Technical Note

Balancing Colored Points on a Line by Exchanging Intervals

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Abstract: Assume that $2a$ red points, $2b$ blue points and $2c$ green points lie on a line, and they are bisected into a left part I and a right part J by a point so that each of them contains $a + b + c$ points. Then we show that there exist a point set $X \subset I$ and a point set $Y \subset J$ such that both X and Y consist of consecutive points, $|X| = |Y|$, and each of $I - X + Y$ and $J - Y + X$ contains exactly a red points, b blue points and c green points. Moreover we extend this result to multi-colored point sets.

Keywords: red and blue points, three colored points, colored points on a line, balanced set, moment curve, ham-sandwich theorem.

1. Introduction

Various topics on red and blue points in the plane have been studied [1], and results on colored points on a line play important role in the proofs of some theorems [1], [2]. In this paper, we consider some problems of 3-colored points and multi-colored points on a line.

Assume that colored $2n$ points lie on a line and they are bisected into a left part I and a right part J by a point so that both I and J contain precisely n points each. If the number of points of each color is even and both I and J contain the same number of points of each color, then we say that I and J are *balanced*.

In this paper, we shall prove the following three theorems, which say that the above I and J can be balanced by exchanging two subsets $X \subset I$ and $Y \subset J$ consisting of a small number of intervals of $I \cup J$. Moreover, their proofs give polynomial-time algorithms for finding such subsets X and Y.

Note that if X and Y are disjoint sets, we often write $X + Y$ for $X \cup Y$. Moreover, if Z is a subset of X, we often write $X - Z$ for $X \setminus Z$.

Theorem 1 Assume that $2a$ red points, $2b$ blue points and $2c$ green points lie on a line, where a, b, c are positive integers, and they are bisected into a left part I and a right part J by a point so that each of them contains precisely $a + b + c$ points. Then there exist a point set $X \subset I$ and a point set $Y \subset J$ such that both X and Y consist of consecutive points, $|X| = |Y|$, and both $I - X + Y$ and $J - Y + X$ are balanced (see **Fig. 1**).

If two colored points lie on a line, we can obtain a slightly stronger result as follows:

Theorem 2 Assume that $2a$ red points and $2b$ blue points lie on a line, where a and b are positive integers, and they are bisected into the left part I and the right part J by a point so that each of them contains precisely $a + b$ points. Then there exist a point set $X \subset I$ and a point set $Y \subset J$ such that both X and Y consist of consecutive points starting at the right end-point of I and J respectively, $|X| = |Y|$, and both $I - X + Y$ and $J - Y + X$ are balanced (see **Fig. 2**).

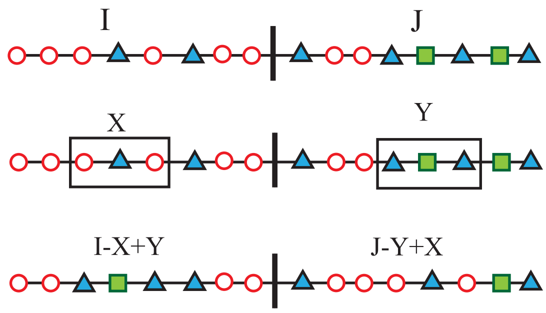

Fig. 1 Red, blue and green points lying on a line, two point sets $X \subset I$ and $Y \subset J$, and two balanced sets $I - X + Y$ and $J - Y + X$.

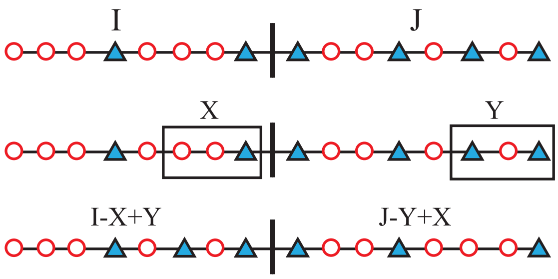

Fig. 2 Red and blue points lying on a line, two point sets $X \subset I$ and $Y \subset J$, and two balanced sets $I - X + Y$ and $J - Y + X$.

If the number of colors is more than three, we can obtain balanced sets by exchanging two or more intervals of $I \cup J$.

Theorem 3 Let $r \geq 2$ be an integer, and let $c_1, c_2, \ldots, c_r$ be r colors. Assume that $2n_i$ points of color c_i lie on a line for every $1 \leq i \leq r$. Furthermore they are bisected into a left part I and a right part J by a point so that each of them contains precisely

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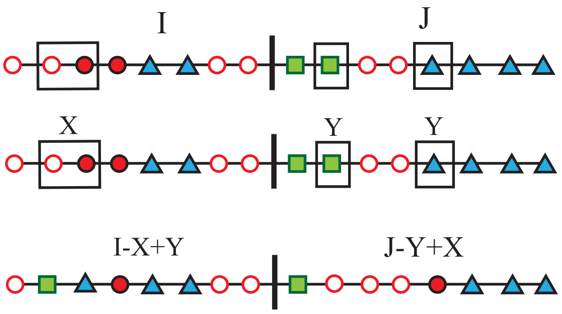

Fig. 3 Four colored points lying on a line, two point sets $X \subset I$ and $Y \subset J$, and two balanced sets $I - X + Y$ and $J - Y + X$.

$n_1 + n_2 + \ldots + n_r$ points. Then there exist point sets $X \subset I$ and $Y \subset J$ such that X and Y consist of at most $\lfloor (r+2)/2 \rfloor$ intervals together, $|X| = |Y|$, and both $I - X + Y$ and $J - Y + X$ are balanced (see **Fig. 3**). Moreover, the bound $\lfloor (r+2)/2 \rfloor$ is sharp.

2. Proofs of Theorems

For a positive integer d, we denote by $\mathbf{R}^d$ the d-dimensional Euclidean space. Note that $\mathbf{R}^1$ is often written $\mathbf{R}$. For a positive number α, the curve $\{(t, t^2, \ldots, t^d) : 0 \le t \le \alpha\}$ in $\mathbf{R}^d$ is called *the moment curve*. The moment curve has the following property.

Lemma 4 (Lemma 1.6.4 of Ref. [3]) Every hyperplane of $\mathbf{R}^d$ intersects the moment curve in $\mathbf{R}^d$ at most d points.

The next theorem is well known.

Theorem 5 (Ham-sandwich theorem [3], [4]) d point sets in $\mathbf{R}^d$ each of which contains even number of points can be simultaneously bisected by a hyperplane. Moreover, there is a polynomial-time algorithm for finding such a hyperplane.

We are ready to prove theorems. We first prove Theorem 1.

Proof of Theorem 1. We may assume that the points of $I \cup J$ are contained in the interval $[0, 1]$ of $\mathbf{R}$. We replace the consecutive points of $I \cup J$ along the moment curve $\gamma = \{(t, t^2, t^3) : 0 \le t \le 1\}$ in the space $\mathbf{R}^3$. By Ham-sandwich theorem, there exists a plane h in the space that simultaneously bisects these three colored points and intersects the moment curve γ in at most three points.

First assume that the hyperplane h intersects γ at three points. Let P_1, P_2, P_3, P_4 denote the four sets of colored points on γ divided by h. By symmetry of I and J, we may assume that $P_1 \cup P_2 \cup Q = I$ and $R \cup P_4 = J$, where $Q = P_3 \cap I$ and $R = P_3 \cap J$ and it may occur that Q or R is an empty set. Moreover, each of $P_1 \cup P_3$ and $P_2 \cup P_4$ contains exactly a red points, b blue points and c green points since h is a bisector.

Let $X = P_2$ and $Y = R$. Then $I - X + Y = P_1 \cup P_3$ and $J - Y + X = P_2 \cup P_4$ are balanced. This implies $|I| - |X| + |Y| = |J| - |Y| + |X|$, and thus $|X| = |Y|$.

We next consider the case where the plane h intersects γ at two points. Let P_1, P_2, P_3 denote the three sets of points on γ divided by h. By symmetry, we may assume that $P_1 \cup Q = I$ and $R \cup P_3 = J$, where $Q = P_2 \cap I$ and $R = P_2 \cap J$. Moreover, each of $P_1 \cup P_3$ and P_2 contains exactly a red points, b blue points and c green points since h is a bisector. Then $I - Q + P_3 = P_1 \cup P_3$ and $J - P_3 + Q = P_2$ are balanced, and this also implies $|Q| = |P_3|$. If the plane h intersects γ at one point, then I and J are balanced, and so the theorem holds for $X = Y = \emptyset$. Consequently Theorem 1 is proved. □

Proof of Theorem 2. We may assume that the points of $I \cup J$ are contained in the interval $[0, 1]$ of $\mathbf{R}$. We place the consecutive points of $I \cup J$ along the the moment curve $\gamma = \{(t, t^2) : 0 \le t \le 1\}$ in the plane $\mathbf{R}^2$. By Ham-sandwich theorem, there exists a line ℓ that simultaneously bisects these red and blue points and intersects γ at most two points. If ℓ intersects γ at one point, then I and J are balanced. Thus we may assume that ℓ and γ intersect at two points.

Let P_1, P_2, P_3 denote the three sets of colored points on γ divided by ℓ. By symmetry, we may assume that $P_1 \cup Q = I$ and $R \cup P_3 = J$, where $Q = P_2 \cap I \neq \emptyset$ and $R = P_2 \cap J \neq \emptyset$. Moreover, each of $P_1 \cup P_3$ and P_2 contains exactly a red points and b blue points since ℓ is a bisector. Hence it follows that $I - Q + P_3 = P_1 \cup P_3$ and $J - P_3 + Q = P_2$ are balanced, and $|Q| = |P_3|$. Furthermore, Q and P_3 contain the right end-points of I and J, respectively. Consequently Theorem 2 is proved. □

Proof of Theorem 3. We may assume that the points of $I \cup J$ are contained in the interval $[0, 1]$ of $\mathbf{R}$. We place the consecutive points of $I \cup J$ along the moment curve $\gamma = \{(t, t^2, \ldots, t^r) : 0 \le t \le 1\}$ in $\mathbf{R}^r$. By Ham-sandwich theorem, there exists a hyperplane h that simultaneously bisects these r colored points and intersects γ at most r points.

Let $P_1, P_2, \ldots, P_s$ denote s sets of colored points on γ divided by h, where h intersects γ at $s - 1$ points and $2 \le s \le r + 1$. If $s = 2$, then $P_1 = I$ and $P_2 = J$ are balanced, and so we may assume $3 \le s \le r+1$. Since h is a bisector, $P_1 \cup P_3 \cup \ldots \cup P_s$ *(or P_{s-1})* and $P_2 \cup P_4 \cup \ldots \cup P_{s-1}$ *(or P_s)* are balanced. In particular, they contain the same number of points of every color. Moreover, we can write $I = P_1 \cup P_2 \cup \ldots \cup P_{t-1} \cup Q$ and $J = R \cup P_{t+1} \cup \ldots \cup P_s$, where $Q = I \cap P_t$, $R = J \cap P_t$, $2 \le t < s$ and it may occur that Q or R is an empty set.

We first assume that t is even. Let $X = P_2 \cup P_4 \cup \cdots \cup P_{t-2} \cup Q$ and $Y = P_{t+1} \cup P_{t+3} \cup \cdots \cup P_s$ *(or P_{s-1})*. Then $I - X + Y = P_1 \cup P_3 \cup \cdots \cup P_s$ *(or P_{s-1})* and $J - Y + X = P_2 \cup P_4 \cup \cdots \cup P_{s-1}$ *(or P_s)* are balanced. This implies $|I| - |X| + |Y| = |J| - |Y| + |X|$ and thus $|X| = |Y|$. Moreover, X and Y consists of

$$\frac{t}{2} + \frac{s-(t+1)}{2} + 1 = \frac{s+1}{2} \quad \text{or}$$
$$\frac{t}{2} + \frac{s-1-(t+1)}{2} + 1 = \frac{s}{2}$$

intervals together. Then X and Y consists of at most $\lfloor (r+2)/2 \rfloor$ intervals by $s \le r + 1$.

Next assume that t is odd. Let $X = P_2 \cup P_4 \cup \cdots \cup P_{t-1}$ and $Y = R \cup P_{t+2} \cup P_{t+4} \cup \cdots \cup P_s$ *(or P_{s-1})*. Then $I - X + Y = P_1 \cup P_3 \cup \cdots \cup P_s$ *(or P_{s-1})*, and $J - Y + X = P_2 \cup P_4 \cup \cdots \cup P_{s-1}$ *(or P_s)* are balanced. This implies $|I| - |X| + |Y| = |J| - |Y| + |X|$ and thus $|X| = |Y|$. Moreover, X and Y consists of

$$\frac{t-1}{2} + \frac{s-t}{2} + 1 = \frac{s+1}{2} \quad \text{or}$$
$$\frac{t-1}{2} + \frac{s-1-t}{2} + 1 = \frac{s}{2}$$

intervals together. Then X and Y consists of at most $\lfloor (r+2)/2 \rfloor$ intervals by $s \le r + 1$. Consequently the existence of X and Y in Theorem 3 is proved.

We finally show that the bound $\lfloor (r+2)/2 \rfloor$ is sharp. Consider the following colored point configuration on a line, in which the

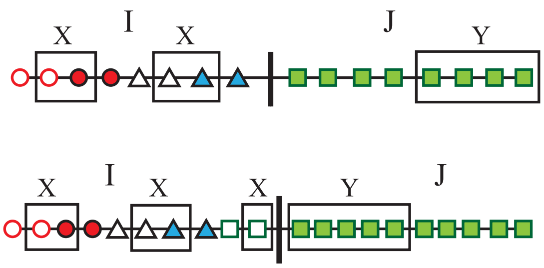

Fig. 4 Five colored points lie on a line, and X and Y consists of 3 intervals together. Six colored points lie on a line, and X and Y consists of 4 intervals together.

number of points of each color c_i, $1 \le i \le r-1$, is two and the number of points of color c_r is $2(r-1)$. The left part I contains all the points of color c_i, $1 \le i \le r-1$, and two points of the same color appear consecutively, and $2(r-1)$ points of color c_r lie on the right part J. Then in order to obtain balanced sets, we have to exchange one point of each color c_i, $1 \le i \le r-1$, of I and $r-1$ points of color c_r of J. Therefore, if r is odd, then X and Y consists of at least $(r-1)/2+1 = (r+1)/2$ intervals of $I \cup J$ (see the first example of **Fig. 4**). If r is even, then X and Y consists of at least $((r-2)/2+1)+1 = (r+2)/2$ intervals together (see the second example of Fig. 4). Hence the bound $\lfloor (r+2)/2 \rfloor$ is sharp. □

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