Electronic Preprint for Journal of Information Processing Vol.25

Regular Paper

On the Rectilinear Local Crossing Number of $K_{m,n}$

Bernardo M. Ábrego[1,a)] Kory Dondzila[1,b)] Silvia Fernández-Merchant[1,c)]
Evgeniya Lagoda[1,†1,d)] Seyed Sajjadi[1,e)] Yakov Sapozhnikov[1,f)]

Received: November 7, 2016, Accepted: May 16, 2017

Abstract: We bound the rectilinear local crossing number of the complete bipartite graph $K_{m,n}$ for every m and n, and completely determine its value when $\min(m, n) \leq 4$.

Keywords: local crossing number, crossing number, bipartite graph

1. Introduction

In this work, we consider rectilinear drawings of the complete bipartite graph $K_{m,n}$, that is, drawings whose vertices are m red and n blue points in the plane, and whose edges are straight line segments joining all pairs of points with different colors. We also assume that any two of these edges share at most one point.

In general, the local crossing number of a graph G was defined by Ringel as follows (see Guy et al. [6], Kainen [7], and Schaefer [12]). The *local crossing number* of a drawing D of a graph G, denoted $\mathrm{lcr}(D)$, is the largest number of crossings on any edge of D. The *local crossing number* of G, denoted $\mathrm{lcr}(G)$, is the minimum of $\mathrm{lcr}(D)$ over all drawings D of G. This is also known as the *cross-index* (Thomassen [15]). The analogous definition for rectilinear drawings is the *rectilinear local crossing number* of G, denoted $\overline{\mathrm{lcr}}(G)$, which is the minimum of $\mathrm{lcr}(D)$ over all rectilinear drawings D of G. Recently, Ábrego and Fernández-Merchant [1] completely determined $\overline{\mathrm{lcr}}(K_n)$ using a separation lemma (See Lemma 2 in Ref. [1]).

The *crossing number* of a graph G, denoted by $\mathrm{cr}(G)$, is the smallest number of crossings among all drawings of G. When this minimum is restricted to rectilinear drawings, we obtain the *rectilinear crossing number* of G, denoted by $\overline{\mathrm{cr}}(G)$. Crossing number problems originated in the 1940s with Turán (see Ref. [3] for more on the history of the brick factory problem) and have been widely studied since then [2], [5], [12]. Crossing numbers of complete graphs are of particular importance because bounds on these numbers give bounds on the crossing number of any graph by using random embeddings [4], [10], [13], [14]. Leighton [10] uses this to bound the VLSI layout area of a graph, while Shahrokhi, Sýkora, Székely, and Vrťo [14] use it to find approximation algorithms of crossing numbers for dense graphs.

The value of $\overline{\mathrm{cr}}(G)$ can be used to bound $\overline{\mathrm{lcr}}(G)$ (as done in Ref. [6] for drawings of K_n on the torus). Namely, adding the number of crossings of every edge over all edges of a graph G counts precisely twice the number of crossings of G. It follows that

$$\overline{\mathrm{lcr}}(K_{m,n}) \geq \frac{2\,\overline{\mathrm{cr}}(K_{m,n})}{mn}.$$

The Zarankiewicz Conjecture (Paul Turán, 1944), states that $\overline{\mathrm{cr}}(K_{m,n}) = \mathrm{cr}(K_{m,n}) = Z(m,n) := \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$, but this has only been proved when $\min(m,n) \leq 6$ [8], and for $m = 7$ or 8 and $n \leq 10$ [16]. The current best published lower bound on $\mathrm{cr}(K_{m,n})$ is $0.86\,Z(m,n)$ by de Klerk et al. [9] and recently, Norine and Zwols [11] announced the lower bound $0.905\,Z(m,n)$, but this has not been published. This would yield

$$\mathrm{lcr}(K_{m,n}) \geq \frac{0.905}{8} mn + o(mn) = 0.113125\, mn + o(mn).$$

If the Zarankiewicz Conjecture were true, we would have

$$\overline{\mathrm{lcr}}(K_{m,n}) \geq \frac{1}{8} mn + o(mn).$$

Turan's drawing of $K_{m,n}$ with $Z(m,n)$ crossings (see **Fig. 1**) has local crossing number $\left(\left\lceil \frac{m}{2} \right\rceil - 1\right)\left(\left\lceil \frac{n}{2} \right\rceil - 1\right)$ showing that

$$\overline{\mathrm{lcr}}(K_{m,n}) \leq \frac{1}{4} mn + o(mn).$$

Clearly, $\overline{\mathrm{lcr}}(K_{2,n})$=0 as Turan's construction for $m = 2$ (Fig. 1) has no crossings. In Section 2, we determine $\overline{\mathrm{lcr}}(K_{m,n})$ for $m = 3$ and 4. More precisely, for any integer $n \geq 2$,

$$\overline{\mathrm{lcr}}(K_{3,n}) = \left\lceil \frac{n-2}{4} \right\rceil \quad \text{and} \quad \overline{\mathrm{lcr}}(K_{4,n}) = \left\lceil \frac{n-2}{2} \right\rceil.$$

In Section 3, we present constructions that improve the upper bound to

$$\overline{\mathrm{lcr}}(K_{m,n}) \leq \frac{3}{14}(m-1)(n-1)$$

for any $m \geq 5$ and $n \geq 5$. Further improvements on this bound are presented for small values of m and any n.

1 California State University, Northridge, CA 91330, USA
†1 Presently with Berlin Mathematical School, Berlin 10623, Germany
a) bernardo.abrego@csun.edu
b) kory.dondzila.981@my.csun.edu
c) silvia.fernandez@csun.edu.Supported by the NSF grant DMS-1400653
d) lagoda@math.tu-berlin.de
e) seyed.sajjadi.947@my.csun.edu
f) yakov.sapozhnikov.473@my.csun.edu

2. Exact Results

We determine $\overline{\mathrm{lcr}}(K_{m,n})$ for $m = 3$ and 4. The following observation will be useful for the presentation of our proofs. Let P be a set of points. We say that $\triangle xyz$ is *P-empty* if the interior of $\triangle xyz$ does not contain points in P.

Observation 1. *For any segment xy and any set of points P, there is a point $z \in P$ such that $\triangle xyz$ is P-empty.*

In fact, z could be a point in P closest to the line xy. We now proceed to prove the two results of this section.

Theorem 2. *For any integer $n \geq 3$,*

$$\overline{\mathrm{lcr}}(K_{3,n}) = \left\lceil \frac{n-2}{4} \right\rceil.$$

Proof. We first prove that $\overline{\mathrm{lcr}}(K_{3,n}) \geq \lceil \frac{n-2}{4} \rceil$. Consider any drawing D of $K_{3,n}$. As usual, the two vertex-classes R and B are colored red and blue. So R has 3 red points, which we label a, b, and c; B has n blue points; and every red point is joined to every blue point by a straight line-segment. We prove that D must have an edge that is crossed at least $\lceil \frac{n-2}{4} \rceil$ times. We consider several cases according to how the blue points are distributed among the regions determined by the red points. In each case, we identify 2 or 4 edges that must be crossed a combined total of at least $\frac{n-2}{2}$ or $n - 2$ times, respectively.

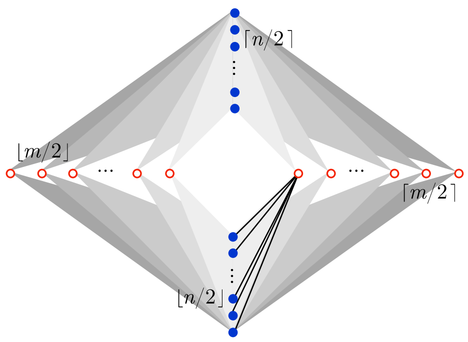

Fig. 1 Zarankiewicz drawing of $K_{m,n}$ with $Z(m,n)$ crossings.

Case 1. The points a, b, and c are collinear. Let l be the line through a, b, and c and suppose that b is between a and c on l. Consider a blue point on each side of l, say x and y, such that $\triangle acx$ and $\triangle acy$ are B-empty (See **Fig. 2** (a)). Then any edge of the form zb, where z is a blue point, crosses one of the edges ax, xc, ay, or yc. So one of these four edges must be crossed at least $\lceil \frac{n-2}{4} \rceil$ times. If all blue points are on the same side of l, then the $\lceil \frac{n-2}{4} \rceil$ can be improved to $\lceil \frac{n-1}{2} \rceil$.

Case 2. The points a, b, and c are noncollinear. In this case, a, b, and c are the vertices of a triangle. The lines ab, bc, and ca divide the plane into 7 regions as shown in Fig. 2 (b). We partition the set of blue points into two parts: The set B_1 of blue points in $R_a \cup R_b \cup R_c$, and the set B_2 of all other blue points. Let $n_1 = |B_1|$ and $n_2 = |B_2|$ so that $n = n_1 + n_2$.

We first look at the points in B_1. If B_1 is nonempty, consider any point $x \in B_1$, say in R_a, closest to $\triangle abc$ in the sense that no other points in B_1 are in the interior of the quadrilateral $xbac$ (See Fig. 2 (c)). Then for any blue point $z \in B_1$, the edge za crosses either xb or xc. Thus one of the edges xb or xc is crossed at least $\lceil \frac{n_1-1}{2} \rceil$ times.

Now look at the points in B_2. If B_2 is nonempty, we have three subcases.

Case 2.1 At least two of the regions R_{ab}, R_{bc}, and R_{ac}, say R_{ab} and R_{ac}, have blue points. Let $x \in R_{ab}$ be a blue point such that there are no other blue points in the intersection of R_{ab} and the sector acx (See **Fig. 3** (a)). Similarly, let $y \in R_{ac}$ be a blue point such that there are no other blue points in the intersection of R_{ac} and the sector yba. Then any point $z \in B_2 \setminus \{x, y\}$ creates at least one crossing with xc or yb. Namely, if $z \in R_{ab}$ then zc crosses yb; if $z \in R_{ac}$ then zb crosses xc; and if $z \in R_{bc}$ then za crosses both

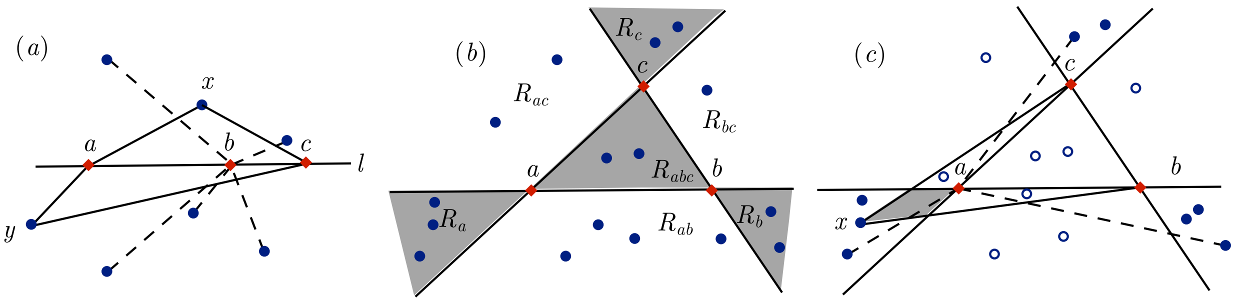

Fig. 2 (a) The red points a, b, and c are collinear. (b) The regions determined by the red points when they are in general position. (c) The blue points in B_1 are solid. The blue points in B_2 are hollow. The shaded region is empty.

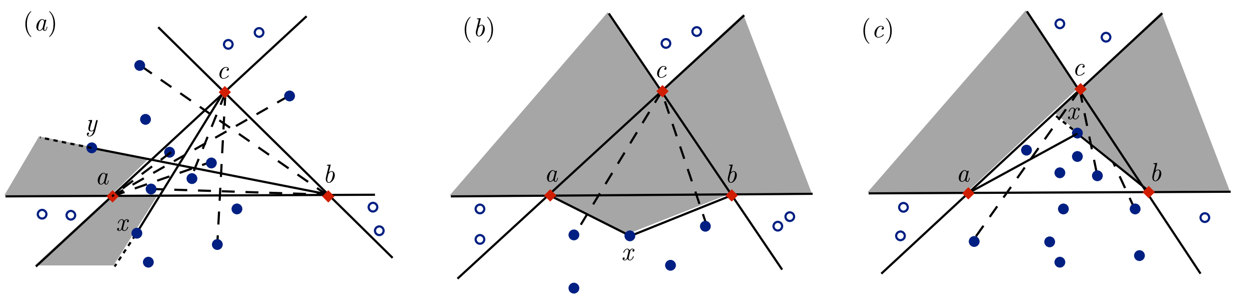

Fig. 3 All shaded regions are B-empty.

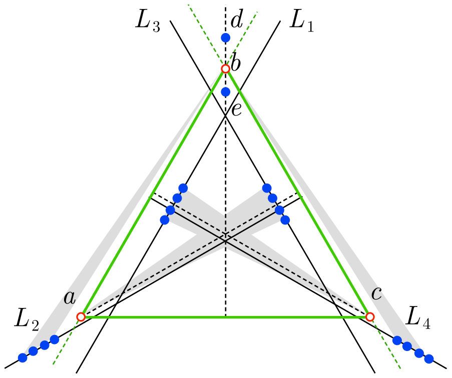

Fig. 4 An optimal construction for $\overline{\mathrm{lcr}}(K_{3,n})$.

xc and yb. Also, xc and yb cross each other, adding one more crossing to each xc and yb. Thus one of the edges xc or yb is crossed at least $\lceil \frac{n_2-2}{2} \rceil + 1 = \lceil \frac{n_2}{2} \rceil$ times.

Case 2.2 Exactly one of the regions R_{ab}, R_{bc}, and R_{ac}, say R_{ab}, has blue points but R_{abc} does not. Let $x \in R_{ab}$ be a blue point such that $\triangle axb$ is B_2-empty (See Fig. 3 (b)). Then for any point $z \in B_2 \setminus \{x\}$, the edge zc crosses xa or xb. Thus one of the edges xa or xb is crossed at least $\lceil \frac{n_2-1}{2} \rceil$ times.

Case 2.3 At most one of the regions R_{ab}, R_{bc}, and R_{ac} has blue points and R_{abc} has blue points. Without loss of generality suppose that R_{bc}, and R_{ac} have no blue points. Let $x \in R_{abc}$ be a blue point such that there are no other blue points in the sector cbx (See Fig. 3 (c)). Then for any point $z \in B_2 \setminus \{x\}$ the edge zc or zb crosses xa or xb, which means that one of the edges xa or xb is crossed at least $\lceil \frac{n_2-1}{2} \rceil$ times.

All cases imply that there is an edge crossed at least $\max\left(\left\lceil \frac{n_1-1}{2} \right\rceil, \left\lceil \frac{n_2-1}{2} \right\rceil\right) \geq \left\lceil \frac{\lceil n/2 \rceil -1}{2} \right\rceil = \left\lceil \frac{n-2}{4} \right\rceil$, and therefore $\overline{\mathrm{lcr}}(D) \geq \lceil \frac{n-2}{4} \rceil$ for any drawing D of $K_{3,n}$. Hence, $\overline{\mathrm{lcr}}(K_{3,n}) \geq \lceil \frac{n-2}{4} \rceil$.

We now prove that $\overline{\mathrm{lcr}}(K_{3,n}) \leq \lceil \frac{n-2}{4} \rceil$ by presenting a drawing of $K_{3,n}$ with local crossing number $\lceil \frac{n-2}{4} \rceil$, see **Fig. 4**. Let $k = \lceil \frac{n-2}{4} \rceil$. The red points are the vertices of an equilateral triangle and are labeled a, b, and c. The lines L_1 and L_2 are right below and parallel to ab and the perpendicular bisector to bc, respectively. The lines L_3 and L_4 are the reflections of L_1 and L_2 about the perpendicular bisector to ac. There are two special blue points: d above b and e below b, both above the lines L_1 and L_3. The rest of the blue points are (almost) evenly distributed among L_1, L_2, L_3, and L_4; then some lines (at least one) have k blue points on them and the rest $k-1$. The blue points on L_1 and L_3 should be below the intersection of L_1 and L_3 such that the edges from these points to c pass below all blue points on L_3; and the edges from the blue points on L_3 to a pass below all blue points on L_1. The points on L_2 and L_4 should be outside of $\triangle abc$, below ac, and so that the edges from these points to b do not cross $\triangle abc$. Note that none of the edges crosses the sides of $\triangle abc$. So exterior edges do not cross interior edges. For L_1 and L_2, the edges to c make no crossings with the edges to a and b. A similar situation applies to L_3 and L_4 by symmetry.

First look at the interior edges. Let p be the i^{th} blue point on L_1 from top to bottom. Then there are at most $k-i$ blue points on L_1 below p and $i-1$ above. The edges from b to the points below p cross the edge pa, giving $k-i \leq k-1 < k$ crossings with pa. The edges from a to e and to the points above p cross the edge pb, giving $i \leq k$ crossings with pb. The edges from a to the points on L_3 cross the edge pc, giving at most k crossings with pc. Note that no other edges cross pa, pb, or pc. A similar argument applies for the i^{th} point on L_3. The edge ea is crossed only by the edges from b to points on L_1, giving at most k crossings with ea. Symmetrically, ec is crossed at most k times. Finally, the edge eb is not crossed, completing the proof that any interior edge is crossed at most k times.

Now look at the exterior edges. Take the i^{th} point q on L_2 from top to bottom. Then there are $i-1$ points on L_2 above q and at most $k-i$ below. The edges from b to the points above q cross the edge qa, giving $i-1 \leq k$ crossings with qa. The edges from a to the points below q and the edge da cross the edge qb, giving at most $k-i+1 \leq k$ crossings with qb. The edges from a to the points on L_4 cross the edge qc, giving at most k crossings with qc. Note that no other edges cross qa, qb, or qc. A similar argument applies for the i^{th} point on L_4. The edge da is crossed only by the edges from b to points on L_2, giving at most k crossings with ea. Symmetrically, dc is crossed at most k times. Finally, the edge db is not crossed, completing the proof that any exterior edge is crossed at most k times.

Note that if there are k points on L_1, L_2, L_3, or L_4, then the edge ea, da, ec, or dc, respectively, is crossed exactly k times. □

Theorem 3. *For any integer $n \geq 4$,*

$$\overline{\mathrm{lcr}}(K_{4,n}) = \left\lceil \frac{n-2}{2} \right\rceil.$$

Proof. The Turán's construction of $K_{4,n}$ for $m = 4$ and any n has local crossing number $\left\lceil \frac{n-2}{2} \right\rceil$ (see Fig. 1), proving that $\overline{\mathrm{lcr}}(K_{4,n}) \leq \left\lceil \frac{n-2}{2} \right\rceil$.

We now prove that $\overline{\mathrm{lcr}}(K_{4,n}) \geq \lceil \frac{n-2}{2} \rceil$. Consider any drawing D of $K_{4,n}$. As usual, the two vertex-classes R and B are colored red and blue. So R has 4 red points, B has n blue points, and every red point is joined to every blue point by a straight line-segment. By simplicity, we assume that R is in general position (Otherwise, a small enough perturbation of the points in R to achieve general position would not affect the local crossing number of D). We prove that D must have an edge that is crossed at least $\lceil \frac{n-2}{2} \rceil$ times. Let C be the vertex set of the convex hull of $R \cup B$. We consider several cases according to the number of blue and red points on C. We say that a simple curve is *C-connecting* if both its endpoints are in C. So a segment with endpoints in C or a simple path of D with endpoint in C are called C-connecting segment or path, respectively. If ℓ is a C-connecting curve, then removing ℓ from the convex hull of $R \cup B$ results in two disjoint connected sets S_1 and S_2 called the *sides* of ℓ. We say that two points are *separated* by ℓ, if they are in different sides of ℓ. So we say that *ℓ separates red points* if both S_1 and S_2 contain at least one red point.

Lemma 4. *If there is a C-connecting curve contained in two edges of D and that separates red points, then $\overline{\mathrm{lcr}}(R \cup B) \geq \lceil \frac{n-2}{2} \rceil$.*

Proof. Suppose that the curve separates the red points x and y and is contained in the edges s and t. Let B_x and B_y be the sets of blue points on the same side of the curve as x and y, respectively. Then xz and yw cross the path for any $z \in B_y$ and $w \in B_x$. That is, there are at least $|B_x \cup B_y| \geq n-2$ edges crossing the curve. Each

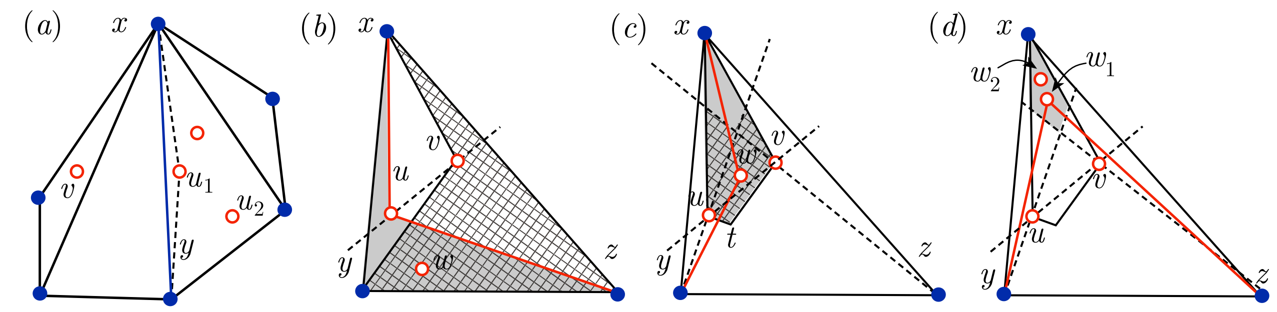

Fig. 5 No red points in C.

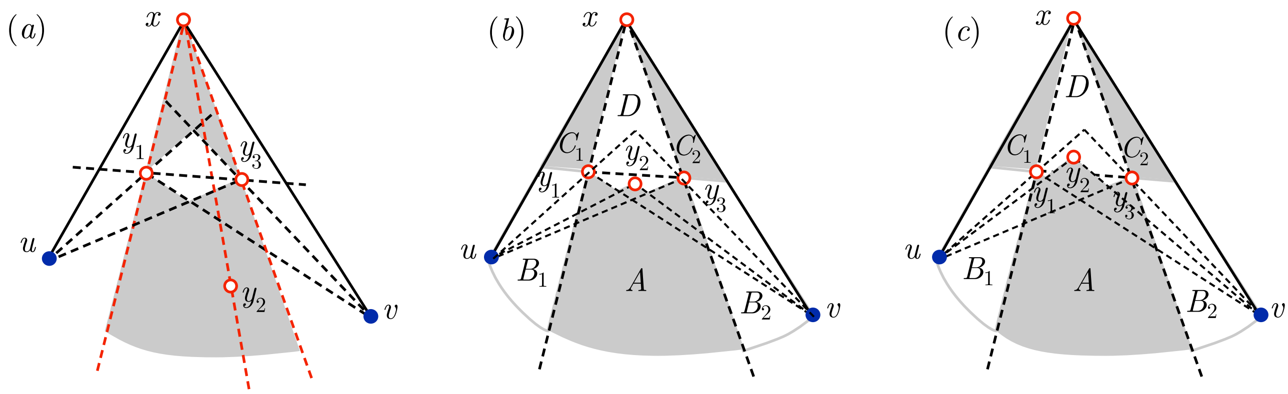

Fig. 6 Exactly one red point in C.

of these segments crosses s or t (or even both), so s or t is crossed at least $\lceil\frac{n-2}{2}\rceil$ times. Thus $\overline{\text{lcr}}(R \cup B) \geq \lceil\frac{n-2}{2}\rceil$. □

Corollary 5. *If there is a C-connecting 2-path of D that separates red points, then* $\overline{\text{lcr}}(R \cup B) \geq \lceil\frac{n-2}{2}\rceil$.

Corollary 6. *If there is a C-connecting segment with blue endpoints that separates red points, then* $\overline{\text{lcr}}(R \cup B) \geq \lceil\frac{n-2}{2}\rceil$.

Proof. If xy is a C-connecting segment with blue endpoints ($x \in B$ and $y \in B$) that separates red points (see **Fig. 5** (a)), then there are at least 2 red points u_1 and u_2 on one of its sides and at least one red point v on the other. By Observation 1, we can assume that $\triangle xyu_1$ does not contain u_2. Then the path xu_1y separates u_2 and v. The result holds by Corollary 5. □

We now return to the proof of Theorem 3.

Case 1. Assume that there are no red points on C. In this case, we can assume that R is contained in a triangle with blue vertices x, y, and z in C, otherwise there is a C-connecting segment with blue end points that separates red points and the result holds by Corollary 6 (See Fig. 5 (a)). Assume that yz is horizontal, y to the left of z, and x is above yz. Let u and v be two red points and assume that the line uv crosses the segments xy and xz, u to the left of v as in Fig. 5 (b)–(d). If quadrilateral $uxyz$ contains a red point w, then the C-connecting path xuz separates the red points v and w and the result holds by Corollary 5 (See Fig. 5 (b)). The same holds for the quadrilateral $vyzx$. So we assume that all red points are in the quadrilateral $xutv$, where t is the point of intersection of yv and zu. If there is a red point w below the line yu, then the C-connecting path xwy separates u and v and the result holds by Corollary 5 (see Fig. 5 (c)). The same holds for the line zv. So we assume that the remaining 2 red points are above the lines yu and zv. By Observation 1, one of these two points, call it w_1, satisfies that $\triangle w_1yz$ does not contain the other, call it w_2 (see Fig. 5 (d)). Then the path yw_1z separates w_2 from u and v and the result holds by Corollary 5.

Case 2. Assume that there are one or two red points on C. Then there are at least two red points *not* on C.

Lemma 7. *If ℓ is a line through two red points not in C and there are blue points in C on both sides of ℓ, then* $\overline{\text{lcr}}(R \cup B) \geq \lceil\frac{n-2}{2}\rceil$.

Proof. Let x be any red point on C. Assume that ℓ is horizontal and passes through the red points y and z not in C with y to the left of z. Suppose that u and v are blue points in C with u below ℓ and v above. If x is to the right of $\overrightarrow{uv}$, then y and x are separated by the path uzv. If x is to the left of $\overrightarrow{uv}$, then x and z are separated by the path uyv. In either case the result follows by Corollary 5. □

Case 2.1 Assume that x is the only red point in C. Let u and v be two blue points in C such that u, x, and v are consecutive (in clockwise order) along C. Label the three remaining red points y_1, y_2, and y_3 according to the order in which the ray $\overrightarrow{xu}$ finds them when continuously rotated around x counterclockwise. By Lemma 7, we can assume that the line y_1y_3 does not cross the segment uv (i.e., y_1y_3 crosses the segments xu and xv, see **Fig. 6** (a)). If y_2 is below vy_1 or uy_3, then y_2 and x are separated by the path uy_1v or uy_3v, respectively. If y_2 is above the line uy_1 or vy_3, then the path uy_2v separates x from y_1 or y_3, respectively. In either case, the result holds by Corollary 5. Assume then that y_2 is above vy_1 and uy_3 and below the lines uy_1 and vy_3.

Define the regions A, B_1, B_2, C_1, C_2, and D as in Fig. 6 (b) and Fig. 6 (c). So A is the region between rays xy_1 and xy_2, below the path uy_2v, and below vy_1 or uy_3; D is the region between rays xy_1 and xy_2 and above region A; B_1 and C_1 are the regions below and above the line uv, respectively, and to the right of ray xy_1; and B_2 and C_2 are the regions below and above the line uv, respectively, and to the left of ray xy_3. If there is a blue point w in region A,

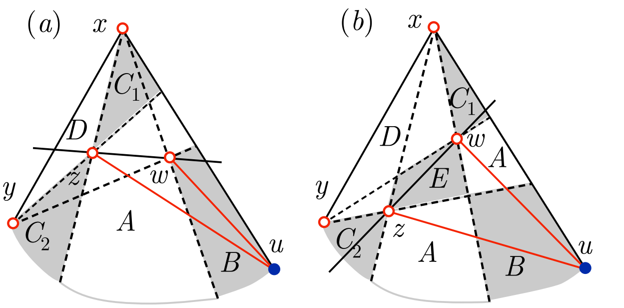

Fig. 7 Exactly two red points in C. Note that region E is empty in (a) and the region A is disconnected in (b).

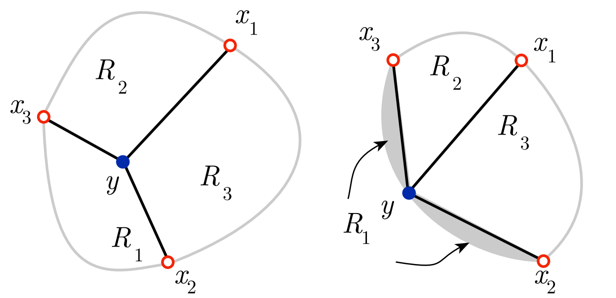

Fig. 8 At least 3 red points on C. Note that when y is on the convex hull C, one of the regions R_1, R_2, or R_3 is disconnected.

then xw, crosses one of the edges vy_1, uy_2, vy_2 or uy_3. Then the union of xw and such an edge contains a simple curve that separates y_1 and y_3 and the result holds by Lemma 4. Then we can assume that there are no blue points in region A.

If y_2 is below the line y_1y_3 (Fig. 6 (b)), then the edges from each blue point in B_1 to y_2 and y_3 cross the edge vy_1, and the edges from each blue point in B_2 to y_1 and y_2 cross the edge uy_3. If y_2 is above the line y_1y_3 (Fig. 6 (c)), then the edges from each blue point in B_1 to y_3 and the edges from each blue point in b_2 to y_1 cross the edge vy_2. Thus, if there are at least $\frac{n-2}{2}$ blue points in $B_1 \cup B_2$, then one of the edges vy_1, uy_3, or vy_2 is crossed at least $\lceil \frac{n-2}{2} \rceil$ times. Then we can assume that there are less than $\frac{n-2}{2}$ blue points in $B_1 \cup B_2$, that is, there are at least $\frac{n-2}{2}$ blue points in $C_1 \cup C_2 \cup D$.

Rotate the ray $\overrightarrow{xu}$ counterclockwise around x until finding the first blue point w in $C_1 \cup C_2 \cup D$. If $w \in C_1$, then the edge wy_3 is crossed at least $\lceil \frac{n-2}{2} \rceil$ times. This is because the edge from any blue point in $C_1 \cup C_2 \cup D$ to either x or y_1 crosses wy_3. A similar argument holds for region C_2. So we now assume that both C_1 and C_2 do not contain blue points, that is, there are at least $\lceil \frac{n-2}{2} \rceil$ blue points in D.

This time rotate the ray $\overrightarrow{y_1v}$ counterclockwise around y_1 until finding the first blue point w in D. If w is below the line y_1y_3, then the edge xw is crossed at least $\lceil \frac{n-2}{2} \rceil$ times. This is because the edge from any blue point in D to either y_1 or y_3 crosses xw. Assume that all the blue points in D are above the line y_1y_3.

Rotate the ray $\overrightarrow{y_1x}$ clockwise around y_1 until finding the first blue point w in D. Then one of the edges wy_2 or y_2v is crossed at least $\lceil \frac{n-2}{2} \rceil$ times. This is because the edge from any blue point in $D \cup B_1 \cup B_2$ to either y_1 or y_3 crosses the path wy_2y_3.

Case 2.2. Assume that there are exactly two red points on C. If these two points, x and y are not consecutive along C, then they are separated by a C-connecting segment with blue endpoints and the result holds by Corollary 6. Assume that u, x, and y are consecutive in counterclockwise order along C, so u is a blue point. Label the other two red points w and z. If the line wz separates two blue points in C, then Lemma 7 implies the result. Then we can assume that either the line wz crosses the segment xy and one of the neighboring convex hull edges (we can assume it crosses xu, **Fig. 7** (a)), or the line wz crosses both edges in the convex hull neighboring xy (Fig. 7 (b)). Consider the regions A, B, C_1, C_2, D, and E as shown in Fig. 7. If there is a blue point t in region A, then the path xty separates w and z and the result holds by Corollary 5. Assume that there are no blue points in region A. If there are at least $\frac{n-2}{2}$ blue points in region B, then one of the edges uw or uz is crossed at least $\lceil \frac{n-2}{2} \rceil$ times. This is because for any blue point t in B, the edges tx and ty determine 2 crossings with uw and uz. Assume then that there are less than $\frac{n-2}{2}$ blue points in B, that is, there are at least $\frac{n-2}{2}$ blue points in $C_1 \cup C_2 \cup D \cup E$.

Rotate the ray $\overrightarrow{xu}$ clockwise around x until finding the first blue point t in $C_1 \cup C_2 \cup D \cup E$. If $t \in C_1$, then the edge ty is crossed at least $\lceil \frac{n-2}{2} \rceil$ times. This is because the edge from any blue point in $C_1 \cup C_2 \cup D \cup E$ to either x or z crosses ty. A similar argument holds for region C_2. So we now assume that both C_1 and C_2 do not contain blue points, that is, there are at least $\lceil \frac{n-2}{2} \rceil$ blue points in $D \cup E$.

When the line wz crosses the segment xy (Fig. 7 (a)), rotate the ray $\overrightarrow{yx}$ clockwise around y until finding the first blue point t in D. Then the path tzu is crossed at least $n-2$ times (the edge from any blue point, other than u or t, to y or w crosses the path tzu), and so one of the edges tz or zu is crossed at least $\lceil \frac{n-2}{2} \rceil$ times.

Now consider the case when the line wz and the segment xy do not cross (Fig. 7 (b)). If the interior of $\triangle yxw$ contains blue points, then rotate the ray $\overrightarrow{yx}$ clockwise around y until finding the first blue point t in D. Then t is in the interior of $\triangle yxw$ and so the edge tz is crossed at least $\lceil \frac{n-2}{2} \rceil$ times (the edge from any blue point in $D \cup E$, other than t, to y or w crosses the edge tz). Similarly, if the interior of $\triangle xyz$ contains blue points, then rotate the ray $\overrightarrow{xy}$ counterclockwise around x until finding the first blue point t' in D. Then t' is in the interior of $\triangle xyz$ and so the edge $t'w$ is crossed at least $\lceil \frac{n-2}{2} \rceil$ times (the edge from any blue point in $D \cup E$, other than t, to x or z crosses the edge tw). Thus we can assume that there are no blue points in region D. This means that there are at least $\frac{n-2}{2}$ blue points in E. Let p be the intersection of the lines xw and yz. This time rotate the ray $\overrightarrow{wp}$ clockwise around w until finding the first blue point t in E. Then the edge xt is crossed at least $\lceil \frac{n-2}{2} \rceil$ times (the edge from any blue point in E, other than t, to y or w crosses the edge tx), concluding the proof.

Case 3. Assume that there are at least three red points on C. Let x_1, x_2, and x_3 be three red points in clockwise order along C and let y be any blue point. The rays $\overrightarrow{yx_1}, \overrightarrow{yx_2}$, and $\overrightarrow{yx_3}$ partition the plane into three regions R_1, R_2, and R_3 (see **Fig. 8**). The fourth red point z is in one of these regions. If z is in R_i, then the path x_jyx_k separates x_i and z, where $\{i, j, k\} = \{1, 2, 3\}$ and the result holds by Corollary 5. □

3. General Upper Bounds

Let $P = R \cup B$ be a set of red (R) and blue (B) points in the

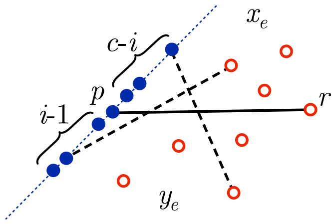

Fig. 9 The c blue points shown here are replacing a blue point $b \in B$, p is the i^{th} point along s_b. There are $x_e + y_e + 1$ red points on the same side of s_b as r.

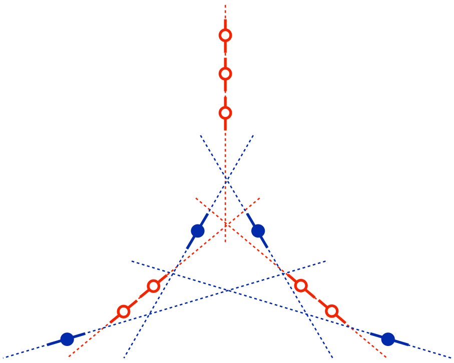

Fig. 10 Initial construction for Theorem 10.

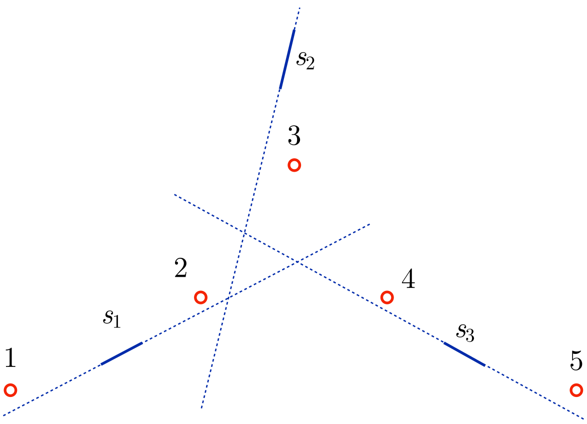

Fig. 11 Initial construction for $m = 5$. The n blue points are almost equally distributed among the three blue segments.

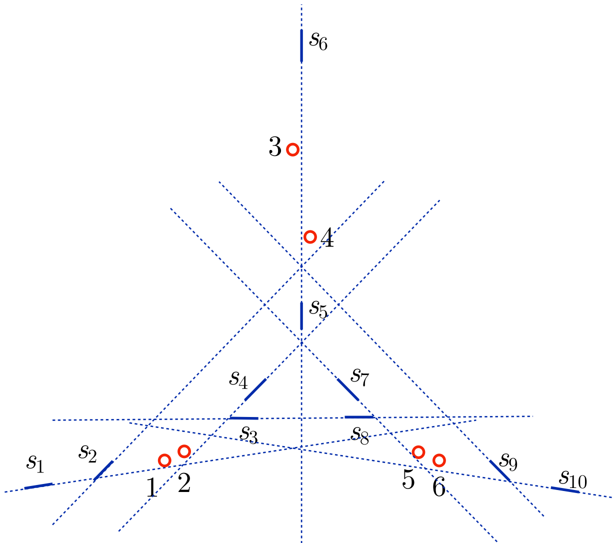

Fig. 12 Initial construction for $m = 6$. The n blue points are almost equally distributed among the ten blue segments.

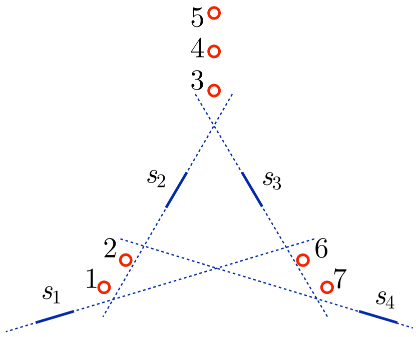

Fig. 13 Initial construction for $m = 7$. The n blue points are almost equally distributed among the four blue segments.

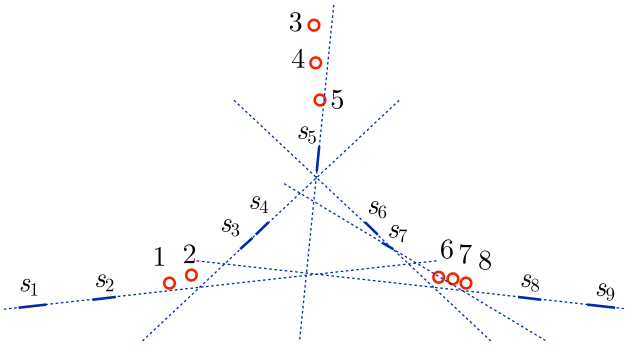

Fig. 14 Initial construction for $m = 8$. The n blue points are almost equally distributed among the nine blue segments.

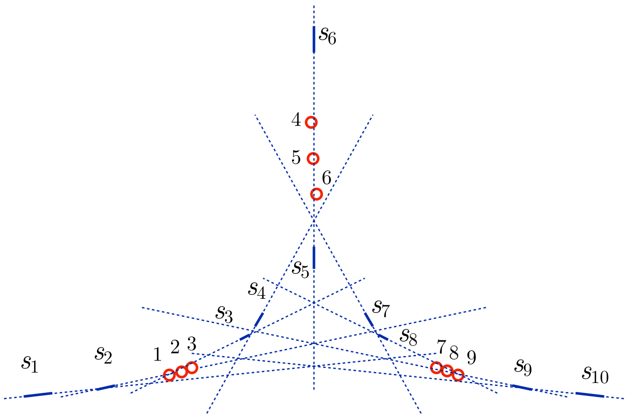

Fig. 15 Initial construction for $m = 9$. The n blue points are almost equally distributed among the ten blue segments.

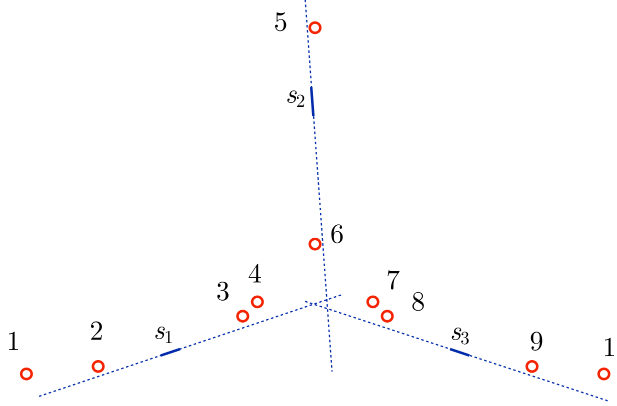

Fig. 16 Initial construction for $m = 10$. The n blue points are almost equally distributed among the three blue segments.

plane. For each blue point $b \in B$, let s_b be a segment centered at b and small enough so that the triangle formed by s_b and any red point $r \in R$ does not contain any points of P in its interior or boundary, except for b and r. Let $S = \{s_b : b \in B\}$. We say that S is a set of valid blue segments for P (We can analogously define a set of valid red segments for P). For any positive integer c, denote by $P_{S,c}$ the set of points obtained from P by replacing each blue point $b \in B$ by c blue points evenly placed along s_b.

Lemma 8. *Let D be a drawing of $K_{m,n}$ with vertex-classes R and B colored red and blue. Let $P = R \cup B$, S a set of valid blue segments for P, and c a positive integer. Let $D_{S,c}$ be the drawing of $K_{m,cn}$ with vertex set $P_{S,c}$. For each edge $e = rb$ (with $r \in R$ and $b \in B$), let x_e and y_e be the number of red points on each side of e and on the same side of s_b as r. Then*

$$\overline{\mathrm{lcr}}(D_{S,c}) \leq \max_{e\ \text{edge of}\ D} (c \cdot \mathrm{cr}(e) + (c-1)\max(x_e, y_e)),$$

where $\mathrm{cr}(e)$ is the number of edges crossing e in D.

Table 1 Upper bounds for $\overline{\mathrm{lcr}}(K_{m,n})$ for $5 \leq m \leq 10$.

m	5	6	7	8	9	10
$\overline{\mathrm{lcr}}(K_{m,n}) \leq$	$\frac{2}{3}n + O(1)$	$\frac{4}{5}n + O(1)$	$n + O(1)$	$\frac{4}{3}n + O(1)$	$\frac{7}{5}n + O(1)$	$\frac{5}{3}n + O(1)$

Table 2 Upper bounds on $\overline{\mathrm{lcr}}(K_{m,n})$ for $3 \leq m \leq 10$.

Initial set for m | Value of $\mathrm{cr}(e)$ from a base point on s_i to the point j | Value of $\max(x_e, y_e)$ from a base point on s_i to the point j

$m = 5$, Fig. 11

$\mathrm{cr}(e)$	1	2	3	4	5
s_1	0	0	1	1	1
s_2	0	1	0	1	0
s_3	1	1	1	0	0

$\max(x_e, y_e)$	1	2	3	4	5
s_1	1	2	1	2	1
s_2	2	1	1	1	2
s_3	1	1	2	1	2

$m = 6$, Fig. 12

$\mathrm{cr}(e)$	1	2	3	4	5	6
s_1	3	6	2	6	7	4
s_2	0	5	5	7	6	5
s_3	5	0	6	6	5	6
s_4	6	3	5	3	4	7
s_5	7	6	4	0	6	7
s_6	6	7	0	4	7	6
s_7	7	4	6	2	3	6
s_8	6	5	7	5	0	5
s_9	5	6	6	6	5	0
s_{10}	4	7	3	5	6	3

$\max(x_e, y_e)$	1	2	3	4	5	6
s_1	3	2	3	2	1	1
s_2	2	3	1	1	2	3
s_3	3	2	1	1	3	2
s_4	2	3	2	3	1	1
s_5	1	2	2	2	2	1
s_6	2	1	2	2	1	2
s_7	1	1	2	3	3	2
s_8	2	3	1	1	2	3
s_9	3	2	1	1	3	2
s_{10}	1	1	3	2	2	3

$m = 7$, Fig. 13

$\mathrm{cr}(e)$	1	2	3	4	5	6	7
s_1	0	1	2	1	0	3	2
s_2	1	0	0	1	2	2	3
s_3	3	2	0	1	2	0	1
s_4	2	3	2	1	0	1	0

$\max(x_e, y_e)$	1	2	3	4	5	6	7
s_1	4	3	2	3	4	1	1
s_2	3	4	4	3	2	1	1
s_3	1	1	4	3	2	4	3
s_4	1	1	2	3	4	3	4

$m = 8$, Fig. 14

$\mathrm{cr}(e)$	1	2	3	4	5	6	7	8
s_1	4	6	0	4	8	10	7	4
s_2	0	4	4	6	8	8	7	6
s_3	4	0	8	8	8	6	7	8
s_4	6	4	8	6	4	4	7	10
s_5	8	8	8	4	0	10	10	10
s_6	9	6	9	6	3	5	7	9
s_7	8	7	10	9	8	0	4	8
s_8	7	8	5	6	7	8	4	0
s_9	6	9	0	3	6	9	7	5

$\max(x_e, y_e)$	1	2	3	4	5	6	7	8
s_1	4	3	4	3	2	2	1	2
s_2	4	3	4	3	2	2	1	2
s_3	3	4	2	3	4	2	1	2
s_4	3	4	2	3	4	2	1	2
s_5	3	4	2	3	4	2	1	2
s_6	1	1	3	4	5	5	4	3
s_7	1	2	2	3	4	2	4	3
s_8	1	1	5	4	3	3	4	5
s_9	1	1	5	4	3	3	4	5

$m = 9$, Fig. 15

$\mathrm{cr}(e)$	1	2	3	4	5	6	7	8	9
s_1	5	8	11	3	7	11	12	9	6
s_2	0	5	10	8	10	12	10	9	8
s_3	10	5	0	10	10	10	8	9	10
s_4	11	8	5	9	7	5	6	9	12
s_5	12	11	10	8	4	0	10	11	12
s_6	10	11	12	0	4	8	12	11	10
s_7	12	9	6	11	7	3	5	8	11
s_8	10	9	8	12	10	8	0	5	10
s_9	8	9	10	10	10	10	10	5	0
s_{10}	6	9	12	5	7	9	11	8	5

$\max(x_e, y_e)$	1	2	3	4	5	6	7	8	9
s_1	5	4	3	5	4	3	2	1	2
s_2	3	4	3	3	2	2	2	3	4
s_3	3	4	3	2	2	3	4	3	2
s_4	3	4	5	3	4	5	2	1	2
s_5	2	3	4	3	4	3	3	2	2
s_6	3	2	2	3	4	3	2	3	4
s_7	2	1	2	3	4	5	5	4	3
s_8	2	2	3	2	3	4	3	4	3
s_9	4	3	2	3	2	2	3	4	3
s_{10}	2	1	2	5	4	3	3	4	5

$m = 10$, Fig. 16

$\mathrm{cr}(e)$	1	2	3	4	5	6	7	8	9	10
s_1	1	0	0	1	2	2	2	2	2	2
s_2	1	2	2	1	0	0	1	2	2	1
s_3	2	2	2	2	2	2	1	0	0	1

$\max(x_e, y_e)$	1	2	3	4	5	6	7	8	9	10
s_1	4	5	5	4	3	3	3	2	2	3
s_2	3	2	2	3	5	5	4	3	3	4
s_3	3	2	2	3	3	3	4	5	5	4

Proof. Let $b \in B$ and $r \in R$. Then b is replaced by c blue points along the segment s_b. Let p be the i^{th} point on s_b. We compute $\mathrm{cr}(pr)$. Let $e = br$ and $e' = pr$. Originally, e is crossed by $\mathrm{cr}(e)$ edges of D. Each of these $\mathrm{cr}(e)$ edges is replaced by c edges in $D_{S,c}$ creating $c \cdot \mathrm{cr}(e)$ crossings with e'. The only other edges crossing e' are those joining points on s_b with red points on the opposite side on e' and on the same side of s_b as r. Say that e' separates $x(e)$ such red points from $i-1$ blue points on s_b, and y_e such red points from the remaining $c-i$ blue points on s_b as shown in **Fig. 9**. Then $\mathrm{cr}(e') = c \cdot \mathrm{cr}(e) + (i-1)x_e + (c-i)y_e \leq c\cdot\mathrm{cr}(e)+(i-1+c-i)\max(x_e, y_e) = c\cdot\mathrm{cr}(e)+(c-1)\max(x_e, y_e)$. □

As a corollary of Lemma 8, we obtain the following result.

Theorem 9. *For any positive integers m, n, and c,*

$$\overline{\mathrm{lcr}}(K_{m,cn}) \leq c\,\overline{\mathrm{lcr}}(K_{m,n}) + (c-1)\left(\left\lceil \frac{m}{2} \right\rceil - 1\right).$$

Proof. Let D be a drawing of $K_{m,n}$ with vertex-classes R and B colored red and blue, respectively, and such that $\overline{\mathrm{lcr}}(D) = \overline{\mathrm{lcr}}(K_{m,n})$. Let $P = R \cup B$. We choose a set S of blue valid segments for P such that for each $b \in B$ the line containing the segment s_b separates the red points in almost half (i.e., there are $\lfloor \frac{m}{2} \rfloor$ on one side of s_b and $\lceil \frac{m}{2} \rceil$ on the other). Then $\max(x_e, y_e) \leq \lceil \frac{m}{2} \rceil - 1$ for any edge e of D. By Lemma 8, $\overline{\mathrm{lcr}}(D_{S,c}) \leq c\,\overline{\mathrm{lcr}}(D) + (c-1)\left(\left\lceil \frac{m}{2} \right\rceil - 1\right) = c\,\overline{\mathrm{lcr}}(K_{m,n}) + (c-1)\left(\left\lceil \frac{m}{2} \right\rceil - 1\right)$. □

Theorem 10. *For any integers $m \geq 5$ and $n \geq 5$,*

$$\overline{\mathrm{lcr}}(K_{m,n}) \leq \frac{3}{14}(m-1)(n-1).$$

Proof. We first deal with the case when $7|m$ and $4|n$. Consider the set $P = R \cup B$ of 11 points together with the valid set of blue segments $S(B)$ and the valid set of red segments $S(R)$ shown in **Fig. 10**. It can be checked that for every edge e in D (the drawing of $K_{7,4}$ with vertex-classes R and B), $1 \leq \max(x_e, y_e) \leq 4$ and $\mathrm{cr}(e) \leq 4 - \max(x_e, y_e)$. Let $P' = P_{S(B),n/4}$ and $P'' = P'_{S(R),m/7}$ (Note that $S(R)$ is also a valid set of red segments for P'). By Lemma 8,

$$\begin{aligned}\overline{\mathrm{lcr}}(P') &= \max_{e \text{ edge of } D}\left(\frac{n}{4}\cdot\mathrm{cr}(e) + \left(\frac{n}{4}-1\right)\max(x_e, y_e)\right)\\ &\leq \left(\frac{n}{4}\cdot(4-\max(x_e, y_e)) + \left(\frac{n}{4}-1\right)\max(x_e, y_e)\right) \leq n-1.\end{aligned}$$

For every red point r, the line containing s_r separates the blue points of P' in half. Thus $\max(x_e, y_e) \leq 2\cdot\frac{n}{4} - 1 = \frac{n}{2} - 1$ for any edge e in D'. Using Lemma 8 again,

$$\begin{aligned}\overline{\mathrm{lcr}}(K_{m,n}) &\leq \overline{\mathrm{lcr}}(P'')\\ &= \max_{e \text{ edge of } D'}\left(\frac{m}{7}\cdot\mathrm{cr}(e) + \left(\frac{m}{7}-1\right)\max(x_e, y_e)\right)\\ &\leq \frac{m}{7}(n-1) + \left(\frac{m}{7}-1\right)\left(\frac{n}{2}-1\right)\\ &= \frac{3}{14}mn - \frac{2m}{7} - \frac{n}{2} + 1\\ &\leq \frac{3}{14}(m-1)(n-1).\end{aligned}$$

For the general case when $7|m$ and $4|n$ do not necessarily hold, let $m' = 7\lceil \frac{m}{7} \rceil \leq m+6$ and $n' = 4\lceil \frac{n}{4} \rceil \leq n+3$, which could add only a $O(m+n)$ term as

$$\begin{aligned}\overline{\mathrm{lcr}}(K_{m,n}) &\leq \overline{\mathrm{lcr}}(K_{m',n'}) \leq \frac{3}{14}m'n' - \frac{2m'}{7} - \frac{n'}{2} + 1\\ &\leq \frac{3}{14}(m-1)(n-1).\end{aligned}$$

□

The technique used in the proof of Theorem 10 can be used more carefully for specific values of m. **Figures 11–16** show initial sets for each $5 \leq m \leq 10$ that improve the upper bound in Theorem 10. These improvements are summarized in **Table 1**. The upper bounds are obtained by using Lemma 8. The values of $\mathrm{cr}(e)$ and $\max(x_e, y_e)$ for each e are included in **Table 2**.

Acknowledgments The authors thank the reviewers for their comments and suggestions that have helped improved the paper. We also thank all helpful conversations with the members of the CSUN Crossing Numbers Research Group. Silvia Fernández-Merchant is supported by the NSF grant DMS-1400653.

References

[1] Ábrego, B.M. and Fernández-Merchant, S.: The Rectilinear Local Crossing Number of K_n, arXiv:1508.07926v2.

[2] Brass, P., Moser, W. and Pach, J.: *Research Problems in Discrete Geometry*, chapter 8, Springer-Verlag, New York (2005).

[3] Beineke, L. and Wilson, R.: The early history of the brick factory problem, *Math. Intelligencer*, Vol.32, pp.41–48 (2010).

[4] Chung, F.R.K., Leighton, F.T. and Rosenberg, A.L.: Embedding graphs in books: A layout problem with applications to VLSI design, *SIAM J. Alg. Disc. Math.*, Vol.8, pp.33–58 (1987).

[5] Guy, R.K.: A combinatorial problem, *Bull. Malayan Math. Soc.*, Vol.7, pp.68–72 (1960).

[6] Guy, R.K., Jenkyns, T. and Schaer, J.: The Toroidal Crossing Number of the Complete Graph, *J. Combinatorial Theory*, Vol.4, pp.376–390 (1968).

[7] Kainen, P.C.: Thickness and Coarseness of Graphs, *Abh. Math. Sem. Univ. Hamb.*, Vol.39, pp.88–95 (1973).

[8] Kleitman, D.J.: The crossing number of $K_{5,n}$, *J. Combin. Theory*, Vol.9, pp.315–323 (1970).

[9] de Klerk, E., Maharry, J., Pasechnik, D.V., Richter, R.B. and Salazar, G.: Improved Bounds for the Crossing Numbers of $K_{m,n}$ and K_n, *SIAM J. Discrete Math.*, Vol.20, No.1, pp.189–202 (2006).

[10] Leighton, F.T.: *Complexity Issues in VLSI: Optimal Layouts for the Shuffle–Exchange Graph and Other Networks*, MIT Press (1983).

[11] Norine, S. and Zwols, Y.: Turan Brickyard Problem and Flag Algebras, Video from 13w5091: Geom. and Top. G. Theory Workshop, A WEB Page, BIRS, available from ⟨http://www.birs.ca/events/2013/5-day-workshops/13w5091/videos/watch/201310011538-Norin.html/⟩ (accessed 2013-10-01).

[12] Schaefer, M.: The Graph Crossing Number and its Variants: A Survey, *E. J. Combinatorics* DS21 (2014).

[13] Shahrokhi, F. and Shi, W.: Efficient deterministic algorithms for embedding graphs on books, *Computing and Combinatorics*, pp.162–168, Hong Kong (1996), *Lecture Notes in Comput. Sci.*, Vol.1090, Springer, Berlin (1996).

[14] Shahrokhi, F., Sýkora, O., Székely, L.A. and Vrťo, I.: Crossing numbers: Bounds and applications, *Intuitive Geometry*, pp.179–206, Budapest (1995), *Bolyai Soc. Math. Stud.*, 6, János Bolyai Mathematical Society, Budapest (1997).

[15] Thomassen, C.: Rectilinear Drawings of Graphs, *J. Graph Theory*, Vol.12, No.3, pp.335–341 (1988).

[16] Woodall, D.R.: Cyclic-order graphs and Zarankiewicz's crossing-number conjecture, *J. Graph Theory*, Vol.17, No.6, pp.657–671 (1993).

Bernardo M. Ábrego was born in 1973. He is a Professor of Mathematics at California State University, Northridge. After earning his undergraduate degree from the National Autonomous University of Mexico in 1995, he moved to the US, where he received his Ph.D. from Rutgers University in 2001. His main areas of research are Convex and Discrete Geometry, and Combinatorics. He was the Math Magazine Problems Editor from 2009 to 2015.

Kory Dondzila was born in 1990. He received his double B.F.A in Animation and Game Development from Savannah College of Art and Design in 2013. He is completing his M.S. in Computer Science at California State University Northridge. He works at Atomic Object as a Software Consultant and Developer.

Silvia Fernádez-Merchant is a Professor of Mathematics at California State University, Northridge. She was born in Mexico City in 1973, where he received her B.S. in Mathematics from the Universidad Nacional Autónoma de México (UNAM). She then moved to the United States where received her Ph.D. in Mathematics from Rutgers University in 2001. Her current research interest is in discrete geometry, combinatorics, and geometric and topological graph theory.

Evgeniya Lagoda received her B.S. degree in Mathematics from California State University, Northridge, USA, in 2016, and then moved to attend a graduate program at Berlin Mathematical School, Germany.

Seyed Sajjadi was born in 1994 in Tehran. He moved to the U.S. at age 18 to attend California State University, Northridge. He is pursuing Computer engineering with double minor in mathematics and computer science. He is currently the president and chapter chair of ACM and an active member of IEEE. His research interests are artificial intelligence and mathematical modeling.

Yakov Sapozhnikov was born in 1985. He received his M.S. in Mathematics from California State University, Northridge in 2017. His research interests are algebra, combinatorics, and number theory.