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On Contractible Edges in Convex Decompositions

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Abstract: Let Π be a convex decomposition of a set P of $n \geq 3$ points in general position in the plane. If Π consists of more than one polygon, then either Π contains a deletable edge or Π contains a contractible edge.

Keywords: convex decomposition, convex deformation, contractible edge

1. Introduction

Let P be a set of $n \geq 3$ points in general position in the plane. A *convex decomposition* of P is a set Π of convex polygons with vertices in P and pairwise disjoint interiors such that their union is the convex hull $CH(P)$ of P and that no point in P lies in the interior of any polygon in Π. A *geometric graph* with vertex set P is a graph G, drawn in the plane in such a way that every edge is a straight line segment with ends in P.

Let Π be a convex decomposition of P. We denote by $G(\Pi)$ the *skeleton graph* of Π, that is the plane geometric graph with vertex set P in which the edges are the sides of all polygons in Π. An edge e of Π is an *interior edge* if e is not an edge of the boundary of $CH(P)$.

An interior edge e of Π is *deletable* if the geometric graph $G(\Pi) - e$, obtained from $G(\Pi)$ by deleting the edge e, is the skeleton graph of a convex decomposition of P. Neumann-Lara et al. [6] proved that if a convex decomposition Π of a set P of n points consists of more that $(3n-2k)/2$ polygons, where k is the number of vertices of $CH(P)$, then Π has at least one deletable edge.

An interior edge $e = uv$ of Π is *contractible* from u to v if the geometric graph $G(\Pi)/\vec{uv} = (G(\Pi) - \{x_1u, x_2u, \ldots, x_mu, uv\}) + \{x_1v, x_2v, \ldots, x_mv\}$ is a skeleton graph of a convex decomposition of $P \setminus \{u\}$, where $x_1, x_2, \ldots, x_m$ are the remaining vertices of $G(\Pi)$ which are adjacent to u.

A *simple convex deformation* of Π is a convex decomposition Π' obtained from Π by moving a single point x along a straight line segment, together with all the edges incident with x, in such a way that at each stage we have a convex decomposition of the corresponding set of points. Deformations of plane graphs have been studied by several authors, both theoretically and algorithmically, see for instance Refs. [3], [4], [7] and [1], [2], [5], respectively.

Let P_1 and P_2 be sets of $n \geq 3$ points in general position in the plane. A convex decomposition Π_1 of P_1 and a convex decomposition Π_2 of P_2 are *isomorphic* if there is an isomorphism of $G(\Pi_1)$ onto $G(\Pi_2)$, as abstract plane graphs, such that the boundaries of $CH(P_1)$ and $CH(P_2)$ correspond to each other with the same orientation.

Thomassen [7] proved that if Π_1 and Π_2 are *isomorphic* convex decompositions, then Π_2 can be obtained from Π_1 by a finite sequence of simple convex deformations. As a tool, Thomassen proved that if Π is a convex decomposition with at least two polygons, then there is an isomorphic convex decomposition Π' that can be obtained from Π by a finite number of simple convex deformations that preserve the boundary and such that Π' contains either a deletable edge or a contractible edge. In this note we prove that every convex decomposition Π with at least two polygons contains an edge which is deletable or contractible. Furthermore, if P contains at least one interior point, then Π contains a contractible edge.

2. Preliminary Results

Let Π be a convex decomposition of P containing no deletable edges. For every interior edge e of $G(\Pi)$, the graph $G(\Pi) - e$ has an internal face Q_e which is not convex and at least one end of e is a reflex vertex of Q_e.

We define an abstract directed graph $\overrightarrow{G(\Pi)}$ with vertex set P in which $\vec{uv} \in A\left(\overrightarrow{G(\Pi)}\right)$ if and only if u is a reflex vertex of Q_{uv}. Notice that for each interior edge uv of $G(\Pi)$, the directed graph $\overrightarrow{G(\Pi)}$ contains at least one of the arcs $\vec{uv}$ and $\vec{vu}$ (see **Fig. 1**).

Remark 1.

(1) The outdegree of every vertex u of $\overrightarrow{G(\Pi)}$ is at most 3.

(2) The outdegree of every vertex u in the boundary of $CH(P)$ is 0.

(3) An interior vertex u of Π has outdegree 3 in $\overrightarrow{G(\Pi)}$ if and only if u has degree 3 in $G(\Pi)$.

(4) If $\vec{uv}, \vec{uw} \in A\left(\overrightarrow{G(\Pi)}\right)$, then uv and uw lie in a common face of $G(\Pi)$.

For two points α and β in the plane, we denote by $r(\alpha\beta)$ the ray, with origin α, that contains the segment $\alpha\beta$.

Lemma 2. *An edge uv of Π is not contractible from u to v if and only if there are edges yx and xu, lying in a common face of $G(\Pi)$*

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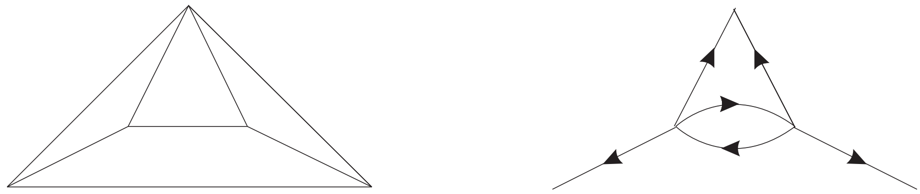

Fig. 1 A Convex partition Π and the corresponding directed graph $\overrightarrow{G(\Pi)}$.

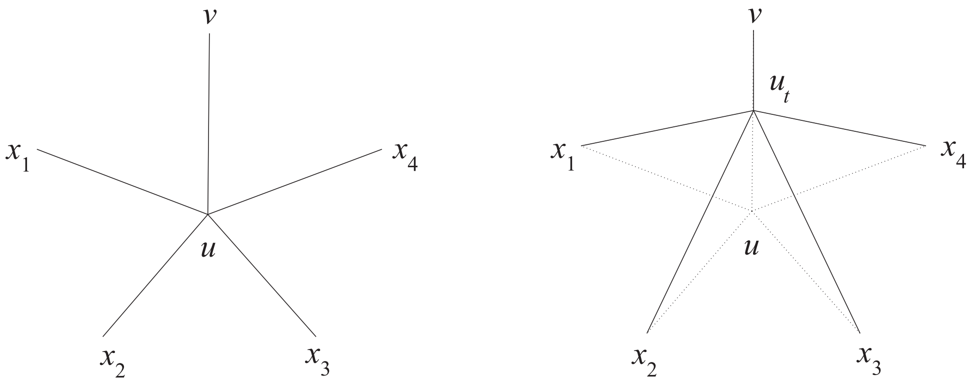

Fig. 2 Contracting an edge uv continuously.

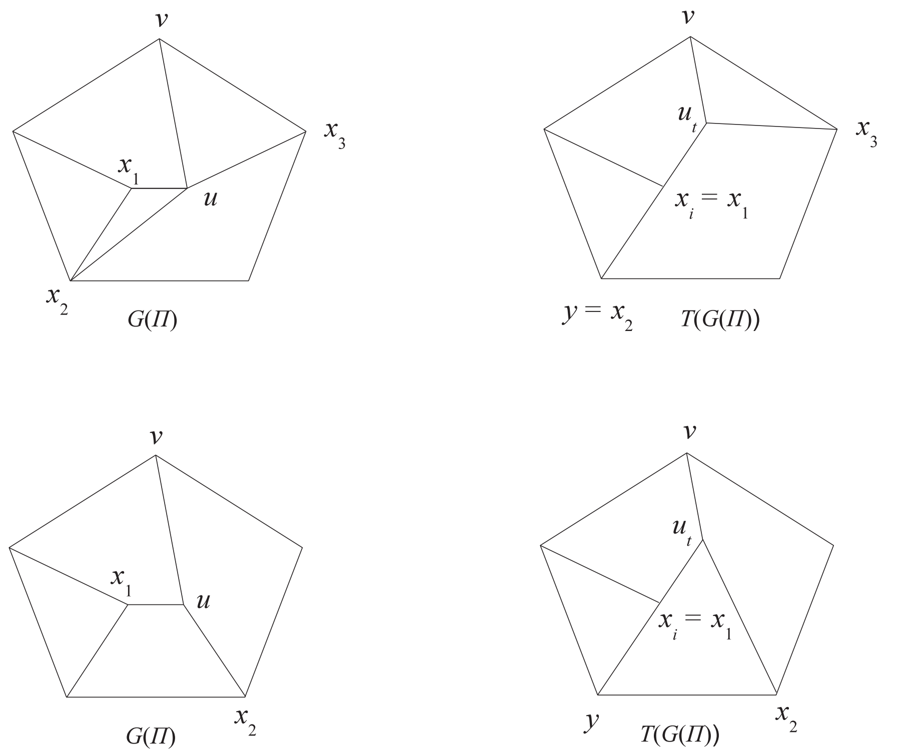

Fig. 3 Edges yx and xu_t become collinear.

that contains vertex u, such that the ray $r(yx)$ meets the edge uv at point u_t, with $u \neq u_t \neq v$, and that the triangular region defined by x, u_t and u contains no point of P in its interior.

Proof. It is easy to see that the existence of such edges yx and xu implies that the edge uv cannot be contracted from u to v. We proceed to prove the remaining part of the lemma. Let uv be an interior edge of Π with u not lying in the boundary of $CH(\Pi)$ and let $x_1, x_2, \ldots, x_m$ be the remaining vertices of $G(\Pi)$ which are adjacent to u. We contract the edge uv in a continuous way as follows: Slide the point u along the ray $r(uv)$, together with the edges $x_1u, x_2u, \ldots, x_mu$ (see **Fig. 2**).

If uv is not contractible from u to v, then either the transformed graph $T(G(\Pi))$ becomes non planar or one of its faces becomes non convex. This implies that we must reach a point $u_t = u + t(v - u)$, with $0 < t < 1$, such that there are two edges yx_i and x_iu_t lying in a common face, which become collinear in $T(G(\Pi))$ (see **Fig. 3**).

Notice that two or more different pairs of edges yx_i, x_iu_t and $y'x_j$, x_ju_t may become collinear simultaneusly; in such a case we may choose any of those pairs and proceed with the proof.

The triangular region defined by x_i, u_t and u is the region swept by the edge x_iu_s, $0 \leq s \leq t$ and therefore it contains no point of P in its interior. The lemma follows since the edges yx_i and x_iu lie in a common face of $G(\Pi)$ and the ray $r(yx_i)$ meets the edge uv at

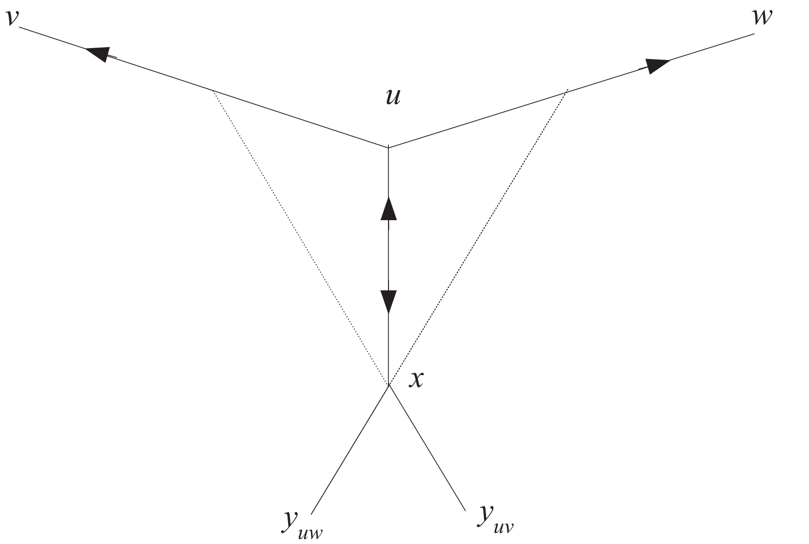

Fig. 4 $f(uv) = f(uw) = \overrightarrow{xu}$.

the point u_t. □

Let N denote the set of arcs $\overrightarrow{uv}$ of $\overrightarrow{G(\Pi)}$ such that the edge uv is not contractible from u to v in Π. For each $\overrightarrow{uv} \in N$ let $y = y_{uv}$, $x = x_{uv}$ and u_t be as in Lemma 2. Since the edges $y_{uv}x_{uv}$ and $x_{uv}u$ lie in a common face of $G(\Pi)$ and the triangular region, defined by x_{uv}, u_t and u, contains no point of P in its interior, the geometric graph $G(\Pi) - x_{uv}u$ contains a face $Q_{x_{uv}u}$ in which x_{uv} is a reflex vertex and therefore $\overrightarrow{x_{uv}u} \in A\left(\overrightarrow{G(\Pi)}\right)$. This defines a function

$$f : N \longrightarrow A\left(\overrightarrow{G(\Pi)}\right)$$

given by $f(\overrightarrow{uv}) = \overrightarrow{x_{uv}u}$.

Notice that the arcs $f(\overrightarrow{uv})$ and $\overrightarrow{uv}$ form a directed path in $\overrightarrow{G(\Pi)}$ with length 2 and middle vertex u. This implies that if $f(\overrightarrow{u_1v_1}) = f(\overrightarrow{u_2v_2})$, then $u_1 = u_2$. Moreover, if uv_1, uv_2 and uv_3 are distinct arcs such that $f(\overrightarrow{uv_1}) = f(\overrightarrow{uv_2}) = f(\overrightarrow{uv_3}) = \overrightarrow{xu}$, then u is adjacent in $G(\Pi)$ to v_1, v_2, v_3 and to x, which is not possible by Remark 1, since u has outdegree 3 in $\overrightarrow{G(\Pi)}$. It follows that there are no three arcs in N with the same image under the function f and therefore $|\mathrm{Im}(f)| = |N| - |U|$, where U is the set of points u of P for which there is a pair of arcs $\overrightarrow{uv}, \overrightarrow{uw} \in N$ such that $f(\overrightarrow{uv}) = f(\overrightarrow{uw})$.

Lemma 3. *Let Π be a convex decomposition of P with no deletable edges. If $U \neq \emptyset$, then there is a function*

$$g : U \rightarrow A\left(\overrightarrow{G(\Pi)}\right)$$

such that for each $u \in U$, $g(u)$ is not in the image of the function f.

Proof. Let $u \in U$ and let v, w and $x = x_{uv} = x_{uw}$ be points in P such that $f(\overrightarrow{uv}) = f(\overrightarrow{uw}) = \overrightarrow{xu}$. If u has degree larger than 3 in $G(\Pi)$, let $z \notin \{v, w, x\}$ be such that uz is an edge of $G(\Pi)$. By Remark 1, the outdegree of u in $\overrightarrow{G(\Pi)}$ is at most 2, therefore $\overrightarrow{uz}$ is not an arc of $\overrightarrow{G(\Pi)}$. It follows that $\overrightarrow{zu}$ must be an arc of $\overrightarrow{G(\Pi)}$. In this case $g(u) = \overrightarrow{zu} \notin \mathrm{Im}(f)$ since $z \neq x$ and $\overrightarrow{xu}$ is the unique arc in $Im(f)$ that ends at u.

If u has degree 3 in $G(\Pi)$, then u has outdegree 3 in $\overrightarrow{G(\Pi)}$, by Remark 1 and therefore $\overrightarrow{ux}$ is an arc $\overrightarrow{G(\Pi)}$. We claim that in this case $g(u) = \overrightarrow{ux} \notin \mathrm{Im}(f)$. Let l_{ux} denote the line containing the edge ux, and let y_{uv} and y_{uw} be points in P and such that the rays $r(y_{uv}x)$ and $r(y_{uw}x)$ intersect the edges uv and uw, respectively.

Without loss of generality we assume that l_{ux} is a vertical line such that v and y_{uw} lie to the left of l_{ux} and w and y_{uv} lie to the right of l_{ux} (see **Fig. 4**). Clearly the angles $\measuredangle y_{uv}xu$ and $\measuredangle y_{uw}xu$ are smaller than π, it is easy to see that $\measuredangle y_{uw}xy_{uv}$ is also smaller than π.

Therefore if xz is an edge of Π with $z \notin \{u, y_{uv}, y_{uw}\}$, then $\overrightarrow{xz}$ is not an arc of $\overrightarrow{G(\Pi)}$. This implies that if $\overrightarrow{ux} \in \mathrm{Im}(f)$, then $\overrightarrow{ux} = f(\overrightarrow{xy_{uv}})$ or $\overrightarrow{ux} = f(\overrightarrow{xy_{uw}})$ since $f(\overrightarrow{a})$ and $\overrightarrow{a}$ form a directed path of length 2 for each arc $\overrightarrow{a} \in N$.

Suppose $\overrightarrow{ux} = f(\overrightarrow{xy_{uv}})$. By the definition of f, there is an edge $y_{xy_{uv}}u$ such that the ray $r(y_{xy_{uv}}u)$ intersects the edge xy_{uv}. Since v and w are the only vertices different from x which are adjacent to u in $G(\Pi)$, one of them must be the vertex $y_{xy_{uv}}$. Since both edges uw and xy_{uv} lie in the right halfplane defined by l_{ux} then $r(wu)$ cannot intersect the edge xy_{uv} and therefore $y_{xy_{uv}} \neq w$. Finally, since $r(y_{uv}x)$ intersects the edge uv, $r(vu)$ cannot intersect the edge xy_{uv}. Therefore $\overrightarrow{ux} \neq f(\overrightarrow{xy_{uv}})$; analogously $\overrightarrow{ux} \neq f(\overrightarrow{xy_{uw}})$. □

3. Main Results

In this section we prove our main results.

Theorem 4. *Let P be a set of points in general position in the plane. If Π is a convex decomposition of P consisting of more than one polygon, then either Π contains a deletable edge or Π contains a contractible edge.*

Proof. Assume the result is false and Π contains no deletable edges and no contractible edges. Define the directed graph $\overrightarrow{G(\Pi)}$ as in the previous section, notice that $A\left(\overrightarrow{G(\Pi)}\right) \neq \emptyset$ since Π contains at least two polygons. Since Π contains no contractible edges, $N = A\left(\overrightarrow{G(\Pi)}\right)$.

Let $B = B\left(\overrightarrow{G(\Pi)}\right)$ be the set of arcs of $\overrightarrow{G(\Pi)}$ of the form $\overrightarrow{uw}$, with w in the boundary of $CH(P)$, and let $\overrightarrow{uw} \in B$. By Remark 1, w has outdegree 0 in $\overrightarrow{G(\Pi)}$ which implies $\overrightarrow{uw} \notin \mathrm{Im}(f)$.

If $U = \emptyset$, then

$$\mathrm{Im}(f) \subset A\left(\overrightarrow{G(\Pi)}\right) \backslash B,$$

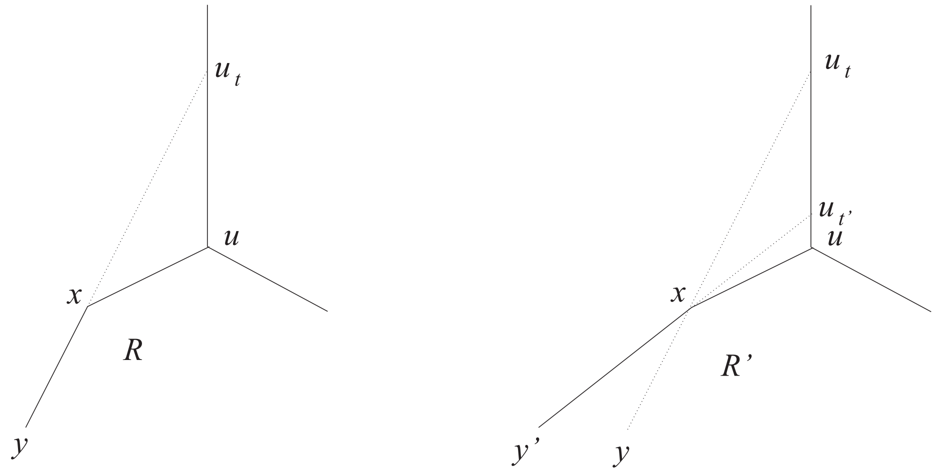

Fig. 5 Left: Ray $r(yx)$ meets edge uv at the point u_t. Right: Ray $r(y'x)$ meets edge uv at an interior point $u_{t'}$.

therefore

$$|N| = |\operatorname{Im}(f)| \leq \left|A\left(\overrightarrow{G(\Pi)}\right)\backslash B\right| \leq \left|A\left(\overrightarrow{G(\Pi)}\right)\right| - 3,$$

which is not possible since Π contains no deletable edges and $|B| \geq 3$.

And if $U \neq \emptyset$, by Lemma 3 no arc in $\operatorname{Im}(g)$ lies in $\operatorname{Im}(f)$, therefore

$$\operatorname{Im}(f) \subset A\left(\overrightarrow{G(\Pi)}\right)\backslash(\operatorname{Im}(g) \cup B).$$

In this case

$$|\operatorname{Im}(f)| \leq \left|A\left(\overrightarrow{G(\Pi)}\right)\right| - |\operatorname{Im}(g)| - |B|,$$

since $g(u) \notin B$. This is a contradiction since $A\left(\overrightarrow{G(\Pi)}\right) = N$, $|\operatorname{Im}(g)| = |U|$, $|B| \geq 3$ and $|\operatorname{Im}(f)| = |N| - |U|$. □

Corollary 5. *Let Π be a convex decomposition of a set of points P in general position in the plane. If P contains at least one interior point, then Π contains at least one contractible edge.*

Proof. Let Π' be a convex decomposition of P obtained from Π by removing deletable edges, one at a time, until no such edges remain, and let $\overrightarrow{G(\Pi')}$ be the corresponding directed abstract graph. Since P contains an interior point, Π' contains at least one interior edge.

By Theorem 4, there is an arc $\overrightarrow{uv} \in A\left(\overrightarrow{G(\Pi')}\right)$ such that uv is contractible from u to v in Π'. If uv is not contractible in Π, then by Lemma 1 there are edges yx and xu lying in a common face of $G(\Pi)$ such that the ray $r(yx)$ meets the edge uv at an interior point u_t and that the triangular region yu_tu contains no point of P in its interior. This implies that the geometric graph $G(\Pi) - xu$ contains a face Q_x in which x is a reflex vertex and therefore xu is not deletable in Π and $\overrightarrow{xu}$ is an arc of $\overrightarrow{G(\Pi)}$.

Let R be the face of $G(\Pi)$ which contains both edges yx and xu. Since Π' is obtained from Π by deleting edges but no points, then there is a face R' of $G(\Pi')$ which contains the edge xu and the region bounded by R, let $y' \in P$ be such that $y'x$ is an edge of R'. Notice that $y' \neq y$ otherwise uv could not be a contractible edge of Π' because the ray $r(yx)$ meets the edge uv at the point u_t (**Fig. 5**, left). Nevertheless, since the face R' contains the edge xu and the region bounded by R, the ray $r(y'x)$ also meets the edge uv at an interior point $u_{t'}$ (Fig. 5, right) which again is a contradiction. □

Corollary 6. *Let Π be a convex decomposition of a set of points P in general position in the plane and Q be the set of points in the boundary of $CH(P)$. There is a sequence $P = P_0, P_1, \ldots, P_m = Q$ of subsets of P, and a sequence $\Pi_0, \Pi_1, \ldots, \Pi_m$ of convex decompositions of $P_0, P_1, \ldots, P_m$, respectively, such that $\Pi_0 = \Pi$, Π_m consists of the boundary of $CH(P)$ and for $i = 0, 1, \ldots, k$, Π_{i+1} is obtained from Π_i by contracting an edge and for $i = k+1, k+2, \ldots, m-1$, Π_{i+1} is obtained from Π_i by deleting an edge.*

Proof. By Corollary 5, if P_i contains interior points, then Π_i has a contractible edge. If P_i contains no interior points, then each interior edge of Π_i is a deletable edge. □

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