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Continuous Flattening of α-Trapezoidal Polyhedra

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Abstract: It was proved that any orthogonal polyhedron is continuously flattened by using a property of a rhombus. We investigated the method precisely, and found that there are infinitely many ways to flatten such polyhedra. We prove that the infimum of the area of moving creases is zero for α-trapezoidal polyhedra, which is a generalization of semi-orthogonal polyhedra. Also we prove that, for any integer n, there exists a continuous flattening motion whose area of moving creases is arbitrarily small for any n-gonal pyramid with a circumscribed base and a top vertex being just above the incenter of the base. As a by-product we provide a continuous flattening motion whose area of moving creases is arbitrarily small for more general types of polyhedra.

Keywords: polyhedron, continuous flattening, moving crease, zig-zag belt

1. Introduction

We use the terminology *polyhedron* for a polyhedral surface in three-dimensional Euclidean space which is permitted to touch itself but not self-intersect, and a polyhedron always can be folded by creases like a piece of paper. Note that, in this paper, polyhedra with higher genus are allowed, so long as no three faces of any polyhedron meet at an edge. A *flat folding* of a polyhedron is a folding by creases into a multilayered flat folded state without self-intersection. It is known that any polyhedron with genus zero has a multilayered flat folded state [2], [6]. However, it remains an open problem to find a continuous motion of the surface down to a multilayered flat folded state for any polyhedron.

The original problem of continuous flattening of polyhedra is in Ref. [5] and the existence of a continuous motion has been proven in Refs. [1], [8] for any convex polyhedron. In Ref. [8] the authors gave a method by use of the cut locus and Alexandrov gluing theorem. On the other hand, in Ref. [1] the authors provided a surprisingly simple method by use of the straight skeleton gluing. In addition, in literature there are several ways of continuous flattening for special classes of convex polyhedra (see Ref. [7] for example).

An important limitation to continuous flattening is the Bellows Theorem [3]: the volume of any polyhedron with rigid faces is invariant even if it can flex, where the terminology "polyhedron" is used for a polyhedral surface which is not permitted to touch itself but we can still apply this result to our cases. Flattening a polyhedron necessarily changes the volume (from nonzero to zero), so some faces cannot be rigid, e.g., by changing their shapes continuously by infinitely moving creases. In this paper we propose the open question to find a method with small area of moving creases for a given polyhedron, and it is interesting because the area of moving creases should be made by special materials for some products. In this paper, we focus on the area of moving creases for continuous flattening of polyhedra, and show the existence of a continuous flattening motion whose area of moving creases is arbitrarily small for some types of polyhedra. Note that what is meant by arbitrarily small area of moving creases is that the ratio of the area supporting moving creases to polyhedron surface area can be made smaller than any positive constant. As far as the authors know, the idea of considering the area of moving creases is introduced here for the first time, and it is our original problem to find a method with small area of moving creases for a given polyhedron.

A polyhedron is called *orthogonal* if the dihedral angle of each edge is 90° or 270° (cf. Ref. [6]). By an appropriate choice of x, y, z axes for Euclidean space, we can equivalently define a polyhedron to be orthogonal if every face is orthogonal to the x, y, or z axis.

More generally, a polyhedron is called *semi-orthogonal* if every face is orthogonal to the z axis or the xy-plane (cf. Ref. [6]). It was proved in Ref. [4] that every semi-orthogonal polyhedron can be continuously flattened so that all orthogonal faces to the z axis are rigid, that is, there are no creases on them.

Furthermore, a more general type of polyhedra is defined here. A polyhedron is called *α-trapezoidal* if every face is orthogonal to the z axis or a trapezoid forming the dihedral angle α (or $180° - \alpha$) with the xy-plane, where $0° < \alpha \leq 90°$. Note that any 90°-trapezoidal polyhedron is just a semi-orthogonal polyhedron, and truncated regular pyramids are α-trapezoidal.

Our main results are the continuous flattening of any α-trapezoidal polyhedron (not necessarily convex or of genus zero) and its applications as follows.

Theorem 1. *For any α-trapezoidal polyhedron, there exists a continuous flattening motion whose area of moving creases is arbitrarily small.*

Theorem 2. *Let $n \geq 3$. For any n-gonal pyramid with a circumscribed base and a top vertex being just above the incenter of the*

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base, there exists a continuous flattening motion whose area of moving creases is arbitrarily small.

Furthermore, we provide a continuous flattening motion whose area of moving creases is arbitrarily small for more general types of polyhedra in Section 6.

2. Zig-Zag Belts

For two points u, v in the three-dimensional space $\mathbb{R}^3$, we denote by uv the line segment joining points u and v, and also denote by $|uv|$ the Euclidean distance between those two points. Let P be a plane parallel to the xy-plane. Then an α *zig-zag belt* B_α, $0° < \alpha \leq 90°$, in three-dimensional Euclidean space is a set of trapezoids (called faces) with same height, called the *width* of B_α, satisfying the following conditions:

(i) the interiors of faces do not intersect each other,
(ii) every top edge is on the plane P,
(iii) every bottom edge is on the xy-plane,
(iv) every two trapezoids do not share a top or bottom edge,
(v) all top edges form a cycle on P and all bottom edges also form a cycle on the xy-plane, and
(vi) every face of B_α forms a dihedral angle α (or $180° - \alpha$) with the xy-plane.

Then top edges and bottom edges of an α zig-zag belt are called *zig-zag sides*. Note that any B_α with $\alpha = 90°$ coincides with a zig-zag belt in Ref. [4].

Especially, α zig-zag belts appear on truncated regular pyramids or parallelepipeds; see **Fig. 1** (a), (b), (c). We can also think of an α zig-zag belt as an unfolded state of a polygon formed by n trapezoids $u_iv_iv_{i+1}u_{i+1}$, $0 \leq i \leq n-1$; see Fig. 1 (d), (e), (f) for examples.

For any α zig-zag belt formed by n trapezoids $u_iv_iv_{i+1}u_{i+1}$, $0 \leq i \leq n-1$, two adjacent trapezoids are not always congruent, but we have the following formulas by the above condition (vi):

$$\angle u_iv_iv_{i-1} = \angle u_iv_iv_{i+1}$$

or

$$\angle u_iv_iv_{i-1} = 180° - \angle u_iv_iv_{i+1}$$

for each $1 \leq i \leq n-1$. Hence any two adjacent faces can be classified into three types by the angles adjacent to the edge u_iv_i. In an α zig-zag belt formed by n trapezoids $u_iv_iv_{i+1}u_{i+1}$, $0 \leq i \leq n-1$, the types of two adjacent faces with $\angle u_iv_iv_{i-1} = \angle u_iv_iv_{i+1} \leq 90°$, $\angle u_iv_iv_{i-1} = 180° - \angle u_iv_iv_{i+1}$ and $\angle u_iv_iv_{i-1} = \angle u_iv_iv_{i+1} > 90°$ are called *Inside-Inside*, *Outside-Inside* and *Outside-Outside*, respectively. For example, the types of Fig. 1 (a), (b) and (c) are Inside-Inside, Outside-Inside and Outside-Outside, respectively. Also the side containing the dihedral angle α is called the α*-side* for each face. Note that the α-side of each face is defined relative to the α angle formed by the bottom face and not the top face.

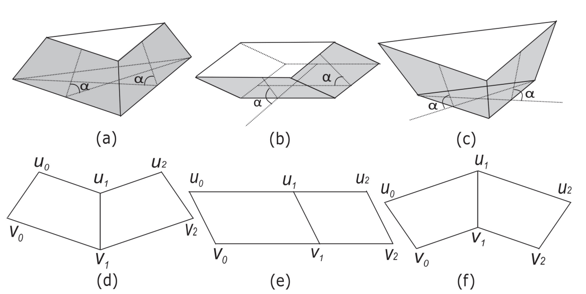

Fig. 1 Three examples of α zig-zag belts (a) on a truncated tetrahedron; (b) on a parallelepiped; (c) on a reverse truncated tetrahedron, and unfolded states of α zig-zag belts (d) on a truncated tetrahedron; (e) on a parallelepiped; (f) on a reverse truncated tetrahedron.

Furthermore, since $\angle u_{i-1}u_iu_{i+1} = \angle v_{i-1}v_iv_{i+1}$ for each $1 \leq i \leq n-1$, we can put $\angle u_{i-1}u_iu_{i+1} = \angle v_{i-1}v_iv_{i+1} = \theta_i$ with $0° < \theta_i \leq 180°$ in $\mathbb{R}^3$. Now, in two adjacent faces formed by two trapezoids $u_{i-1}v_{i-1}v_iu_i$ and $u_iv_iv_{i+1}u_{i+1}$, the relation between θ_i and α ($0° < \alpha < 90°$) is provided as follows.

Proposition 1. *For cases of Inside-Inside, Outside-Inside with* $0° < \angle u_iv_iv_{i-1} < 90°$ *and Outside-Outside, we have*

$$\tan\angle u_iv_iv_{i-1}\cos\alpha = \frac{|u_iv_i|\sin\angle u_iv_iv_{i-1}\cos\alpha}{|u_iv_i|\cos\angle u_iv_iv_{i-1}} = \tan\frac{\theta_i}{2}, \quad (1)$$

$$\tan\angle u_iv_iv_{i-1}\cos\alpha = \tan\frac{180° - \theta_i}{2} \quad (2)$$

and

$$\tan(180° - \angle u_iv_iv_{i-1})\cos\alpha = \tan\frac{\theta_i}{2}, \quad (3)$$

respectively.

Proof. For cases of Inside-Inside and Outside-Inside with $0° < \angle u_iv_iv_{i-1} < 90°$, let h_1 be a point on $v_{i-1}v_i$ so that $\angle u_ih_1v_i = 90°$. Also let h_2 be a point on the xy-plane so that $\angle u_ih_2v_i = \angle u_ih_2h_1 = 90°$ and $\angle u_ih_1h_2 = \alpha$ in $\mathbb{R}^3$. Then it follows that $|h_1h_2| = |u_iv_i|\sin\angle u_iv_iv_{i-1}\cos\alpha$, $|h_1v_i| = |u_iv_i|\cos\angle u_iv_iv_{i-1}$ and $\angle h_1v_ih_2$ is a bisector of $\angle v_{i-1}v_iv_{i+1}$. Hence Egs. (1) and (2) can be obtained. Similarly, since $\angle v_{i-1}v_iu_i = 180° - \angle u_{i-1}u_iv_i$ and two faces of Outside-Outside are reverse two faces of Inside-Inside, (3) can also be shown for a case of Outside-Outside. □

Sice $0° < \alpha < 90°$, Proposition 1 shows the following result.

Proposition 2. *For cases of Inside-Inside, Outside-Inside with* $0° < \angle u_iv_iv_{i-1} < 90°$ *and Outside-Outside, we have*

$$0° < \theta_i < 2\angle u_iv_iv_{i-1}, \quad (4)$$

$$180° - 2\angle u_iv_iv_{i-1} = -\angle u_iv_iv_{i-1} + \angle u_iv_iv_{i+1} < \theta_i < 180° \quad (5)$$

and

$$0° < \theta_i < 360° - 2\angle u_iv_iv_{i-1}, \quad (6)$$

respectively.

3. Folding Kites

In Ref. [4] the authors investigated the rhombus property, and then it was shown that any 90° zig-zag belt can be continuously flattened by use of the rhombus property. And also it was shown that there exists a continuous flattening motion of any semi-orthogonal polyhedron (cf. Ref. [4]).

As a generalization of the rhombus foldings, in Ref. [9] the author gives the kite foldings.

Definition 1. *Let* $K = abcd$ *be a kite with* $|ab| = |ad|$ *and* $|bc| = |dc|$ *(see* ***Fig. 2*** *(a)). Let* r *be a point on* ac*. For a point* q *on* br*, fold* K *by mountain creases on* aq*,* bq*,* cq *and* dr*, and valley creases on* qr*,* ar *and* cr*. Then we obtain a figure as shown in Fig. 2 (b) which is flexible. We call such figure a folded kite*

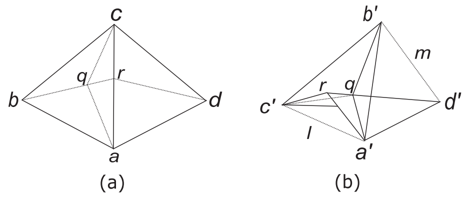

Fig. 2 (a) A kite $K = abcd$ and (b) a folded kite $K_r(l, m)$ with a', b', c', d'.

with wing-shape.

For a kite K and a point r, if we choose the distances $|a'c'| = l$ and $|b'd'| = m$, a point q and the folded kite with wing-shape are uniquely determined and denoted by $K_r(l, m)$, where a', b', c' and d' are corresponding vertices of $K_r(l, m)$ to a, b, c and d.

In Ref. [9] the author presents the flattening methods for some polyhedra by use of the following lemma.

Lemma 1. [9] *Let $K = abcd$ be any kite and r be a point on ac. For any two folded kites with wing-shape $K_r(l, m)$ and $K_r(l', m')$, there exists a continuous folding process from $K_r(l, m)$ to $K_r(l', m')$.*

Lemma 1 is very useful and will play an important role in the proof of an Outside-Outside case in Lemma 2.

4. Proof of Theorem 1

In this section, we show one of our main results by use of the following lemma.

Lemma 2. *An α zig-zag belt B_α with two trapezoids can be continuously flattened with the area of moving creases being proportional to square of the width so that the two zig-zag sides remain rigid and translate only in the z direction, and moreover, each face is squashed to the direction of the α-side for the face.*

Proof. Assume that two tetragons $aegc$ and $acwu$ are two adjacent trapezoids forming a dihedral angle α with the xy-plane in $\mathbb{R}^3$, two edges ea and au are parallel to the xy-plane and two edges gc and cw are on the xy-plane. Then it is sufficient to prove for the three cases: Inside-Inside, Outside-Inside and Outside-Outside. Also let $\angle eau = \theta$, $0° < \theta < 180°$, in $\mathbb{R}^3$. Here the two adjacent trapezoids with θ are called a (α, θ) zig-zag belt.

Now we consider the flattening so that the zig-zag sides remain rigid and translate only in the z direction.

At first the creases parallel to the xy-plane are considered. Since the two zig-zag sides can translate only in the z direction and each face forms a dihedral angle α with the xy-plane, it is shown that there must exist creases bf and bv parallel to the xy-plane so that

$$|ab| : |bc| = |ef| : |fg| = |uv| : |vw| = (1 - \cos\alpha) : (1 + \cos\alpha), \tag{7}$$

where b, f and v are on ac, eg and uw, respectively.

Now, we give the flattening motion for each case.

Case of Outside-Inside

Let a (α, θ) zig-zag belt with two faces be formed by two trapezoids $aegc$ and $acwu$ with $\angle acg < \angle acw$ as in **Fig. 3** (a) and (b). Then, by Eg. (7), we can deside the creases parallel to zig-zag sides as in Fig. 3 (b) so that $|ab| : |bc| = |ef| : |fg| = |uv| : |vw| = 1 - \cos\alpha : 1 + \cos\alpha$. Also we can decide the point h on bf and the area $\triangle ach$ of moving creases so that

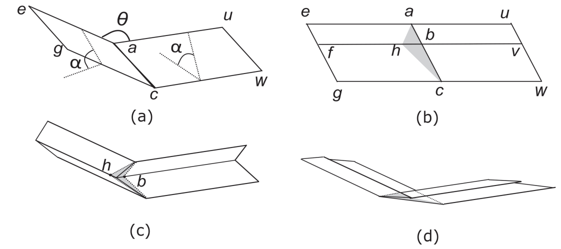

Fig. 3 (a) An Outside-Inside (α, θ) zig-zag belt in $\mathbb{R}^3$, (b) an unfolding of the zig-zag belt, (c) a moving motion, (d) a multilayered flat folded state, where the region of moving creases is shown in gray triangles.

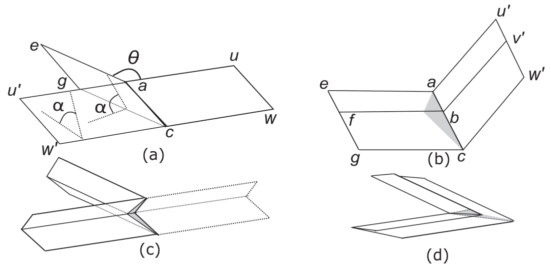

Fig. 4 (a) An Inside-Inside (α, θ) zig-zag belt in $\mathbb{R}^3$, (b) an unfolding of the zig-zag belt, (c) a moving motion, (d) a multilayered flat folded state, where the region of moving creases is shown in gray triangles.

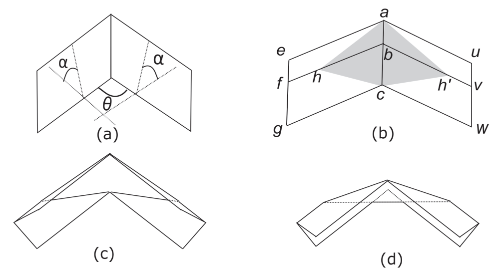

Fig. 5 (a) An Outside-Outside (α, θ) zig-zag belt in $\mathbb{R}^3$, (b) an unfolding of the zig-zag belt, (c) a moving motion without two triangles $\triangle ahc$ and $\triangle ah'c$, (d) a multilayered flat folded state, where the region of moving creases is shown in gray triangles.

$$\angle ach = \frac{\theta + \angle acg - \angle acw}{2}. \tag{8}$$

Then there exists a moving motion with moving creases at and ct, where t is a moving point on the bh; see Fig. 3 (c). Hence it follows that any Outside-Inside zig-zag belt can be continuously flattened.

Case of Inside-Inside

In Fig. 3 (a), let u', v' and w' be symmetrical points with u, v and w about a, b and c, respectively, as in **Fig. 4** (a) and (b). Then two trapezoids $aegc$ and $acw'u'$ form the $(\alpha, 180° - \theta)$ zig-zag belt of the Inside-Inside type with the area $\triangle ach$ of moving creases satisfying Eg. (8). By folding the trapezoid $acw'u'$ similar to the trapezoid $acwu$ in the case of Outside-Inside, it follows that any Inside-Inside zig-zag belt can be continuously flattened; see Fig. 4 (c) and (d).

Case of Outside-Outside

We can decide the creases bf and bv satisfying Eg. (7). On the other hand, the region of moving creases can be decided as in **Fig. 5** (b) so that

$$\angle ach = \frac{2\angle acg + \theta}{4}, \ \angle ach' = \frac{2\angle acw + \theta}{4}. \tag{9}$$

Then we can flatten the two pentagons $aegch$ and $auwch'$ so that

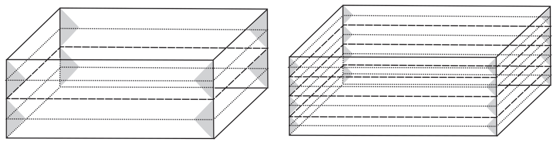

Fig. 6 (a) A box with one slice, (b) a box with three slice, where the region of moving creases are shown in gray triangles.

two zig-zag sides eau and gcw remain rigid and each face is squashed to a direction of the α-side for the face; see Fig. 5 (c) and (d). Also it follows that $|hh'|$ in (b) is greater than $|hh'|$ in (a), (c) and equal to $|hh'|$ in (d). Hence, by Lemma 1 with $K = ahch'$ and a point r being a foot of a perpendicular from h to ac, two adjacent triangles $\triangle ach$ and $\triangle ach'$ can be continuously flattened from (a) to (d). Thus it follows that any Outside-Outside zig-zag belt can be continuously flattened so that the area of moving creases is $\triangle ach + \triangle ach'$ at most.

By Egs. (4), (5) and (6), we can decide the point h on the line bf for any case. If the point h is not on the face or global interactions occur in the flattening motion, then the zig-zag belt can be sliced thin enough by the planes parallel to the xy-plane so that the point h of each part is on the face and global interactions do not occur. Since the shape of the triangles ach and ach' depends on $\angle acg$, α and θ, the area of moving creases, $\triangle ach$ and $\triangle ach + \triangle ach'$, is proportional to square of the width. Thus the proof is complete. □

Lemma 2 is very useful for continuous flattening of some types of polyhedra.

Next the proof of Theorem 1 is presented.

Proof of Theorem 1. For any α-trapezoidal polyhedron, all of side faces form some α zig-zag belts. Also we can slice each of the α zig-zag belts into α zig-zag belts with sufficiently small width by planes orthogonal to the z axis. Since, in each α zig-zag belt, the creases parallel to the xy-plane appear in the same height on each face by Eg. (7), the α zig-zag belt can be continuously flattened by Lemma 2.

When each two adjacent faces of original α zig-zag belts with the area $\triangle ach$ or $\triangle ach + \triangle ach'$ of moving creases is sliced into n parts, the sum of the area of moving creases is $\triangle ach \times \frac{1}{n^2} \times n$ or $(\triangle ach + \triangle ach') \times \frac{1}{n^2} \times n$. Hence it is shown that there exists a continuous flattening motion whose area of moving creases is arbitrarily small. □

For example, an orthogonal polyhedron can be continuously flattened by creases and moving creases as shown in **Fig. 6**.

5. Proof of Theorem 2

For any n-gonal pyramid, its top part is not an α zig-zag belt, even if we slice the pyramids into many parts. However, it is known that there exists a continuous flattening motion of n-gonal pyramids as follows.

Lemma 3. [1] *For any convex n-gonal pyramid with $n \geq 3$, there exists a continuous flattening so that the n-gonal base have no crease.*

Note that Lemma 3 gives the continuous flattening motion whose moving creases cover all side faces and the area is not small. On the other hand, Theorem 2 shown by use of Lemmas 2 and 3 gives the continuous flattening motion whose area of moving creases is arbitrarily small as follows.

Proof of Theorem 2. Any n-gonal pyramid with a circumscribed base and a top vertex being just above the incenter of the base is convex. Also each side face of the pyramid forms a dihedral angle α with the base. Let the pyramid be equally sliced by planes orthogonal to the z axis. Then the top part can be flattened by Lemma 3. On the other hand, since other parts compose α zig-zag belts, each part can be flattened by Lemma 2. When the top part is sufficiently small, the area of moving creases on the top part is also sufficiently small. Since we can slice the n-gonal pyramid into sufficiently small parts, the proof is complete. □

6. Concluding Remarks

We finally note that Lemma 2 can be applied to more general polyhedra. At first we provide the following result.

Theorem 3. *For any regular antiprism, there exists a continuous flattening motion whose area of moving creases is arbitrarily small.*

Proof. Any regular antiprism contains the closed side faces consisting of similar triangles, and also the triangles form a dihedral angle α, $0° < \alpha < 90°$, with the bottom face. We call the belt formed trapezoids or triangles an *α zig-zag quasi-belt*. Furthermore, in the α zig-zag quasi-belt on a regular antiprism, any two adjacent faces are type of Outside-Inside, where the type is same meaning as in Section 2.

Let the α zig-zag quasi-belt be sliced equally two parts, and u_iu_{i+1} and w_iw_{i+1}, $0 \leq i \leq n-1$, be edges of the top n-gonal face and the bottom n-gonal face, respectively. Then triangles and trapezoids appear alternately on each part. Also let v_i, v'_i be the intersections as shown in **Fig. 7**.

Then, for the bottom face, the α-side of each $\triangle v'_iw_{i+1}v_{i+1}$, $0 \leq i \leq n-1$, is the inside of the antiprism. On the other hand, for the top face, the α-side of each $\triangle v_iu_iv'_i$, $0 \leq i \leq n-1$, is also the inside of the antiprism. Hence the two parts can be continuously flattened by same manner of a case of Outside-Inside in Lemma 2 so that triangles $\triangle v'_iw_{i+1}v_{i+1}$ and $\triangle v_iu_iv'_i$ have no moving crease.

Furthermore, we can slice the α zig-zag quasi-belt into parts with sufficiently small width, and continuously flatten each α zig-zag quasi-belt by choosing sides for the Outside-Inside type consistently so that triangles $\triangle v'_iw_{i+1}v_{i+1}$ and $\triangle v_iu_iv'_i$ have no moving crease. Thus the proof is complete. □

For example, a triangular antiprism can be continuously flattened by creases and moving creases as shown in **Fig. 8**. Note that no triangle face of any α zig-zag quasi-belt have moving crease.

Now, Platonic solids are considered.

Lemma 4. *A regular dodecahedron can be partitioned into two regular truncated pentagonal pyramids, each with no base and an α zig-zag belt. A regular icosahedron can be partitioned into two regular pentagonal pyramids, each with no base and an α zig-zag belt.*

Proof. Let the bottom face of a regular dodecahedron be on the xy-plane and both a top vertex and a bottom vertex of a regular icosahedron be on the z axis. Also let S be a set of all planes parallel to the xy-plane and containing some vertices for each polyhedron. Then it follows that each polyhedron can be partitioned

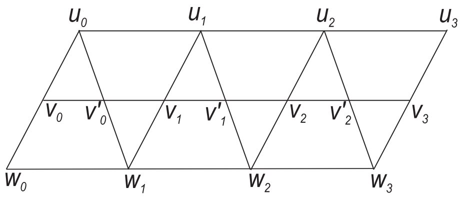

Fig. 7 A development of α zig-zag belt of an antiprism.

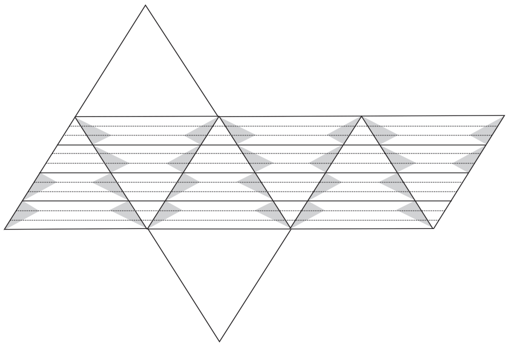

Fig. 8 A development of a triangular antiprism, where the region of moving creases are shown in gray triangles.

by planes in S into the required parts. □

Finally, we present the continuous flattening motion of Platonic solids as follows.

Theorem 4. *For any Platonic solid, there exists a continuous flattening motion whose area of moving creases is arbitrarily small.*

Proof. A cube, a regular tetrahedron and a regular octahedron can be continuously flattened by Theorems 1, 2 and 3, respectively. On the other hand, by Lemma 4, a regular dodecahedron and a regular icosahedron can be partitioned into some truncated regular pyramids with no base, some regular pyramids with no base and some α zig-zag belts on regular antiprisms. The regular truncated pyramids, the regular pyramids and the α zig-zag belts can be continuously flattened by the same methods in the proofs of Theorems 1, 2 and 3, respectively, so that top edges and bottom edges remain rigid in each part. Hence, it follows that a regular dodecahedron and a regular icosahedron can be continuously flattened. Furthermore, in each part, there exists a continuous flattening motion whose area of moving creases is arbitrarily small. Since the number of parts is finite, the sum of the area of moving creases is arbitrarily small. □

The method given in this paper will be applied to non α-trapezoidal polyhedra, for example Archimedean polyhedra. This will be discussed in a forthcoming paper.

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