equal to $c_1^1(\alpha_\beta)$, as it reads in Lemma 2. Finally, when $\beta > \gamma$ the induced cost would be

$$\begin{aligned}
 &\max\{c_1^1(2\pi - \beta), c_2^1(2\pi - \beta)\} \\
 &= 1 + 2\pi + \max\{4 \sin(\pi - \beta/2) - \beta, 2 \sin(\pi - \beta/2) - 2\pi + \beta\} \\
 &= 1 + 2\pi + 2 \sin(\beta/2) + \max\{2 \sin(\beta/2) - \beta, -2\pi + \beta\} \\
 &= 1 + \beta + 2 \sin(\beta/2)
 \end{aligned}$$

where the last equality is due to that $\beta \geq \gamma$, the definition of γ and the fact that $-2\pi + \beta$ is increasing in β . This proves Theorem 1. ■

Notably, the proof of Theorem 1 suggests that for the best strategy α as a function of β , we have that $c_0^1 \leq c_2^1(\alpha) \leq c_1^1(\alpha)$. This was expected, since having the basic landing outside the fence is intuitively more favourable than having it inside the fence and without needing the remedy phase, which is more favourable than needing the remedy phase. It is also interesting to note that for $\beta \leq \gamma$, the best jump choice α_β attains values close to $\beta/2$. This suggests an alternative approach to the problem that does not require the ability to solve technical trigonometric equations, and that will be explored later. As a result, there arises a nice recursive relation between costs c_0^1, c_1^1 that is soon to be generalized for k-Jump Algorithms.

4. Generic Description of k -Jump Algorithms

Analogously to the previous sections, the k -Jump Algorithm has parameters $\alpha_1, \dots, \alpha_k$ and runs similarly to the $(k-1)$ -Jump Algorithm, except from the case that the last landing point of the $(k-1)$ -Jump Algorithm (which happens to be the $(k-1)$ st-jump landing point, if this is realized) is within the fence. If that happens, the k -Jump Algorithm makes an additional counterclockwise jump corresponding to arc length α_k . If the k th-jump landing point is in the fence, then it runs Algorithm 1. Otherwise, the k th-jump landing point is outside the fence, and the robot applies the remedy phase from Algorithm 2 in Section 3.

Algorithm 3 k -Jump Algorithm

- 1: Walk to the perimeter of the disk
 - 2: $i \leftarrow 0$
 - 3: **while** landing point is inside the fence & $i < k$ **do**
 - 4: $i \leftarrow i + 1$
 - 5: make a ccw chord jump of arc length α_i
 - 6: Perform steps 3–7 of Algorithm 1.
-

It is clear from the discussion above that any k -Jump Algorithm is specified by the jump steps $\alpha_1, \alpha_2, \dots, \alpha_k$, where the i th jump is realized only if the basic landing point, along with the landing points of the previous $i - 1$ jumps, fall within the fence. In order to preclude the possibility that a jump passes over the area that holds the treasure and bring the robot back to the fence, we require that $\alpha_i \leq \beta$. Moreover, for the jumps to be in counterclockwise direction (and to break the symmetry), we also require that $\alpha_i \leq \pi$.

Similarly, for the 1-Jump Algorithm we required that $\alpha_1 \leq \min\{\pi, 2\pi - \beta\}$. However, according to Theorem 1, the optimal jump step is less than β (for all β), meaning that the correctness condition for choosing the jump step could have been replaced by $\alpha_1 \leq \min\{\beta, 2\pi - \beta\}$. Indeed, our intuition tells us that an algorithm, which after the basic landing point within the fence makes a jump more than the length of the fence, will land outside the fence and subsequently will unavoidably need to apply the (suboptimal) remedy phase. Motivated by this observation, we require the following condition regarding the step sizes of k -Jump Algorithms: $\alpha_i \leq \min\{\pi, 2\pi - \beta\}$, $i = 1, \dots, k$.

The next lemma generalizes Lemma 2 and provides a handy recurrence description of the cost of the k -Jump Algorithm with jump steps $\alpha_1, \dots, \alpha_k$ depending on the first landing point outside the fence. In this direction, we denote by c_t^k to be the worst case cost of the k -Jump Algorithm when the basic landing point along with the landing points of the first $t - 1$ jumps fall all inside the fence and the robot lands outside the fence in the t th jump, which is shown in **Fig. 3**. Let us observe that, c_0^k is the cost of the case when the basic landing point is outside the fence, while c_{k+1}^k corresponds to the case that the landing points of all k jumps, as well as the basic landing point, fall inside the fence.

Lemma 3 For any β , let $\alpha_0 = \beta$. Depending on the landing points, the cost of the k -Jump Algorithm with jump steps $\alpha_1, \dots, \alpha_k$ is

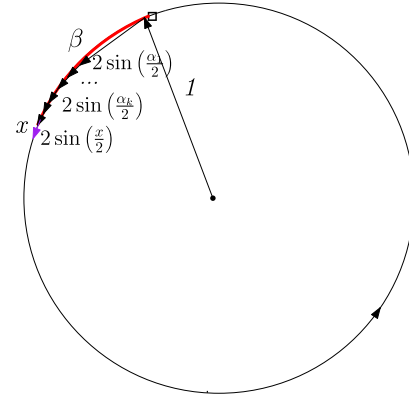


Fig. 3 k -Jump Algorithm.

$$\begin{cases} c_0^k & := 1 + 2\pi - \alpha_0 + 2 \sin(\alpha_0/2) \\ c_t^k & := c_{t-1}^k + 4 \sin(\alpha_t/2) - 2 \sin(\alpha_{t-1}/2) \\ c_{k+1}^k & := 1 + 2\pi - \sum_{i=1}^k (\alpha_i - 2 \sin(\alpha_i/2)) \end{cases} \quad (3)$$

with the understanding that c_t^k are functions on β and $\alpha_1, \dots, \alpha_{t-1}$, for $t = 1, \dots, k + 1$.

Proof. As previously mentioned, when the basic landing point is outside the fence, the cost is indeed $c_0^k = c_0^0$. Furthermore, when all landing points, including the basic one, fall within the fence, then similarly to the cost c_2^1 of Lemma 2, the worst positioning of the fence makes the basic landing point inside and arbitrarily close to the counterclockwise endpoint of the fence. Meanwhile, the treasure is arbitrarily close to the same endpoint but outside the fence. Effectively, the robot in this case will traverse the entire disk counterclockwise, saving from each jump exactly $\alpha_i - 2 \sin(\alpha_i/2)$, $i = 1, \dots, k$.

For the most interesting case, we need to compare the costs c_t^k , c_{t-1}^k , for some $t \in \{1, \dots, k\}$. In both cases, the worst positioning of the treasure is arbitrarily close to the clockwise endpoint of the fence. The worst positioning of the basic landing point should bring the robot inside, as well as arbitrarily close to the counterclockwise endpoint of the fence, so as to induce the maximum possible remedy phase cost. Note that for the case of cost c_t^k , the robot traverses twice the chord of length $2 \sin(\alpha_t/2)$, but only once the chord of length $2 \sin(\alpha_{t-1}/2)$. Other than that, in both cases, the robot performs exactly the same jumps, and search exactly the same sub-perimeter of the disk. This proves Lemma 3. ■

5. The Halving Heuristic k -Jump Algorithm

In this section, we present a simple heuristic that is meant to approximate the optimal jump steps without relying on solutions of trigonometric optimization problems. Most importantly, our algorithm requires very limited memory and does not need to perform numerical operations other than simple algebraic manipulations. In fact, only constant many operations are needed to determine every possible jump size. Moreover, parameter k , i.e., the number of jumps, may not necessarily be determined in advance, and is allowed to be even infinite. First, we present the algorithm and analyze it. Then, in Section 6, we contrast it to the Optimal k -Jump Algorithm (for certain values of k).

Closely examining the optimal solution for the 1-Jump Algo-

rithm in Section 3, we are tempted to choose an alternative first jump step equal to $\beta/2$, which is a good approximation to α_β . This choice is valid, as long as the jump does not exceed $2\pi - \beta$, and indeed for large enough values of β , i.e., for $\beta \geq 4.041$, as per Theorem 1, the best choice for just one jump is $2\pi - \beta$. Note that changing the first jump from α_β to $\beta/2$ results to a new threshold value $\frac{4}{3}\pi \approx 4.188$ after which the first jump should become $2\pi - \beta$. Interestingly, the pattern also repeats in the optimal k-Jump Algorithms (see Section 6).

The previous observation suggests a natural heuristic for k-Jump Algorithms. First, go to an arbitrary point on the disk. While on (some unknown position on) the fence, make a valid jump (i.e., no more than π , $2\pi - \beta$) equal to half of the unexplored fence, unless this value exceeds $2\pi - \beta$ in which case the jump should be $2\pi - \beta$. Formally, the description of the heuristic follows if we can determine the length of the chord-jump α_i in every i -th jump, and then invoke Algorithm 3.

Algorithm 4 Halving Heuristic jumps

```

1: explored  $\leftarrow$  0
2: temp  $\leftarrow$   $\frac{\beta - \text{explored}}{2}$ 
3: if temp  $\leq$   $2\pi - \beta$  then
4:   jump  $\leftarrow$  temp
5: else
6:   jump  $\leftarrow$   $2\pi - \beta$ 
7: explored  $\leftarrow$  explored + jump
    
```

Note that the calculations of jumps α_i can be incorporated within Algorithm 3 and do not need to be computed in advance. As the maximum number of jumps can be part of the input, Algorithm 4 can be performed only for k many landings within the fence (see Algorithm 3), or as long as the the jump step does not drop below a given threshold. Interestingly, the definition of step sizes on the fly by Algorithm 4 even allows for $k = \infty$. That would correspond to the theoretical case that the robot makes an infinite number of jumps for which all landings happen within the fence. Still, the time for the robot to reach the endpoint of the fence would be finite (Zeno's paradox).

The process above fully determines the jump step of the t -th jump as a function of β , for every $t = 1, \dots, k$, and for every k . In what follows we provide an analytic description of these values so that we can analyze the performance of the algorithm. The lemma below will allow us to derive later a nicer closed formula for the jump steps of the halving Algorithm.

Lemma 4 Let $h_t = \frac{2\pi(t+1)}{t+2}$ for $t \geq 1$ and $h_0 = 0$. For any $\beta \in (0, 2\pi)$, the value of the i -th jump in the Halving Algorithm equals

$$\alpha_i = \begin{cases} \frac{j\beta - (j-1)2\pi}{2^{i-j+1}} & \text{if } \beta \leq h_i \text{ and } \beta \in (h_{j-1}, h_j] \\ 2\pi - \beta & \text{if } \beta > h_i \end{cases}$$

Proof. We will derive the promised formulas from scratch, without relying on the statement of the lemma. First, note that the process above defines natural threshold values h_i for β , after which the i -th jump step α_i becomes $2\pi - \beta$. In particular, the value of α_i will depend on which interval $(h_{j-1}, h_j]$ value β belongs to, where $j = 1, \dots, k$, and with the understanding that $\alpha_i = 2\pi - \beta$ if $\beta > h_i$.

Therefore, it is natural to introduce notation

$$A_{i,j} := \alpha_i, \quad \text{when } \beta \in (h_{j-1}, h_j]$$

It is easy to see that if $A_{i,j} = 2\pi - \beta$ then $A_{r,j} = 2\pi - \beta$ for all $r = 1, \dots, i-1$, and in general that $A_{i,j} = 2\pi - \beta$ whenever $j \geq i+1$. In other words, $A(i, i)$ is the last expression (in β) for α_i before it becomes $2\pi - \beta$, while all previous jump steps should be equal to $2\pi - \beta$. Since at every step, the algorithm attempts a jump of half the unexplored fence, right before the i -th jump there has been explored a total of $(i-1)(2\pi - \beta)$ part of the fence. Hence,

$$A_{i,i} = \frac{\beta - (i-1)(2\pi - \beta)}{2} = \frac{j\beta - (i-1)2\pi}{2}$$

The threshold h_i is determined by requiring that $A_{i,i} \leq 2\pi - \beta$, from which we obtain that

$$h_i = \frac{2\pi(i+1)}{i+2}$$

which is indeed increasing in i .

Our next claim is that

$$A_{i,j} = \frac{j\beta - (j-1)2\pi}{2^{i-j+1}}, \quad \text{for all } i \geq j.$$

The proof is by induction on $i-j$. Indeed, the claim is true when $i = j$. So assume that $i = j+t$ for some $t \geq 1$. The explored part of the fence up to the first $(i-1)$ jumps is equal to

$$\begin{aligned} \sum_{r=1}^{i-1} A_{r,j} &= \sum_{r=1}^{j-1} A_{r,j} + \sum_{r=j}^{j+t-1} A_{r,j} \\ &= (j-1)(2\pi - \beta) + \sum_{r=j}^{j+t-1} \frac{j\beta - (j-1)2\pi}{2^{r-j+1}} \\ &= (j-1)(2\pi - \beta) + \left(1 - \frac{1}{2^t}\right)(2\pi(j-1) - j\beta) \\ &= \frac{\beta(2^t - j) + 2\pi(j-1)}{2^t}. \end{aligned}$$

According to the Halving Algorithm, the i -th jump step will be exactly half of the unexplored fence, if that value does not exceed h_j . Indeed, the candidate step size is

$$\begin{aligned} \frac{\beta - \frac{\beta(2^t - j) + 2\pi(j-1)}{2^t}}{2} &= \frac{j\beta - (j-1)2\pi}{2^{t+1}} \\ &= \frac{j\beta - (j-1)2\pi}{2^{i-j+1}} \\ &= A_{i,j}. \end{aligned}$$

Finally, for this jump to be valid, we need to show that $0 < A_{i,j} \leq 2\pi - \beta$. To that end, recall that $A_{i,j}$ corresponds to the i -th jump when $h_{j-1} < \beta \leq h_j$, i.e., when $\frac{2\pi j}{j+1} < \beta \leq \frac{2\pi(j+1)}{j+2}$. Note that we are in the case where $i \geq j$, and so we have $\beta > \frac{2\pi j}{j+1} > \frac{2\pi(j-1)}{j}$ and hence $A_{i,j} > 0$. Also,

$$\begin{aligned} 2\pi - \beta - A_{i,j} &\geq 2\pi - \frac{2\pi(j+1)}{j+2} - \frac{j\beta - (j-1)2\pi}{2^{i-j+1}} \\ &\geq 2\pi - \frac{2\pi(j+1)}{j+2} - \frac{j\beta - (j-1)2\pi}{2} \\ &= \frac{j(2\pi(j+1) - \beta(j+2))}{2(j+2)}, \end{aligned}$$

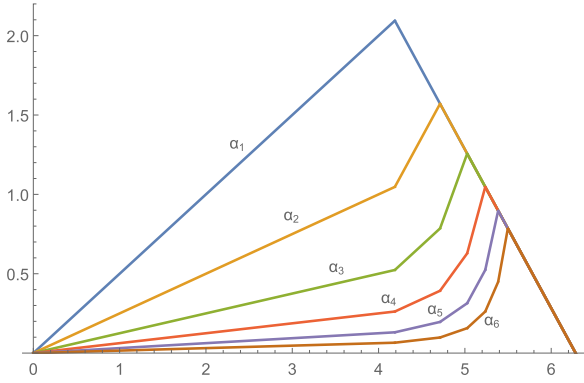


Fig. 4 The plot jumps $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_6$ of the Halving Algorithm as a function of β .

which is again non negative, since $\beta \leq \frac{2\pi(j+1)}{j+2}$, as wanted. This proves Lemma 4. ■

We are now ready to present a closed formula for the jump steps of the Halving Algorithm along with its performance, as a function on the number of jumps.

Theorem 2 For any $\beta \in (0, 2\pi)$, let $\rho_\beta := \max\left\{\frac{2\beta-2\pi}{2\pi-\beta}, 1\right\}$. The value of the i -th jump ($i \geq 1$) in the Halving Algorithm equals

$$\alpha_i = \begin{cases} 2\pi - \beta & \text{if } i < \rho_\beta \\ \frac{2\pi - \lceil \rho_\beta \rceil (2\pi - \beta)}{2^{i - \lceil \rho_\beta \rceil + 1}} & \text{otherwise} \end{cases}$$

Proof. According to Lemma 4, $\beta > h_i$ is satisfied as long as $i < \frac{2\beta-2\pi}{2\pi-\beta}$. Hence, if $i < \rho_\beta$ we have $\alpha_i = 2\pi - \beta$. For the same reason $\beta \leq h_i$ if and only if $i \geq \rho_\beta$, so ρ_β is the smallest integer for which $\beta \leq h_{\rho_\beta}$, meaning that $\beta \in (h_{\rho_\beta-1}, h_{\rho_\beta}]$. Therefore, again by Lemma 4, we set $j = \lceil \rho_\beta \rceil$ (and rearrange the terms) to derive the promised formula. This proves Theorem 2. ■

In **Fig. 4**, we depict the behaviour of the decreasing sequence α_i (in i) as a function of β for a $i = 1, 2, 3, 4, 5$ and 6 . Notably, for every k there is some threshold value of β , after which $\alpha_i = 2\pi - \beta$ for all $i \leq k$.

Finally, we use the closed formula for the jump steps to derive a closed formula for the cost of the Halving k -Jump Algorithm. The main idea for the proof is to show that the worst case for the algorithm is when all k jumps fall within the fence, and that the performance is strictly decreasing in k . This is what the next lemma establishes.

Lemma 5 The Halving k -Jump Algorithm incurs the maximum possible cost when all jump landings (including the basic one) fall within the fence.

Proof. For the values of α_i as defined in Theorem 2, we will show that the worst configuration is when all k jump landings (together with the basic one) fall within the fence. In the language of Lemma 3 we will show that $c_t^k < c_{t+1}^k$ for all $t = 0, k$. Also note that c_{k+1}^k is decreasing in k , in fact no matter what the jump steps are, since $x - \sin(x/2) > 0$, for all $x > 0$, which concludes the lemma.

As already claimed, it is immediate that $c_0^k < c_i$, for all i , since the cost c_0^k is incurred exactly when the basic landing falls outside the fence. Due to the fact that the robot has knowledge of the length β , the robot can fully avoid the fence by jumping over it. Next, according to Lemma 3 we have that

$$c_t^k - c_{t-1}^k = 4 \sin(\alpha_t/2) - 2 \sin(\alpha_{t-1}/2),$$

for all $t = 1, \dots, k$. In particular, for $t < \rho_\beta$ the jump steps remain equal to $2\pi - \beta > 0$, and hence $c_t^k - c_{t-1}^k > 0$ for all $t < \rho_\beta$.

When $t = \lceil \rho_\beta \rceil$ we have

$$\begin{aligned} c_{\lceil \rho_\beta \rceil}^k - c_{\lceil \rho_\beta \rceil - 1}^k &= 4 \sin(\alpha_{\lceil \rho_\beta \rceil}/2) - 2 \sin((2\pi - \beta)/2) \\ &= 4 \sin\left(\frac{\pi}{2} - \frac{\lceil \rho_\beta \rceil}{4}(2\pi - \beta)\right) - 2 \sin(\beta/2) \\ &\geq 4 \sin\left(\frac{\pi}{2} - \frac{\rho_\beta}{4}(2\pi - \beta)\right) - 2 \sin(\beta/2) \\ &\geq 4 \sin\left(\frac{\pi}{2} - \frac{2\beta-2\pi}{4}(2\pi - \beta)\right) - 2 \sin(\beta/2) \\ &= 4 \sin(\beta/2) - 2 \sin(\beta/2) \geq 0. \end{aligned}$$

When $t \geq \lceil \rho_\beta \rceil + 1$, the jump steps drop by a factor of two in each iteration. Since for all $x > 0$ we have that $2 \sin(x/4) - \sin(x/2) > 0$, we obtain easily that $c_t^k - c_{t-1}^k > 0$, for $t = \rho_\beta + 1, \dots, k$.

Hence it remains to show that $c_{k+1}^k > c_k$. To that end assume that $k \geq \lceil \rho_\beta \rceil$. Then we have

$$\begin{aligned} c_{k+1}^k - c_k^k &= \beta - \sum_{i=1}^k \alpha_i - 2 \sin(\alpha_k/2) \\ &= \beta - \left((\lceil \rho_\beta \rceil - 1)(2\pi - \beta) + \sum_{i=0}^{k - \lceil \rho_\beta \rceil} \frac{\alpha_{\lceil \rho_\beta \rceil}}{2^i} \right) \\ &\quad - 2 \sin(\alpha_k/2) \\ &= \beta - \left((\lceil \rho_\beta \rceil - 1)(2\pi - \beta) + 2\alpha_{\lceil \rho_\beta \rceil} \left(1 - \frac{1}{2^{k - \lceil \rho_\beta \rceil + 1}}\right) \right) \\ &\quad - 2 \sin(\alpha_k/2) \\ &= \beta - \left((\lceil \rho_\beta \rceil - 1)(2\pi - \beta) + 2 \left(\pi - \frac{\lceil \rho_\beta \rceil}{2}(2\pi - \beta) \right) \right. \\ &\quad \left. \cdot \left(1 - \frac{1}{2^{k - \lceil \rho_\beta \rceil + 1}}\right) \right) - 2 \sin(\alpha_k/2) \\ &= \frac{2\pi - \lceil \rho_\beta \rceil (2\pi - \beta)}{2^{k - \lceil \rho_\beta \rceil + 1}} - 2 \sin(\alpha_k/2). \end{aligned}$$

Recall that $\alpha_k = \frac{2\pi - \lceil \rho_\beta \rceil (2\pi - \beta)}{2^{k - \lceil \rho_\beta \rceil + 1}}$, so the last expression is non negative since $x - \sin(x/2) > 0$ for all $x > 0$. Finally, for the case that $k < \lceil \rho_\beta \rceil$, and since k is an integer, we have $k \leq \rho_\beta$, and so

$$\begin{aligned} c_{k+1}^k - c_k^k &= \beta - k(2\pi - \beta) - 2 \sin(\beta/2) \\ &\geq \beta - \rho_\beta(2\pi - \beta) - 2 \sin(\beta/2) \\ &= 2\pi - \beta - 2 \sin(\beta/2). \end{aligned}$$

The last expression is non-negative since $\sin(\beta/2) = \sin((2\pi - \beta)/2)$ and $x - 2 \sin(x/2) > 0$, for all $x > 0$. ■

We are ready to conclude with the cost of the Halving k -Jump Algorithm.

Theorem 3 The cost of the Halving k -Jump Algorithm is strictly decreasing with k and it equals

$$\begin{aligned} 1 + 2\pi + \frac{\lceil \rho_\beta \rceil (2\pi - \beta) - 2\pi}{2^{k - \lceil \rho_\beta \rceil + 1}} + 2(\lceil \rho_\beta \rceil - 1) \sin(\beta/2) \\ + 2 \sum_{i=0}^{k - \lceil \rho_\beta \rceil} \sin\left(\frac{\alpha_{\lceil \rho_\beta \rceil}}{2^{i+1}}\right), \end{aligned}$$

where $\rho_\beta = \max\left\{\frac{2\beta-2\pi}{2\pi-\beta}, 1\right\}$ and $\alpha_{\lceil \rho_\beta \rceil} = \pi - \frac{\lceil \rho_\beta \rceil}{2}(2\pi - \beta)$.

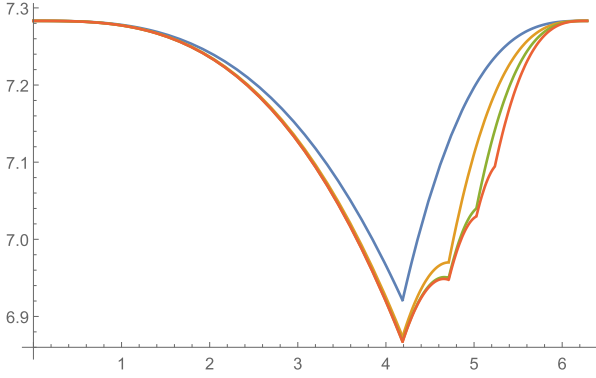


Fig. 5 The performance of the Halving Algorithm for 1, 2, 3 and 4 jumps (decreasing in the number of jumps, respectively) as a function of β .

Proof. Using the terminology of Lemma 3, and by Lemma 5, the cost of the Halving k -Jump Algorithm equals

$$c_{k+1}^k = 1 + 2\pi - \beta - \sum_{i=1}^k (\alpha_i - 2 \sin(\alpha_i/2))$$

where the jump steps are as determined in Theorem 2. From the expression above, it is immediate that the cost is strictly increasing in k , as long as all jump steps are positive. Next, we compute the summation in parts. We have,

$$\begin{aligned} \beta + \sum_{i=1}^k \alpha_i &= \beta + \sum_{i=1}^{\lceil \rho_\beta \rceil - 1} \alpha_i + \sum_{i=\lceil \rho_\beta \rceil}^k \alpha_i \\ &= \beta + (\lceil \rho_\beta \rceil - 1)(2\pi - \beta) + \sum_{i=0}^{k - \lceil \rho_\beta \rceil} \frac{\alpha_{\lceil \rho_\beta \rceil}}{2^i} \\ &= \beta + (\lceil \rho_\beta \rceil - 1)(2\pi - \beta) + \alpha_{\lceil \rho_\beta \rceil} \left(2 - \frac{1}{2^{k - \lceil \rho_\beta \rceil}} \right) \\ &= \beta + (\lceil \rho_\beta \rceil - 1)(2\pi - \beta) + \left(\pi - \frac{\lceil \rho_\beta \rceil}{2} (2\pi - \beta) \right) \\ &\quad \cdot \left(2 - \frac{1}{2^{k - \lceil \rho_\beta \rceil}} \right) \\ &= \frac{\lceil \rho_\beta \rceil (2\pi - \beta) - 2\pi}{2^{k - \lceil \rho_\beta \rceil + 1}}. \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{i=1}^k \sin(\alpha_i/2) &= \sum_{i=1}^{\lceil \rho_\beta \rceil - 1} \sin(\alpha_i/2) + \sum_{i=\lceil \rho_\beta \rceil}^k \sin(\alpha_i/2) \\ &= (\lceil \rho_\beta \rceil - 1) \sin(\beta/2) + \sum_{i=0}^{k - \lceil \rho_\beta \rceil} \sin\left(\frac{\alpha_{\lceil \rho_\beta \rceil}}{2^{i+1}}\right) \end{aligned}$$

Putting the two expression together gives the promised cost. This proves Theorem 3. \blacksquare

Figure 5 summarizes the cost of the Halving k -Jump Algorithm for $k = 1, 2, 3$ and 4.

6. Some Optimal k -Jump Algorithms & Comparison

It is apparent from Lemma 3 that choosing the optimal jump steps $\alpha_1, \dots, \alpha_k$ amounts to solving the involved optimization problem $\min_{\alpha_1, \dots, \alpha_k} \max_{i=1, \dots, k+1} \{c_i^k\}$, where $\alpha_i \leq \min\{\pi, 2\pi - \beta\}$. In this section we contrast the choices and performance of the Halving k -Jump Algorithm with the choices and performance of

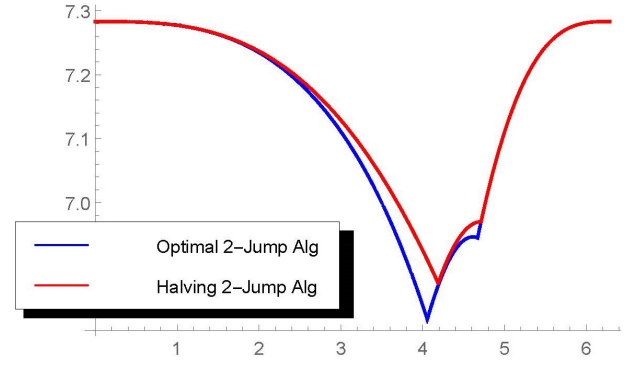


Fig. 6 Performance comparison between the Optimal 2-Jump Algorithm and the Halving 2-Jump Algorithm, as a function of β .

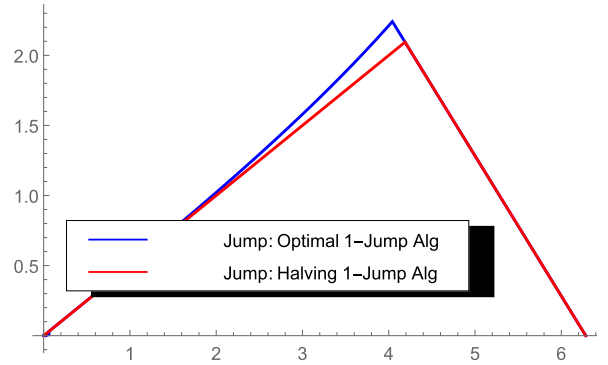


Fig. 7 Comparison of jump choices between the Optimal 1-Jump Algorithm and the Halving 1-Jump Algorithm, as a function of β .

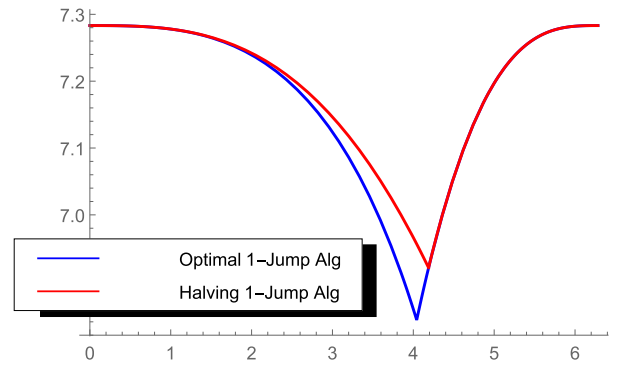


Fig. 8 Performance comparison between the Optimal 1-Jump Algorithm and the Halving 1-Jump Algorithm, as a function of β .

the Optimal k -Jump Algorithm that was obtained using optimization software packages, for $k \leq 3$ (except from $k = 1$ whose formal analysis appears in Section 3). Our findings are summarized in the figures below (1-Jump in **Figs. 6–8**, 2-Jump in **Fig. 9**, and 3-Jump in **Figs. 10–11**).

For $k = 1, 2, 3$ we numerically compute the optimal k -Jump Algorithm (note that for $k = 1$ the rigorous analysis appears in Section 3). Then, we contrast the performance of the optimal and of the Halving algorithm (for the same number of jumps), as well as contrasting their corresponding jump steps. The numerical calculations indicate that the choices of the Halving algorithm are nearly optimal for a wide spectrum of β (with the largest discrepancy for $\beta \approx \gamma$). Interestingly, for larger values of β , the choices of the Halving algorithm are nearly optimal that also reflects on the cost of the two algorithms which becomes nearly identical. More importantly, experiments indicate that for large values of β ,

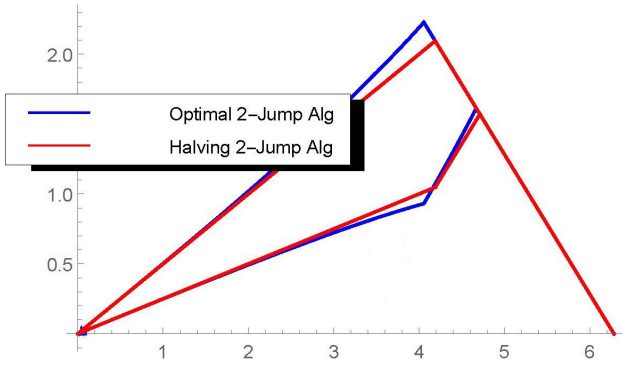


Fig. 9 Comparison of the two jump choices between the Optimal 2-Jump Algorithm and the Halving 2-Jump Algorithm, as a function of β . The first jump of each algorithm is always no smaller than the second one, and eventually they all attain the value $2\pi - \beta$.

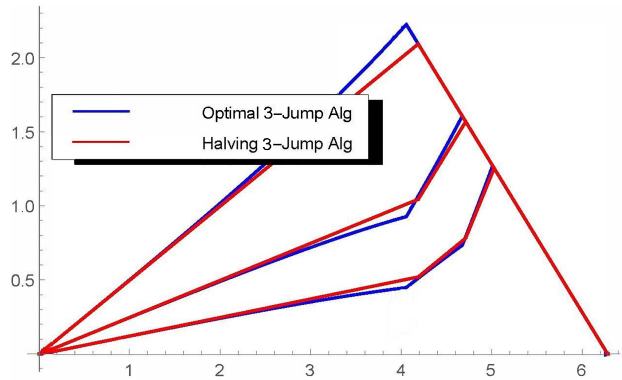


Fig. 10 Comparison of the three jump choices between the Optimal 3-Jump Algorithm and the Halving 3-Jump Algorithm, as a function of β . For both algorithms, the first jump of each algorithm is always no smaller than the second one, which is no smaller than the third one. Eventually they all attain the value $2\pi - \beta$.

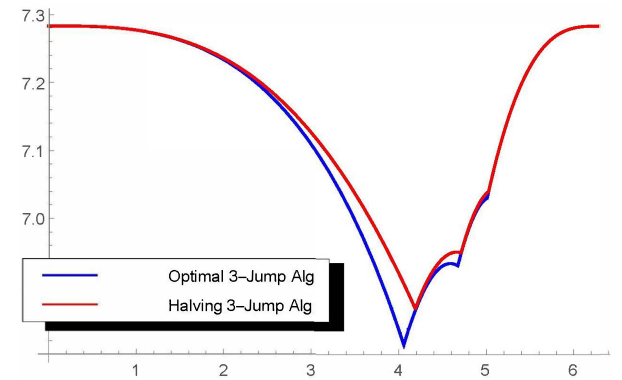


Fig. 11 Performance comparison between the Optimal 3-Jump Algorithm and the Halving 3-Jump Algorithm, as a function of β . Notably, performance is nearly the same for all values of $\beta > 5$. The bigger discrepancy is observed for values of β close to 4, for which also the jump steps between the two algorithms exhibit the larger gaps. See Fig. 10.

the optimal choices for k jump steps is to make all equal to $2\pi - \beta$, which is also the choice of the Halving algorithm. Finally, in the next table we indicate numerically the largest observed gap for $k = 1, 2, 3$, and in each case we also provide the value of β that is responsible for this discrepancy.

| | $k = 1$ | $k = 2$ | $k = 3$ |
|---------------|--------------------|---------------------|---------------------|
| Largest gap | ≈ 0.113696 | ≈ 0.0931841 | ≈ 0.0902863 |
| Worst β | ≈ 4.04196 | ≈ 4.055585 | ≈ 4.05714 |

7. Randomized Algorithms

In this section we explore how randomness can be used to improve the performance of our algorithmic. Intuitively, random algorithmic choices reduce the power of the adversary, and as such, the (expected) worst-case performance will be reduced.

In order to exhibit the power of randomness, we will restrict ourselves to variations of Algorithm 1, i.e., algorithms that perform no jumps. Still, we will be able to improve performance significantly.

7.1 Single Random-Bit Algorithm

Since randomness is considered a computational resource, we are motivated to explore a randomized algorithm, Algorithm 5, that uses a single random bit. One way of exploring a random bit is to run Algorithm 1 with the modification that the exploration of the perimeter of the disk is either clockwise or counterclockwise, each with probability $1/2$. This is what we formally describe next.

Algorithm 5 R-Robot Direction

- 1: Walk to the perimeter of the disk
 - 2: Start walking on the perimeter, either in cw or in ccw direction, each with probability $1/2$.
 - 3: Perform steps 3–7 of Algorithm 1.
-

Theorem 4 The worst case expected termination time of Algorithm 5 is $1 + \pi$, i.e., it is independent of the length β of the fence.

Proof. Observe that the random choice of Algorithm 5 involves only the direction of the robot’s movement. In particular, an adversary can still choose where the robot will land, as well as choosing how far away the treasure is from the fence. The worst case (expected) termination time over these choices will be the algorithm’s performance. For notational convenience, we omit in the calculations the deployment cost 1, i.e., the time it takes the robot to reach the perimeter.

Clearly, the adversary can choose to have the robot land either outside the fence, or inside the fence. In each case we will determine the worst positioning of the treasure, and then we will find the maximum of the two. In that direction, and given any choice of the adversary, let X be the random variable that equals the termination time for the associated experiment, over the random choice of the algorithm.

In the first case, suppose that the adversary chooses to have the robot land outside the fence (See Fig. 12 (a)). Let x denote the clockwise distance between the robot and the fence. Also let d denote the clockwise distance between the treasure and the fence, with $d \geq 0$. Depending on the values of x, d , together with the random choice of the algorithm, the robot will first either encounter the treasure or the fence. Without loss of generality we may assume that $x + d + \beta \leq 2\pi$, and then we may focus only on the random choice of the algorithm. So, let A denote the event that the fence is encountered first (in Fig. 12 (a) that corresponds to moving ccw). It is easy to see that the robot will jump over the fence, avoiding the cost of walking along the fence perimeter. We define A to be the event in which the robot encounters the fence

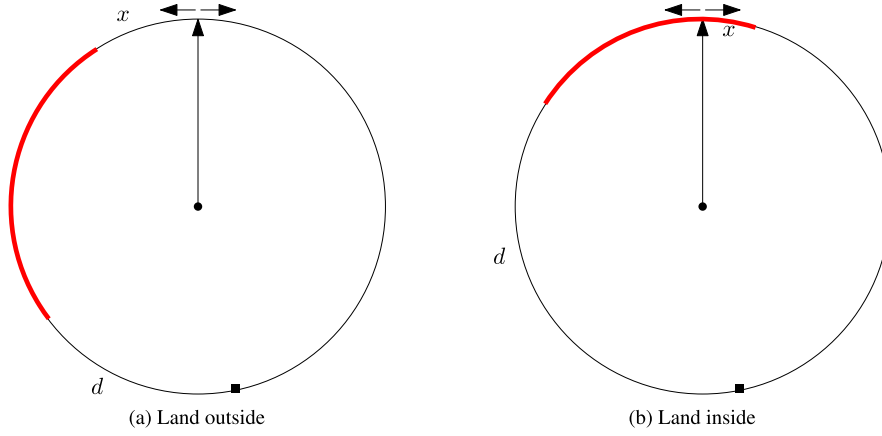


Fig. 12 The Robot randomly chooses a direction to move in.

before the treasure. The termination time is:

$$E[X|A] = x + 2 \sin\left(\frac{\beta}{2}\right) + d \quad (4)$$

For the complementary event, A^c , the robot encounters the treasure first (in Fig. 12 (a) that corresponds to moving cw). The incurred cost in this case is

$$\mathbb{E}[X|A^c] = 2\pi - \beta - d - x \quad (5)$$

Overall, if the adversary makes the robot land outside the fence, then the worst case expected running time is

$$\begin{aligned} \mathbb{E}[X] &= \max_{x,d: x+d+\beta \leq 2\pi, \& \ x,d \geq 0} \{\mathbb{P}[A] \cdot \mathbb{E}[x|A] + \mathbb{P}[A^c] \cdot \mathbb{E}[x|A^c]\} \\ &= \frac{1}{2} \left(2\pi - \beta + 2 \sin\left(\frac{\beta}{2}\right) \right) \end{aligned} \quad (6)$$

In the second case, suppose that the adversary chooses to have the robot land inside the fence (See Fig. 12 (b)). As before, let x denote the ccw distance of the robot to the end of the fence. Let also d be the ccw distance between the fence and the treasure. Clearly, we have that $0 \leq x \leq \beta$, and $0 \leq d \leq 2\pi - \beta$. Let now D denote the event that the robot moves ccw. It is immediate that

$$\mathbb{E}[X|D] = \beta - x + d \quad (7)$$

and that

$$\mathbb{E}[X|D^c] = x + 2\pi - \beta - d \quad (8)$$

So, the worst case expected termination time becomes

$$\begin{aligned} \mathbb{E}[X] &= \max_{x,d: 0 \leq x \leq \beta, \& \ 0 \leq d \leq 2\pi - \beta} \{\mathbb{P}[D] \cdot \mathbb{E}[X|D] + \mathbb{P}[D^c] \cdot \mathbb{E}[X|D^c]\} \\ &= \pi \end{aligned} \quad (9)$$

Therefore it is clear that the worst case expected termination time is computed as the maximum between Eqs. (6) and (9), corresponding to the adversarial choice as to whether the robot lands inside or outside the fence. Since $x \geq 2 \sin(x/2)$ for all x , we see that the dominant expression is Eq. (9) (for every β) and the theorem follows. ■

It is noteworthy that the performance of the algorithm is not dependent on β . Also, the adversary's only concern is to make sure the robot lands within the fence, in which case the actual relative placement of the fence and the treasure become irrelevant.

7.2 Unbounded Random-Bits Algorithms

Our next attempt is toward further improving the worst case expected termination time. Provably, a favourable event/case in the analysis of all our algorithms this far is when the robot lands outside the fence for the first time. Indeed, even for Algorithm 5 the adversary chooses to have the robot land inside the fence. Motivated by this, we introduce a new randomized option, Algorithm 6, i.e., we choose a random initial landing point on the perimeter of the disk (and then run Algorithm 1).

Algorithm 6 R-Fence Walk

- 1: Walk to a point of perimeter of the disk, chosen uniformly at random.
 - 2: Perform steps 3–7 of Algorithm 1.
-

Against such an algorithm, an adversary can only choose the relative distance between the treasure and the fence. On the other hand, the required randomness in order to choose a random point on the disk would be enormous. Nevertheless, we are motivated to examine whether such an approach would improve performance.

Theorem 5 Let β_0 be the unique solution of equation $\beta - \sin(\beta/2) = \pi$ in $[0, 2\pi]$, i.e., $\beta_0 \approx 4.04$. When the length of the fence is β , then the worst case expected termination time of Algorithm 6 is

$$1 + \pi + \begin{cases} \frac{1}{2\pi} \left(\beta - \sin\left(\frac{\beta}{2}\right) - \pi \right)^2, & \text{if } \beta \leq \beta_0 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. As before, we omit in the calculations the deployment cost 1, i.e., the time it takes the robot to reach the perimeter. Let X be the random variable that is equal to the worst case (over the adversarial choices) expected (over the algorithmic choices) termination time of Algorithm 6. We partition the possible outcomes in three events. In all cases, we assume that the treasure is placed at cw distance d from the fence, where $0 \leq d \leq 2\pi - \beta$.

Let A_1 be the event that the robot lands inside the fence (see also Fig. 13). Clearly, we have that $\mathbb{P}[A_1] = \beta/2\pi$. In that case, the cw distance of the robot to the end of the fence is x , where $0 \leq x \leq \beta$, and so

$$\mathbb{E}[X|A_1] = \frac{1}{\beta} \int_0^\beta (\beta - x + d) dx = \frac{\beta}{2} + d. \quad (10)$$

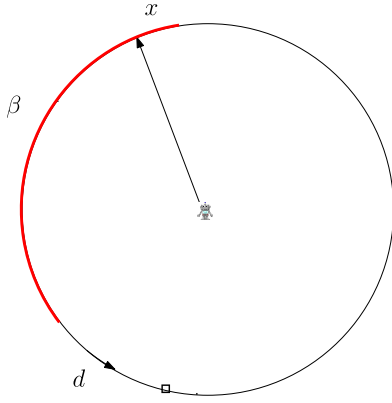


Fig. 13 RFW Case 1 (Random basic landing point).

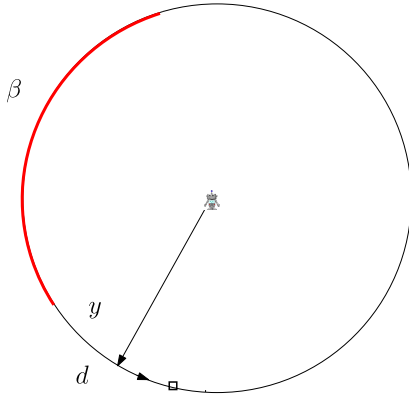


Fig. 14 RFW Case 2 (Random basic landing point).

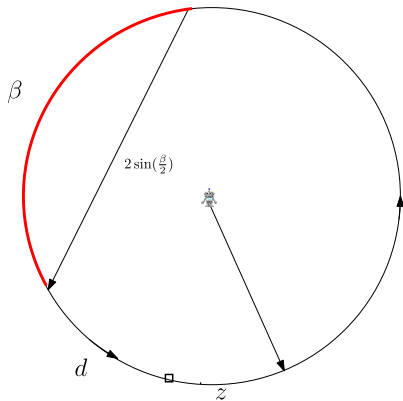


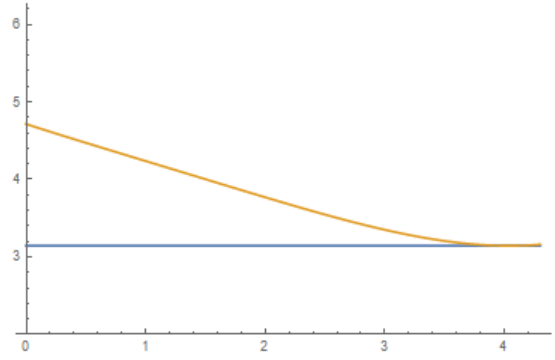
Fig. 15 RFW Case 3 (Random basic landing point).

Let A_2 be the event that the robot lands outside the fence and that it encounters the treasure before it encounters the fence (see Fig. 14). We have that $\mathbb{P}[A_2] = d/2\pi$. Let also y denote ccw distance between the robot and the fence, where $0 \leq y \leq d$. Then

$$\mathbb{E}[X|A_2] = \frac{1}{d} \int_0^d (d-y) dy = \frac{d}{2}. \quad (11)$$

Finally, let A_3 denote the event that the robot lands outside the fence and that it encounters the fence before it encounters the treasure (see Fig. 15). We have $\mathbb{P}[A_3] = (2\pi - \beta - d)/2\pi$. Let also z denote ccw distance between the robot and the treasure, where $0 \leq z \leq 2\pi - \beta - d$ (implying restriction $d \geq 2\pi - \beta$ for the adversary). Then

$$\mathbb{E}[X|A_3] = \frac{1}{2\pi - \beta - d} \int_0^{2\pi - \beta - d} \left(2\pi - z - \beta + 2 \sin\left(\frac{\beta}{2}\right) + d\right) dz$$


 Fig. 16 Performance of Algorithm 6 (yellow line) compared to the performance of Algorithm 5 (blue line) as a function of the fence length $\beta \leq \beta_0$.

$$= \frac{3d}{2} + \pi - \frac{\beta}{2} + 2 \sin\left(\frac{\beta}{2}\right). \quad (12)$$

Expressions Eqs. (10), (11) and (12), along with elementary calculations, allow us to conclude that

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{P}[A_1] \cdot \mathbb{E}[X|A_1] + \mathbb{P}[A_2] \cdot \mathbb{E}[X|A_2] + \mathbb{P}[A_3] \cdot \mathbb{E}[X|A_3] \\ &= \frac{(d - 2\pi + \beta)(-d + \beta - 2 \sin(\beta/2)) + 2\pi^2}{2\pi}. \end{aligned} \quad (13)$$

Now, we determine the adversarial choice for d which would induce the worst case expected termination time, i.e., that would maximize Eq. (13). Observe that Eq. (13) is a degree 2 polynomial in d with negative leading coefficient. Hence, the expression is maximized when the derivative of the expression is 0. Elementary calculations show that the maximizer is $d_0 = \pi - \sin(\frac{\beta}{2})$, and in particular, when $d < d_0$ expression Eq. (13) is strictly increasing and when $d > d_0$ expression Eq. (13) is strictly decreasing.

Notably, we have explicitly assumed above that $d \leq 2\pi - \beta$, so d_0 is a valid adversarial choice only if $d_0 \leq 2\pi - \beta$. Given the value of d_0 , we conclude that the adversary would choose $d = d_0$ as long as $\beta - \sin(\beta/2) \leq \pi$, i.e., as long as $\beta \leq \beta_0$ (where β_0 is as in the statement of the theorem). For these values of β , the worst case expected performance is obtained by replacing $d = \pi - \sin(\frac{\beta}{2})$ in Eq. (13). If on the other hand $\beta > \beta_0$, then due to the monotonicity of Eq. (13), the adversarial choice inducing the worst case expected performance is when $d = 2\pi - \beta$. Again, substituting in Eq. (13) together with elementary calculations derive the promised formula. ■

A comparison between the performances of Algorithm 5 and 6, is depicted in Fig. 16. Surprisingly, Algorithm 6 is strictly worse than Algorithm 5, for all fence lengths $\beta < \beta_0 \approx 4.04$. From the analysis of Algorithm 6 we learned that although the adversary cannot choose the robot's landing point, the fact that the robot may land inside the fence, and since the robot's movement direction is known, the position of the treasure can be chosen so that expected termination cost is high. At the same time, we should recall that the power of Algorithm 5 lies in that the movement direction of the robot is random, and in particular that the relative positioning of the fence and the treasure are irrelevant. Therefore we are motivated in combining the randomized choices of Algorithms 5 and 6. This is what we explore with Algorithm 7.

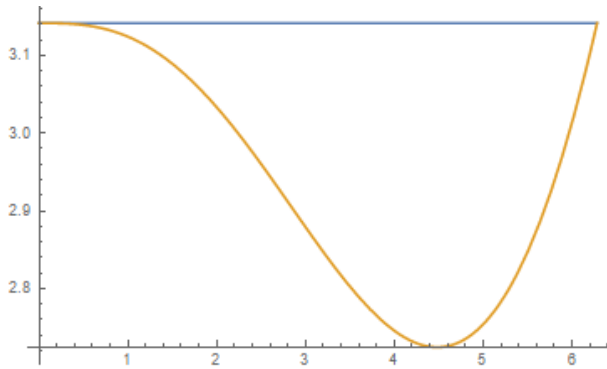


Fig. 17 Algorithm 7 performance (yellow line) compared to Algorithm 5 performance (blue line) for all fence lengths $0 \leq \beta \leq 2\pi$.

Algorithm 7 RD-Fence Walk

- 1: Walk to point of perimeter of the disk, chosen uniformly at random.
- 2: Start walking on the perimeter, either in cw or in ccw direction, each with probability 1/2.
- 3: Perform steps 3–7 of Algorithm 1.

Theorem 6 The worst case expected termination time of Algorithm 7 is

$$1 + \frac{\beta}{2} + \left(1 - \frac{\beta}{2\pi}\right) \left(\pi - \frac{\beta}{2} + \sin\left(\frac{\beta}{2}\right)\right).$$

Proof. The analysis is very similar to the one we did for Algorithm 5, i.e., the proof of Theorem 4. We examine two cases as to whether the robot lands inside or outside the fence. If we condition on the assumption that the robot lands inside the fence (which happens with probability $\beta/2\pi$), then the expected performance of the algorithm is π , independently of where the treasure is placed. If we condition on the assumption that the robot lands outside the fence (which happens with probability $1 - \beta/2\pi$), then the expected performance of the algorithm is $\pi - \beta/2 + \sin(\beta/2)$, again independently of the positioning of the treasure. Multiplying the conditional expectations with the probabilities of the corresponding events results in the promised formula. ■

As expected, Algorithm 7 strictly improves Algorithm 5, and this is depicted in **Fig. 17**.

8. Conclusion

In this paper we investigated a new search problem for a mobile robot to find a stationary target placed at an unknown location at distance 1, in the presence of a fence placed on the perimeter of a unit disk. First we determined the optimal 1-Jump algorithm for the robot to find the target. Then we provided a generic description of k -Jump algorithms and analyzed their cost depending on the jump landings. Subsequently, we analyzed the Halving k -Jump algorithms, where k is the max number of jumps the robot makes so as to overcome the fence and find the target. Finally, we successfully explored the role of randomness toward improving the performance of our algorithms. Several interesting questions remain open, when e.g., there are multiple fences on the perimeter of the disc, and the robot's speed changes when traversing a fence.

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